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Multivalued monotone mappings are almost everywhere single-valued

by

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Abstract. Let E be a Banach space and let E' be its conjugate space. If $T: E \to E'$ is a maximal monotone (multivalued) mapping from E to its conjugate space E' with domain $D(T) = E$, then T is an upper semi-continuous multivalued mapping from E to E' endowed with the weak* topology. If in addition E' has an equivalent strongly convex norm, then the set $\{x \in E: T(x)$ has more than one element$\}$ is of the first category, i.e., it is a countable union of nowhere dense subsets. Thus "almost everywhere" in the title of the note means "except for a first category set". As an application another proof is given of the theorem of Asplund [2] that every continuous convex function defined on E is differentiable in the sense of Gâteaux almost everywhere.

Monotone mappings have been the subject of much research in the last few years. They were studied by many authors in different directions (for information see Browder [3], [4] and [5]). Our aim here is to prove the results announced in [7].

Let E be a Banach space and let E' be its conjugate (the set of all bounded linear functionals on E). The value of the functional $y \in E'$ at the point $x \in E$ will be denoted by $\langle x, y\rangle$. The subset G of $E \times E'$ is said to be a *monotone set* if, for each pair (x_1, y_1) and (x_2, y_2) in G, we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. Such a set is said to be a *maximal monotone* one if it is not properly contained in any other monotone subset of $E \times E'$. The multivalued mapping $T: E \to E'$ is said to be *monotone* (*maximal monotone*) if its graph $G(T) = \{(x, y) \in E \times E' : y \in T(x)\}$ is a monotone (maximal monotone) subset of $E \times E'$.

By using Zorn's lemma, it is not difficult to prove that every monotone set is contained in a maximal one. Thus, for every monotone mapping $T: E \to E'$, there exists a maximal monotone one, $\tilde{T}: E \to E'$, such that $T(x) \subset \tilde{T}(x)$ whenever $x \in E$.

In what follows we shall consider that $T(x) \neq \emptyset$ for all $x \in E$.

The following theorem will play an important role in our arguments.

THEOREM 1 (Rockafellar [11]). *Every maximal monotone mapping T: $E \to E'$ with $T(x) \neq \emptyset$ for all $x \in E$ is locally bounded, i.e., for every $x_0 \in E$,* *there exists an open* $V \ni x_0$ *such that* $T(V) = \bigcup (T(x)\colon x \in V)$ *is a bounded subset of* E'.

By $\sigma(E', E)$ we will denote as usual the weakest topology in E' with respect to which all elements of E are continuous (sometimes this topology is called weak* topology).

PROPOSITION 1. *The graph* $G(T) = \{(x, y) \in E \times E'\colon y \in T(x)\}$ *of every maximal monotone mapping is a closed subset of* $E \times (E', \sigma(E', E))$.

Proof. Let $(x_a, y_a) \in G(T)$ be a convergent net in $E \times (E', \sigma(E', E))$ with $\lim_a (x_a, y_a) = (x_0, y_0)$. By Theorem 1 we can assume $c = \sup \|y_a\| < \infty$. Then for $(x, y) \in E \times E'$ we get

$$|\langle x_a - x, y_a - y\rangle - \langle x_0 - x, y_0 - y\rangle| \leqslant |\langle x_a - x_0, y_a - y\rangle| + |\langle x_0 - x, y_a - y_0\rangle| \leqslant \|x_a - x_0\|(c + \|y\|) + |\langle x - x_0, y_a - y_0\rangle| \to 0.$$

Therefore $\langle x_0 - x, y_0 - y\rangle = \lim_a \langle x_a - x, y_a - y\rangle \geqslant 0$ for $(x, y) \in G(T)$, and $(x_0, y_0) \in G(T)$ by the maximality of T.

COROLLARY 1 (Browder [4]). *Let* $T\colon E \to E'$ *be a maximal monotone mapping with* $T(x) \neq \emptyset$ *whenever* $x \in E$. *Then, for every* $x \in E$, *the set* $T(x)$ *is* $\sigma(E', E)$*-compact and convex.*

Proof. $T(x)$ is $\sigma(E', E)$-closed (Proposition 1) and bounded (Theorem 1). Thus it is $\sigma(E', E)$-compact (Kelley and Namioka [6]). The convexity of $T(x)$ is a result of the maximality of T.

THEOREM 2. *Every maximal monotone mapping* $T\colon E \to (E', \sigma(E', E))$ *with* $T(x) \neq \emptyset$ *is upper semi-continuous, i.e., for every* $x_0 \in E$ *and every open* $U \supset T(x_0)$, *there exists an open neighbourhood* $V \ni x_0$ *such that* $T(x) \subset U$ *for all* $x \in V$.

Proof. Let V be such an open neighbourhood of $x_0 \in E$ that $T(V)$ is bounded (Theorem 1). Then the $\sigma(E', E)$-closure $\overline{T(V)}^{\sigma}$ of the set $T(V)$ is $\sigma(E', E)$-compact. On the other hand (after Proposition 1), the graph of the mapping $T\colon V \to (\overline{T(V)}^{\sigma}, \sigma(E', E))$ is $\sigma(E', E)$-closed. Such a mapping has to be upper semi-continuous at every point $x \in V$. This shows that the mapping T is upper semi-continuous. The proof is finished.

Let us now return to the Banach space $(E, \|\cdot\|)$. Denote by $\|y\|_{E'}$ the $\max_{\|x\| \leqslant 1} |\langle x, y\rangle|$, where y is an arbitrary element of E'. When there is no danger of ambiguity we will use $\|y\|$ instead of $\|y\|_{E'}$.

Let $T\colon E \to E'$ be a maximal monotone mapping with $T(x) \neq \emptyset$ for all $x \in E$. Put $f(x) = \inf(\|y\|\colon y \in T(x)) = \min(\|y\|\colon y \in T(x))$ (the last equality is due to the $\sigma(E', E)$-compactness of $T(x)$).

LEMMA 1. *For every real number* a, *the set* $\{x \in E\colon f(x) > a\}$ *is open.*

Proof. The set $A = \{y \in E'\colon \|y\| \leqslant a\}$ is $\sigma(E', E)$-closed and $E' \setminus A$ is $\sigma(E', E)$-open. Let us choose $x_0 \in E$ so that $f(x_0) > a$; then $T(x_0) \cap A = \emptyset$ and hence $T(x_0) \subset E' \setminus A$.

According to Theorem 2 an open $V \ni x_0$ exists such that $T(x) \subset E' \setminus A$ for every $x \in V$. Then for $x \in V$, $T(x) \cap A = \emptyset$ and $f(x) > a$. The lemma is proved.

Let us now set $H_n = \{x \in E\colon$ in every neighbourhood V of x; there exists a $y \in V$ such that $f(y) > f(x) + 1/n\}$, $n = 1, 2, 3, \ldots$, and $D = E \setminus \bigcup_{n=1}^{\infty} H_n$.

LEMMA 2. *The function* $f(x)$ *is continuous at every point* $x \in D$.

LEMMA 3. *The set* H_n *is nowhere dense.*

The proof of Lemma 2 is a trivial consequence of Lemma 1 and the definition of the set D. In order to prove Lemma 3 we will suppose the contrary: H_n is dense in an open $V \neq \emptyset$. Fix an $x_1 \in H_n \cap V$. By induction we will now construct a sequence $\{x_i\}_{i=1}^{\infty} \subset E$ such that it satisfies the following three conditions for $i = 1, 2, 3, \ldots$: 1) $x_i \in H_n \cap V$; 2) $f(x_{i+1}) > f(x_i) + 1/n$; 3) $\|x_{i+1} - x_i\| < 1/2^{i+1}$. Suppose that $x_1, x_2, x_3, \ldots, x_k$, satisfying 1), 2) and 3) have already been chosen. In particular, $x_k \in H_n \cap V$. Then, as follows from the definitions of the set H_n and the function $f(x)$, there exists a point $x^* \in V$ with $\|x^* - x_k\| < 1/2^{k+2}$ such that $f(x^*) > f(x_k) + 1/n$. Because of Lemma 1 the last inequality holds for every x from some open $V^* \ni x^*$, $V^* \subset V$. As the set H_n is dense in V, the intersection $V^* \cap H_n$ is not empty. Let us choose $x_{k+1} \in V^* \cap H_n$ in such a way that $\|x_{k+1} - x^*\| < 1/2^{k+2}$. Then $x_{k+1} \in V^* \cap H_n \subset V \cap H_n$; $f(x_{k+1}) > f(x_k) + 1/n$ $(x_{k+1} \in V^*)$ and $\|x_{k+1} - x_k\| \leqslant \|x_{k+1} - x^*\| + \|x^* - x_k\| < 1/2^{k+2} + 1/2^{k+2} = 1/2^{k+1}$. The required sequence is constructed. Condition 3) shows that it is a Cauchy sequence and hence it is convergent. Moreover, it follows from the construction that the sets $\bigcup_{i \geqslant n} T(x_i)$, $n = 1, 2, 3, \ldots$, are all unbounded, which contradicts Theorem 1.

THEOREM 3. *Let* $(E, \|\cdot\|)$ *be a Banach space with a strongly convex*(1) *dual norm* $\|\cdot\|_{E'}$ *and suppose that* $T\colon E \to E'$ *is a monotone set-valued mapping with* $T(x) \neq \emptyset$ *for every* $x \in E$. *Then the set* $\{x \in E\colon T(x)$ *has more than one element*$\}$ *is of the first category, i.e., it is a countable union of nowhere dense subsets.*

Proof. Without loss of generality we can consider that $T\colon E \to E'$ is a maximal monotone mapping. The proof will be finished if we prove that $T(x)$ is a single point set for all $x \in D = E' \setminus (\bigcup_{n=1}^{\infty} H_n)$. Let $x_0 \in D$ and

(1) I.e., if $y_1, y_2 \in E'$, $\|y_1\|_{E'} = \|y_2\|_{E'}$ and $y_1 \neq y_2$, then $\|(y_1 + y_2)/2\|_{E} < \|y_1\|_{E'} = \|y_2\|_{E'}$.

suppose that $y, \bar{y} \in T(x_0)$ $y \neq \bar{y}$; we can assume that $f(x_0) = \|y\|$. As $y \neq \bar{y}$, there is an $e \in E$ such that $\varepsilon = \langle e, y-\bar{y}\rangle > 0$. The sequence $\{x_n = x_0 + (1/n)e\}_{n=1}^{\infty} \to x_0 \in D$ and hence (Lemma 2) $f(x_n) \to f(x_0)$. Let us note that $f(x_n) = \|y_n\|$ for some $y_n \in T(x_n)$. As follows from the upper semi-continuity of $T: E \to (E', \sigma(E', E))$ and the $\sigma(E', E)$-compactness of the images $T(x)$, there is a $y_0 \in T(x_0)$ such that every $\sigma(E', E)$-open neighbourhood of y_0 contains infinitely many members of the sequence $\{y_n\}_{n=1}^{\infty}$ (this is a common property of upper semi-continuous mappings with compact images). Let $\delta > 0$; the set $A_\delta = \{\tilde{y} \in E': \|\tilde{y}\| \leqslant \|y\| + \delta\}$ is $\sigma(E', E)$-closed and it contains y_n when n is sufficiently large for $\|y_n\| = f(x_n) \to f(x_0) = \|y\|$. This shows that $y_0 \in A_\delta$, i.e., $\|y_0\| \leqslant \|y\| + \delta$. The last inequality is true for each $\delta > 0$; hence $\|y_0\| \leqslant \|y\| = f(x_0) = \min\{\|y\|: y \in T(x_0)\}$. On the other hand, the norm $\|\cdot\|_{E'}$ is a strongly convex function and it attains its minimum on the convex set $T(x_0)$ at only one element. Thus $y_0 = y$.

Further we will make use of the monotonicity of T:

For all $n = 1, 2, 3, \ldots$ we have $0 \geqslant n\langle x_n - x_0, \bar{y} - y_n\rangle = \langle e, \bar{y} - y_0\rangle + \langle e, y_0 - y_n\rangle = \varepsilon + \langle e, y_0 - y_n\rangle$. This is a contradiction, since $\varepsilon > 0$ and $\{y_n - y_0\}_{n \geqslant 1}$ has 0 as its $\sigma(E', E)$-cluster point.

A Banach space E is called *weakly compactly generated* if it has a $\sigma(E, E')$-compact subset $A \subset E$ such that the linear span $L(A)$ is dense in E. For example, every reflexive Banach space, as well as every separable Banach space, is weakly compactly generated.

THEOREM 4 (Amir and Lindenstrauss [1]). *Let E be a weakly compactly generated Banach space. Then it has an equivalent norm $\|\cdot\|$ such that $\|\cdot\|_{E'}$ is strongly convex.*

COROLLARY 2. *Let E be a weakly compactly generated Banach space and let $T: E \to E'$ be a monotone multivalued mapping with non-empty images $T(x)$. Then the set $\{x \in E: T(x)$ has more than one element$\}$ is of the first category.*

Let us now discuss the connection between monotone mappings and convex functions.

Suppose that $h: E \to R$ (where R is the usual real line) is a convex continuous function. It is well known that, for every $x_0 \in E$, there exists at least one $y_0 \in E'$ such that the inequality $h(x) - h(x_0) \geqslant \langle x - x_0, y_0\rangle$ holds for each $x \in E$. Assigning to each $x_0 \in E$ the non-empty set $\partial(x) = \{y \in E': h(x) - h(x_0) \geqslant \langle x - x_0, y\rangle$ whenever $x \in E\}$, we get a multivalued mapping $\partial: E \to E'$. This mapping is monotone. Indeed, if $y_i \in \partial(x_i)$, $i = 1, 2$, we have $h(x_1) - h(x_2) \geqslant \langle x_1 - x_2, y_2\rangle$ and $h(x_2) - h(x_1) \geqslant \langle x_2 - x_1, y_1\rangle = \langle x_1 - x_2, -y_1\rangle$. Adding these two inequalities, we obtain $0 \geqslant \langle x_1 - x_2, y_2 - y_1\rangle$. R. T. Rockafellar has shown [10] that ∂ is a maximal monotone mapping. This, combined with Theorem 2, enables us to state

COROLLARY 3 (Moreau [9]). *The set-valued mapping $\partial: E \to (E', \sigma(E', E))$ is upper semi-continuous.*

Furthermore, we would like to point out that the continuous convex function $h: E \to R$ is differentiable at the point $x_0 \in E$ in the sense of Gâteaux if the set $\partial(x_0)$ consists of only one element. Thus Theorem 3 implies.

THEOREM 5 (Asplund [2]). *Let E be as defined in Theorem 3 and suppose that $h: E \to R$ is a real continuous convex function. Then the set $\{x \in E$: h is not differentiable at x in the sense of Gâteaux$\}$ is a set of the first category.*

In the case of E being a separable Banach space we obtain the following result of Mazur:

COROLLARY 5 (Mazur [8]). *Let E be a separable Banach space and let $h: E \to R$ be a continuous convex function defined on the whole of E. Then the function $h(x)$ is differentiable in the sense of Gâteaux at every point of E except for the points of a first category set.*

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