For every positive integer n, and every n-tuple $\sigma = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$, let $x^{(n,\sigma)}$ denote the sequence whose ith term is given by

$$x_i^{(n,\sigma)} = \begin{cases} \dfrac{n}{n+1}\varepsilon_i & \text{if} \quad 1 \leqslant i \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The norm of the difference of any two different such sequences is at least $\frac{2}{3}$, and thus the collection A of all such sequences is a closed set in the open unit ball. Since $\sup f(A) = \|f\|$ for every f in c_0^*, $\overline{\text{co}}(A)$ is the closed unit ball of c_0.

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Received January 7, 1975 (925)

STUDIA MATHEMATICA, T. LVI. (1976)

The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group

by

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Abstract. Let a particle perform a Brownian motion $X(t)$ in $\boldsymbol{R}^n$ and $Y(t)$ be another Brownian motion in $\boldsymbol{R}^n$ independent of $X(t)$ interpreted as a random constant field of forces in $\boldsymbol{R}^n$. The joint distribution of X, Y, and the energy E produced by the motion is calculated by interpreting $(X(t), Y(t), E(t))$ as a process on the Heisenberg group G connected with a subelliptic operator on G.

In [4] and [2] a construction of a semigroup of functions p_t, $t > 0$, associated with a subelliptic operator on a Lie group was given. The aim of this note is to point out the role of the functions p_t in random walks on a Lie group which in turn may arise in a very elementary physical problem. The link of the p_t's to a subelliptic operator on the general Heisenberg group enables us to compute the p_t explicitly. This apart from an apparent physical interest shows that certain subelliptic operator on $\boldsymbol{R}^{2d+1}$ is analytic-hypoelliptic.

The author is grateful to Professor H. McKean for a conversation in which he convinced the author that the explicit formula for the p_t's can be obtained though by a different method than the one applied here, as well as to Professor Palle Jørgensen for illuminating remarks concerning analytic-hypoellipticity.

1. Introduction. Let G be a connected Lie group, $U = U^{-1}$ an open neighbourhood of the identity in G such that the exponential map exp: $LG \to G$ is a local diffeomorphism onto U. Let $g_1, \ldots, g_k$ be a set of generators of G, i.e., the smallest analytic subgroup containing $g_1, \ldots, g_k$ is G, or, in other words, if $g_j = \exp X_j$, $j = 1, \ldots, k$, the smallest Lie subalgebra containing $X_1, \ldots, X_k$ is LG. Suppose that μ is a symmetric probability measure equally distributed on $g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}$ and let for $t > 0$ and $n = 1, 2, \ldots$ the probability measure $\mu_{t,n}$ of G be defined by

$$\mu_{t,n}(M) = s(2k)^{-1}, \quad \text{if } s \text{ of } \exp\left[\pm\sqrt{\frac{kt}{n}}\, X_j\right] \in M.$$

* Research partly supported by National Sciences Foundation Grant P038489

It can be easily deduced from [3] that

$$\lim_{n\to\infty} \langle \mu_{t,n}^{*n}, f\rangle = \langle P_t, f\rangle$$

for every continuous function f on G vanishing at infinity, where P_t, $t > 0$ is a semigroup of measures whose infinitesimal generator is

$$\tfrac{1}{2}(X_1^2 + \ldots + X_k^2) = \mathscr{L}.$$

Since $X_1, \ldots, X_k$ generate LG as a Lie algebra, it follows from either [4] or [2] that

$$P_t(M) = \int_M p_t(g)\,dg,$$

where p_t is a C^∞ function and is the fundamental solution of the equation

$$(1)\qquad \left(\frac{\partial}{\partial t} - \mathscr{L}\right) u(t, x) = 0 \quad \text{with } u(0, x) = f(x),\ f \in L^\infty(G), \quad \text{i.e.,} \quad u(t, x) = p_t * f(x).$$

Let $Y(t)$ be a d-dimensional Brownian motion which we interpret as a random force in $\boldsymbol{R}^d$. Suppose a particle performs a Brownian motion $X(t)$ in $\boldsymbol{R}^d$, independent of the force $Y(t)$ and we are interested in the joint distribution of X, Y and the energy E produced by the motion of the particle starting at the time $t = 0$. Clearly enough $E(t)$ is given by the Itô's integral

$$E(t) = \int_0^t Y(s)\,dX(s)$$

and this could be a starting point for our considerations, but the following simple discretisation of the problem makes the theory of Itô's integral unnecessary for our purposes and, on the other hand, shows the relation of the problem to a random walk on the Heisenberg group.

Consider the phase space $\boldsymbol{R}^d \times \boldsymbol{R}^d \times \boldsymbol{R}$, where the coordinates of a point $(\boldsymbol{x}, \boldsymbol{y}, z)$ in the phase space describe the position of the particle, the force and the energy produced, respectively.

Let for a positive number a the position of the particle and the force vary independently and the probability of the transition

$$\boldsymbol{x} \to \boldsymbol{x}', \quad \boldsymbol{y} \to \boldsymbol{y}'$$

be equal to $(4d)^{-1}$ if either

$$\boldsymbol{x}' = \boldsymbol{x} \pm a\boldsymbol{e}_j \quad \text{and} \quad \boldsymbol{y}' = \boldsymbol{y},$$

or

$$\boldsymbol{x}' = \boldsymbol{x} \quad \text{and} \quad \boldsymbol{y}' = \boldsymbol{y} \pm a\boldsymbol{e}_j,$$

where $\boldsymbol{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ in $\boldsymbol{R}^d$, $j = 1, \ldots, d$.

In the first case the energy produced is equal to $\pm a\boldsymbol{e}_j \cdot \boldsymbol{y}$, and in the second case it is equal to zero. This describes a random walk in $\boldsymbol{R}^d \times \boldsymbol{R}^d \times \boldsymbol{R}$ with the transition probability

$$P_a(\boldsymbol{x}, \boldsymbol{y}, z \mid \boldsymbol{x}', \boldsymbol{y}', z') = \begin{cases} (4d)^{-1} & \text{if } \boldsymbol{x}' - \boldsymbol{x} = \pm a\boldsymbol{e}_j,\ \boldsymbol{y}' - \boldsymbol{y} = 0 \text{ and } z' - z = \pm a\boldsymbol{e}_j \cdot \boldsymbol{y}, \\ & \quad \text{or } \boldsymbol{x}' - \boldsymbol{x} = 0,\ \boldsymbol{y}' - \boldsymbol{y} = \pm a\boldsymbol{e}_j \text{ and } z' - z = 0, \\ & \quad j = 1, \ldots, d, \\ 0 & \text{otherwise.} \end{cases}$$

Now let us identify $\boldsymbol{R}^d \times \boldsymbol{R}^d \times \boldsymbol{R}$ with the Heisenberg group G, the multiplication being defined by

$$(2)\qquad gg' = (\boldsymbol{x}, \boldsymbol{y}, z)(\boldsymbol{x}', \boldsymbol{y}', z') = (\boldsymbol{x} + \boldsymbol{x}', \boldsymbol{y} + \boldsymbol{y}', z + z' + \boldsymbol{x}\boldsymbol{y}').$$

Then the unit element e in G is $(\boldsymbol{0}, \boldsymbol{0}, 0)$ and we see that

$$P_a(g \mid g') = P_a(gg'^{-1} \mid e).$$

Now it is clear that if μ_a is the measure defined by

$$\mu_a(M) = \sum_{g \in M} P_a(g \mid e),$$

then the probability of reaching a set M from e in n consecutive steps is $\mu_a^{*n}(M)$ (where $*$ denotes the convolution on the Heisenberg group G). We see that the measure μ_a is equally distributed on the elements $(\pm a\boldsymbol{e}_j, \boldsymbol{0}, 0)$, $(\boldsymbol{0}, \pm a\boldsymbol{e}_j, 0)$, $j = 1, \ldots, d$, and that the Lie algebra of G is generated by $X_1, \ldots, X_d, Y_1, \ldots, Y_d$, where

$$\exp X_j = (\boldsymbol{e}_j, \boldsymbol{0}, 0), \quad \exp Y_j = (\boldsymbol{0}, \boldsymbol{e}_j, 0), \quad j = 1, \ldots, d,$$
$$[X_i, Y_j] = \delta_{ij} Z, \quad \text{with} \quad \exp Z = (\boldsymbol{0}, \boldsymbol{0}, 1).$$

Thus, by what we have said above,

$$\lim_{n\to\infty} \langle \mu_{\sqrt{2dt/n}}^{*n}, f\rangle = \int f(\boldsymbol{x}, \boldsymbol{y}, z)\, p_t(\boldsymbol{x}, \boldsymbol{y}, z)\, d\boldsymbol{x}\,d\boldsymbol{y}\,dz,$$

where p_t is the fundamental solution of (1) with

$$(3)\qquad \mathscr{L} = \tfrac{1}{2} \sum_{j=1}^{d} (X_j^2 + Y_j^2).$$

On the other hand, it is clear that the limit of the joint distribution of the position of the particle and the random force at a moment t has the density

$$\Phi_t(\boldsymbol{x}, \boldsymbol{y}) = \int_{-\infty}^{+\infty} p_t(\boldsymbol{x}, \boldsymbol{y}, z)\,dz = (2\pi t)^{-d} \exp\left[\frac{-|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2}{2t}\right].$$

2. Explicit calculation. In this section we shall find the Fourier transform of the fundamental solution of (1) on the Heisenberg group with $\mathscr{L}$ being defined by (3).

As is well known (cf. e.g. [6]) the group G has the following series of representations T^λ, $\lambda \in \boldsymbol{R} \setminus \{0\}$. For a $g = (\boldsymbol{x}, \boldsymbol{y}, z) \in G$ and a function f on $\boldsymbol{R}^d$ we write

$$(T_g^\lambda f)(\xi) = \exp[i\lambda(\boldsymbol{y}\cdot\xi - z)] f(\xi - \boldsymbol{x}).$$

Of course, T_g^λ is a linear operator on the space of bounded uniformly continuous functions on $\boldsymbol{R}^d$ and it is easy to verify that $T_g T_h = T_{gh}$. Since $p_t \in L^1(G)$, we may write

$$(4) \qquad (T_{p_t}^\lambda f)(\xi) = \int p_t(g)(T_g^\lambda f)(\xi)\,dg = \int p_t(\boldsymbol{x}, \boldsymbol{y}, z)\exp[i\lambda(\boldsymbol{y}\cdot\xi - z)] f(\xi - \boldsymbol{x})\,d\boldsymbol{x}\,d\boldsymbol{y}\,dz = \int p_t(\boldsymbol{x}, -\hat{\lambda}\xi, \hat{\lambda}) f(\xi - \boldsymbol{x})\,d\boldsymbol{x},$$

where for a function $p(\boldsymbol{x}, \boldsymbol{y}, z)$ on $\boldsymbol{R}^d \times \boldsymbol{R}^d \times \boldsymbol{R}$ we write

$$p(\boldsymbol{x}, \hat{\boldsymbol{\beta}}, z) = \int p(\boldsymbol{x}, \boldsymbol{y}, z)\exp[-i\boldsymbol{\beta}\cdot\boldsymbol{y}]\,d\boldsymbol{y}$$

and similarly with respect to other variables.

For a fixed λ let $d_\lambda = \partial T^\lambda$. We then have

$$d_\lambda X_j = -\frac{\partial}{\partial \xi_j}, \qquad d_\lambda Y_j = i\lambda\xi_j,$$

and so

$$d_\lambda \mathscr{L} = L^\lambda = \frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial^2}{\partial \xi_j^2} - \lambda^2 \xi_j^2\right).$$

It is easy to verify that then

$$(5) \qquad L^\lambda [T_{p_t}^\lambda f](\xi) = \frac{\partial}{\partial t}[T_{p_t}^\lambda f](\xi).$$

Let, for an $\boldsymbol{a}$ in $\boldsymbol{R}^d$, $f(\xi) = \exp[i\boldsymbol{a}\cdot\xi]$. Then, by (4),

$$[T_{p_t}^\lambda f](\xi) = \exp[i\boldsymbol{a}\cdot\xi]\hat{p}_t(\boldsymbol{a}, -\lambda\xi, \lambda).$$

Let $\boldsymbol{\beta} = -\lambda\xi$; then, if

$$L_{\beta}^\lambda = \sum_{j=1}^{d}\left(\lambda^2\frac{\partial^2}{\partial \beta_j^2} - \beta_j^2\right)$$

and

$$(6) \qquad u(t, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \hat{p}_t(\boldsymbol{a}, \boldsymbol{\beta}, \lambda)\exp[-i\lambda^{-1}\boldsymbol{a}\cdot\boldsymbol{\beta}],$$

then, by (5),

$$(7) \qquad L_\beta^\lambda u(t, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \frac{\partial}{\partial t}u(t, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) \quad \text{with} \quad u(0, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \exp[-i\lambda^{-1}\boldsymbol{a}\cdot\boldsymbol{\beta}].$$

For $m = 0, 1, 2, \ldots$, let

$$\varphi_m(x) = \exp\left[\frac{x^2}{2}\right](-1)^m \frac{d^m}{dx^m}\exp[-x^2],$$

$$\psi_m(x) = (2^m m!\sqrt{\pi})^{-1/2}\varphi_m(x),$$

$$\psi_m^\lambda(x) = |\lambda|^{-1/4}\psi_m(x|\lambda|^{-1/2}).$$

Then, in virtue of [7], p. 76, we verify that

$$\left(\lambda^2\frac{d^2}{dx^2} - x^2\right)\psi_m^\lambda(x) = -|\lambda|(2m+1)\psi_m^\lambda(x).$$

Consequently, if $\boldsymbol{m} = (m_1, \ldots, m_d)$, $m_j \geqslant 0$, $|\boldsymbol{m}| = \sum_{j=1}^{d} m_j$, $\boldsymbol{x} = (x_1, \ldots, x_d)$ and

$$\psi_{\boldsymbol{m}}^\lambda(\boldsymbol{x}) = \psi_{m_1}^\lambda(x_1)\ldots\psi_{m_d}^\lambda(x_d),$$

then

$$L_x^\lambda\psi_{\boldsymbol{m}}^\lambda(\boldsymbol{x}) = -|\lambda|(2|\boldsymbol{m}|+d)\psi_{\boldsymbol{m}}^\lambda(\boldsymbol{x}).$$

On the other hand, by an easy change of variable in [7], p. 78, we get

$$\int_{\boldsymbol{R}^d}\psi_{\boldsymbol{k}}^\lambda(\boldsymbol{x})\psi_{\boldsymbol{l}}^\lambda(\boldsymbol{x})\,d\boldsymbol{x} = \delta_{\boldsymbol{k},\boldsymbol{l}}.$$

Now let

$$a_{\boldsymbol{m}} = \int_{\boldsymbol{R}^d}\exp[-i\lambda^{-1}\boldsymbol{a}\cdot\boldsymbol{\beta}]\psi_{\boldsymbol{m}}^\lambda(\boldsymbol{\beta})\,d\boldsymbol{\beta} = \prod_{j=1}^{d}\int_{-\infty}^{+\infty}\exp[-i\lambda^{-1}a_j\beta_j]\psi_{m_j}^\lambda(\beta_j)\,d\beta_j = \prod_{j=1}^{d} i^{m_j}(2\pi)^{1/2}|\lambda|^{1/4}\psi_{m_j}(-a_j\sqrt{|\lambda|}\lambda^{-1}) = (2\pi)^{d/2}|\lambda|^{d/4}i^{|\boldsymbol{m}|}\psi_{\boldsymbol{m}}(-\sqrt{|\lambda|}\lambda^{-1}\boldsymbol{a}),$$

cf. [7], p. 81. By an easy regularization process, (7) yields

$$u(t, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \exp[tL_\beta^\lambda]\exp[-i\lambda^{-1}\boldsymbol{a}\cdot\boldsymbol{\beta}],$$

that is,

$$u(t, \boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \sum_{\boldsymbol{m}} a_{\boldsymbol{m}}\exp[-t|\lambda|(2|\boldsymbol{m}|+d)]\psi_{\boldsymbol{m}}^\lambda(\boldsymbol{\beta}) = (2\pi)^{d/2}e^{-t|\lambda|d}\sum_{\boldsymbol{m}} i^{|\boldsymbol{m}|}e^{-2t|\lambda||\boldsymbol{m}|}\psi_{\boldsymbol{m}}\left(-\frac{\sqrt{|\lambda|}}{\lambda}\boldsymbol{a}\right)\psi_{\boldsymbol{m}}\left(\frac{1}{\sqrt{|\lambda|}}\boldsymbol{\beta}\right).$$

Thus, by (6),

$$\hat{p}_t(\boldsymbol{a}, \boldsymbol{\beta}, \lambda) = (2\pi)^{d/2} \prod_{j=1}^{d} \sigma_j, \tag{8}$$

where

$$\sigma_j = e^{-t|\lambda|} \sum_{m=0}^{\infty} i^m e^{-2t|\lambda| m} \psi_m\left(-\frac{\sqrt{|\lambda|}}{\lambda}\,a_j\right) \psi_m\left(\frac{1}{\sqrt{|\lambda|}}\,\beta_j\right) \exp[a_j\beta_j\lambda^{-1}].$$

If

$$s = e^{-2t|\lambda|}, \quad x = -\frac{\sqrt{|\lambda|}}{\lambda}\,a_j, \quad y = \frac{1}{\sqrt{|\lambda|}}\,\beta_j, \tag{9}$$

then, by [7], pp. 77–78,

$$\sigma_j = s^{1/2} e^{ixy} \sum_{m=0}^{\infty} (is)^m \psi_m(x)\psi_m(y)$$
$$= \left[\frac{s}{\pi(1+s^2)}\right]^{1/2} \exp\left[\tfrac{1}{2}(x^2-y^2) - \frac{(x-isy)^2}{1+s^2} + ixy\right],$$

whence, putting back abbreviations (9), we obtain

$$\sigma_j = (2\pi\cosh 2t\lambda)^{-1/2} \exp\left[\frac{-\frac{1}{2}(a_j^2+\beta_j^2)\sinh 2t\lambda + i a_j\beta_j(\sinh t\lambda)^2}{2\lambda\cosh 2t}\right]$$

which yields our final formula

(10)
$$\hat{p}_t(\boldsymbol{a}, \boldsymbol{\beta}, \lambda) = (\cosh 2t\lambda)^{-d/2} \exp\left[\frac{-\frac{1}{2}(|\boldsymbol{a}|^2+|\boldsymbol{\beta}|^2)\sinh 2t\lambda + i\boldsymbol{a}\cdot\boldsymbol{\beta}(\sinh t\lambda)^2}{2\lambda\cosh 2t\lambda}\right].$$

It is clear that even though the formula above was proved under the assumption $\lambda \neq 0$, it still holds for $\lambda = 0$, since

$$\lim_{\lambda\to 0} \hat{p}_t(\boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \exp[-\tfrac{1}{2}(|\boldsymbol{a}|^2+|\boldsymbol{\beta}|^2)t],$$

which, of course, is the characteristic function of the joint distribution of the position of the particle and the force (both Gaussian and independent).

Thus we have proved the following

THEOREM 1. *If $G = \boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$ is the Heisenberg group, $\mathscr{L}$ is defined by* (3), *then the Fourier transform of the fundamental solution p_t of equation* (1) *is given by* (10). *This can also be interpreted as the characteristic function of the joint distribution at the moment t of the position of the particle, the force, and the energy in the Brownian motion in the Gaussian field of forces as described in the introduction.*

Analytic-hypoellipticity. Now we are going to show that (10) implies that $p_t(\boldsymbol{x}, \boldsymbol{y}, z)$ is real analytic in $\boldsymbol{R}^+\times\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$. This is an immediate consequence of the following

THEOREM 2. *There is an open neighbourhood U of $\boldsymbol{R}^+\times\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$ in $\boldsymbol{C}\times\boldsymbol{C}^d\times\boldsymbol{C}^d\times\boldsymbol{C}$ such that $p_t(\boldsymbol{x}, \boldsymbol{y}, z)$ is the restriction to $\boldsymbol{R}^+\times\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$ of a holomorphic function $p_\tau(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta)$ in U.*

Proof. Let for a $\tau = t+is$, $t > 0$,

$$f(\tau, a, \beta, \lambda) = (\cosh\tau\lambda)^{-1/2} \exp\left[\frac{-\frac{1}{2}(a^2+\beta^2)\sinh\tau\lambda + 2ia\beta(\sinh\frac{1}{2}\tau\lambda)^2}{\lambda\cosh\tau\lambda}\right].$$

For fixed a, β, λ, since $\cosh z = 0$ implies $\operatorname{Re} z = 0$, $f(\tau, a, \beta, \lambda)$ is holomorphic in $\{\tau\colon \operatorname{Re}\tau > 0\}$. We also have

$$\hat{p}_\tau(\boldsymbol{a}, \boldsymbol{\beta}, \lambda) = \prod_{j=1}^{d} f(\tau, a_j, \beta_j, \lambda). \tag{11}$$

If $a = r\cos\frac{1}{2}\theta$, $\beta = r\sin\frac{1}{2}\theta$, $\eta = \sin\theta$, then

$$f(\tau, a, \beta, \lambda) = (\cosh\tau\lambda)^{-1/2} \exp\left[-\frac{r^2}{2}\cdot\frac{\sinh\tau\lambda - 2i\eta(\sinh\frac{1}{2}\tau\lambda)^2}{\lambda\cosh\tau\lambda}\right].$$

We are going to evaluate $\operatorname{Re} M$, where

$$M = \frac{\sinh\tau\lambda - 2i\eta(\sinh\frac{1}{2}\tau\lambda)^2}{\lambda\cosh\tau\lambda},$$

from below as a function of τ and λ.

By routine transformation we get

$$M = \frac{2}{\lambda}\,[1+(\cosh\tfrac{1}{2}\tau\lambda)^2(\sinh\tfrac{1}{2}\tau\lambda)^{-2}]^{-1}\cdot[\cosh\tfrac{1}{2}\tau\lambda(\sinh\tfrac{1}{2}\tau\lambda)^{-1} - i\eta]$$
$$= \frac{2(A-i\eta)}{\lambda(1+A^2)},$$

where

$$A = \cosh\frac{\lambda(t+is)}{2}\left[\sinh\left(\frac{\lambda(t+is)}{2}\right)\right]^{-1} = \frac{\sinh t\lambda - i\sin s\lambda}{\cosh t\lambda - \cos s\lambda}.$$

We select a branch for $\operatorname{Arg} z$ and if, say, $|s| < \frac{1}{2}t$, we have $|\operatorname{Arg} A| < c < \frac{1}{2}\pi$ for all such s, t and arbitrary real λ. Hence, since $-1 \leqslant \eta \leqslant 1$, for suitable choice of $\pm$ depending on A

$$|\operatorname{Arg} A| = \left|\operatorname{Arg}\frac{A-i\eta}{1+A^2}\right| = \left|\operatorname{Arg}\frac{1}{1+A^2} + \operatorname{Arg}(A-i\eta)\right|$$
$$\leqslant \left|\operatorname{Arg}\frac{1}{1+A^2} + \operatorname{Arg}(A\pm i)\right| = \left|\operatorname{Arg}\frac{A\pm i}{1+A^2}\right| = \operatorname{Arg}\left|\frac{1}{A\pm t}\right| < c < \tfrac{1}{2}\pi.$$

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Hence

$$\operatorname{Re} M = \frac{2}{\lambda}\cdot\frac{1}{1+A^2} \geqslant \frac{2}{\lambda} A \geqslant c(\lambda)t,$$

where $c(\lambda) > 0$ if only $|s| < \frac{1}{2}t$.

On the other hand,

$$|\cosh \tau\lambda|^2 = \cos^2 s\lambda + \sinh^2 t\lambda \geqslant \tfrac{1}{2}\cosh^2 t\lambda > \tfrac{1}{2}e^{t|\lambda|},$$

if only $|s| < \frac{1}{2}t$.

Consequently, for $|s| < \frac{1}{2}t$

$$|\hat{p}_\tau(\boldsymbol{\alpha}, \boldsymbol{\beta}, \lambda)| \leqslant 2^d \exp\left\{-\frac{|\boldsymbol{r}|^2}{2}c(\lambda)t - |\lambda|\,dt\right\},$$

where

$$|\boldsymbol{r}|^2 = \sum_{j=1}^{d} \alpha_j^2 + \beta_j^2.$$

Let

$$U = \{(\tau, \xi, \boldsymbol{\eta}, \zeta) \in \boldsymbol{C}\times\boldsymbol{C}^d\times\boldsymbol{C}^d\times\boldsymbol{C}\colon |\operatorname{Im}\tau| < \tfrac{1}{2}\operatorname{Re}\tau,\ |\operatorname{Im}\xi| < \tfrac{1}{2}d\operatorname{Re}\tau\}.$$

Then, for a $(\tau, \xi, \boldsymbol{\eta}, \zeta) \in U$, we have

$$|\hat{p}_\tau(\boldsymbol{\alpha}, \boldsymbol{\beta}, \lambda)\exp i[\boldsymbol{\alpha}\cdot\xi + \boldsymbol{\beta}\cdot\boldsymbol{\eta} + \lambda\zeta]|$$
$$\leqslant 2^d \exp\left\{-\frac{|\boldsymbol{r}|^2}{2}\operatorname{Re}\tau\, c(\lambda) + \operatorname{Re}\{\boldsymbol{\alpha}\cdot\xi + \boldsymbol{\beta}\cdot\boldsymbol{\eta}\} - |\lambda|\frac{d\operatorname{Re}\tau}{2}\right\},$$

and consequently the integral

$$(2\pi)^{-2d-1}\int\limits_{\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}} \hat{p}_\tau(\boldsymbol{\alpha}, \boldsymbol{\beta}, \lambda)\exp i[\boldsymbol{\alpha}\cdot\xi + \boldsymbol{\beta}\cdot\boldsymbol{\eta} + \lambda\zeta]\,d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\,d\lambda$$

is absolutely convergent to a holomorphic function $p_\tau(\xi, \boldsymbol{\eta}, \zeta)$, whose restriction to $\boldsymbol{R}^+\times\boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}$ is $p_t(\boldsymbol{x}, \boldsymbol{y}, z)$.

3. Remarks. The following remarks concerning Theorem 2 are due to Professor Palle Jørgensen.

It follows immediately from Theorem 2 that if

$$\partial = \sum_{j=1}^{d} X_j^2 + Y_j^2 = \sum_{j=1}^{d} D_{x_j}^2 + D_{y_j}^2 + (y_j D_z)^2 + 2y_j D_{x_j z} - D_t$$

then $\partial u(\boldsymbol{x}, \boldsymbol{y}, z, t) = 0$ for $\boldsymbol{x}, \boldsymbol{y}, z, t \in \boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}\times\boldsymbol{R}^+$ an for a p, $1 \leqslant p \leqslant \infty$, $\|u(\cdot, t)\|_p < c$ for all $t \in \boldsymbol{R}^+$, then $u(\boldsymbol{x}, \boldsymbol{y}, z, t)$ is real analytic. This should be compared with an example due to M. S. Baouendi and C. Goulaouic [1]. They show that the sublaplacian

$$D_x^2 + D_y^2 + (yD_z)^2$$

is not analytic-hypoelliptic. The vector fields $X = D_x$, $Y = D_y$, $Z = yD_z$ generate a nilpotent Lie algebra, the corresponding group being the product of the Heisenberg group and the real line.

On the other hand T. Matsuzawa showed [5] that $D_x^2 + (xDy)^2$ is analytic-hypoelliptic.

Added in proof. 1. After this paper was submitted for publication B. Gaveau gave an alternative proof of formula (10). This makes use of Paul Lévy formula for $\int_0^s YdX - XdY$, where $X(t)$, $Y(t)$ are independent Brownian motions [cf. B. Gaveau, C. R. Acad. Sc. Paris, 280 (Mars 1975), pp. 571-573].

2. The fact that $\mathscr{L}$ alone is analytic-hypoelliptic can be also deduced from the explicite formula for the solution of $\mathscr{L}u = \sigma_e$ as G. B. Folland has shown to the author in a letter.

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Received January 8, 1975 (930)