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Approximation of commutators of singular integrals

by

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Abstract. We consider approximations of singular integral operators of type $\beta > 1$ and show that certain associated operators satisfy a discrete smoothing property. We also consider approximations of the corresponding continuous smoothing operators, and show that the discrete smoothing operators lead to error estimates stronger than those given by the elementary approximation results. These estimates are suggested by the continuous and discrete smoothing properties of the operators.

1. Introduction. In this paper we continue our investigation of approximations of singular integral operators. We give discrete smoothing results for approximation of operators of type β which are analogous to the theorems on smoothing operators proved by Calderón. We also prove error estimates for approximation of these smoothing operators. Our methods are based on an extension of a theorem of Calderón and Zygmund [2] on boundedness of an operator with a discrete kernel, on some L^p approximation theory using Fourier multipliers, and on our earlier work on approximation of translation invariant operators (see [7] and [8]).

In Section 2 we present notation and define the approximation of a translation invariant operator on $L^p(\boldsymbol{R}^n)$. In Section 3 we apply a result of Rivière in order to obtain the desired extension of a theorem of Calderón and Zygmund. In Section 4 we apply the result of Section 3 in order to prove the discrete smoothing properties associated with operators of type β. The technique is similar to that in the continuous case but becomes more computational, and seems to require stronger assumptions on the original singular integral operators.

In Section 5 we give an interesting error estimate for approximation of the expressions

$$D_j(A^*-A^{\#})u,\quad D_j(AB-A\circ B)u,\quad AAu-AAu.$$

We show for example that for u in H_1^p,

$$\|D_j(A^*-A^{\#})u-\partial_{h,j}(A_h^*-A_h^{\#})u\|_p = O(h^{\varkappa})$$

where $\varkappa$ is (almost) the fraction part of β and $\partial_{h,j}$ is a difference operator. On the other hand, the approximation theorems of [8] show that for u in H_1^p,

$$\|D_j(A^* - A^{\#})u - D_j(A_h^* - A_h^{\#})u\|_p = O(1).$$

Certain restrictions on the operators A and B appear to be necessary for these extended error estimates. The proofs become quite technical but we cannot see any way to avoid this.

2. Notation and definitions. We shall define the Fourier transform $u\hat{}$ of a test function u by

$$u\hat{}(\xi) = (2\pi)^{-n/2}\int u(x)e^{-i\langle x,\xi\rangle}\,dx.$$

$u\check{}$ is the inverse Fourier transform of u. We shall use the standard definitions and notations for multi-indices, differential operators, L^p and l^p spaces, and the Sobolev spaces H_M^p. We denote the norm of u in L^p by $\|u\|_p$ and in l^p by $\|u\|_{l^p}$. The norm in H_M^p is

$$\|u\|_{p,M} = \sum_{|\alpha|\leqslant M}\|D^\alpha u\|_p.$$

The seminorm in H_M^p is written

$$|u|_{p,M} = \sum_{|\alpha| = M}\|D^\alpha u\|_p.$$

We shall often work with the sets

$$\Sigma = \{\xi \in \boldsymbol{R}^n:\ |\xi| = 1\},$$

$$Q' = \{\xi \in \boldsymbol{R}^n:\ -\pi \leqslant \xi_j < \pi,\ j = 1, \ldots, n\},$$

$$Q = \{x \in \boldsymbol{R}^n:\ -1 < 2x_j < 1,\ j = 1, \ldots, n\}.$$

We shall write $\|u\|_{p,\Omega}$ for the norm of u in $L^p(\Omega)$.

We assume that the reader is familiar with standard results on singular integral operators (to save space we shall write SIO hereafter) and Fourier multipliers. The facts we shall use implicitly may be found in [1], [3], [4], and [5]. For earlier results on approximation of SIO's we refer to [7] and [8]. We shall only define the approximation method here. Let η be a fixed C^∞ function which has support in Q' and is one in a neighborhood of the origin. If A is a translation invariant operator on L^p, then

$$Au = T * u$$

for some tempered distribution T. For $h > 0$ define $T_h\hat{}$ to be periodic with period $2\pi/h$ and for $h\xi$ in Q',

$$T_h\hat{}(\xi) = \eta(h\xi)T\hat{}(\xi).$$

Define

$$A_h u = T_h * u.$$

If we assume $T\hat{}$ is positively homogeneous of degree 0, then

$$A_h u(x) = \sum_{\mu\in \boldsymbol{Z}^n} a_\mu u(x - \mu h)$$

where

$$a_\mu = (2\pi)^{-n/2}\int \eta(\xi)T\hat{}(\xi)e^{i\langle\mu,\xi\rangle}\,d\xi.$$

3. Discrete singular operators. Using a result of Rivière [6] for singular integrals on locally compact abelian groups, we shall show that a Calderón–Zygmund kernel truncated near the origin and at infinity, and restricted to the integers, determines a bounded convolution operator on l^p and L^p, $1 < p < \infty$. If the kernel were not truncated, this result would follow from a theorem of Calderón and Zygmund ([2], Section 8). We also obtain boundedness of the operator with kernel truncated only near the origin.

For $\alpha \geqslant 0$ define

$$U_\alpha = \{\mu \in \boldsymbol{Z}^n:\ -\alpha \leqslant \mu_j \leqslant \alpha,\ j = 1, \ldots, n\}$$

and $U'_\alpha = \boldsymbol{Z}^n \setminus U_\alpha$. It is easy to see that $\{U_\alpha, \Phi\}$, where $\Phi(\alpha) = 2\alpha$, is a regular Vitali family as defined in [6], Section 3.

In order to simplify the notation we shall write

$$\Sigma^\gamma f(\mu), \qquad \Sigma^\gamma_\alpha f(\mu)$$

to denote the sum of $f(\mu)$ over μ in U_γ and μ in $U'_\alpha \cap U_\gamma$, respectively. We allow $\gamma = \infty$ and write $U_\infty = \boldsymbol{Z}^n$.

A singular kernel for the regular Vitali family $\{U_\alpha, 2\alpha\}$ is a function k on $\boldsymbol{Z}^n$ satisfying the conditions (3.1) and (3.2).

(3.1) *k is summable over every bounded subset of $\boldsymbol{Z}^n$ which does not contain the origin. Moreover, the sum*

$$\Sigma^\gamma_\alpha f(\mu)$$

is uniformly bounded for $\gamma > \alpha > 0$, and for each γ, its limit exists as $\alpha \to 0$.

(3.2) *The sum $\Sigma^{2\alpha}_\alpha |k(\mu)|$ is bounded uniformly in α.*

Theorem 3.1 is the version of Theorem 4.1 of [6] which we shall apply.

THEOREM 3.1. *Let k be a singular kernel for the regular Vitali family $\{U_\alpha, 2\alpha\}$. Assume that*

(3.3) $$\Sigma^\infty_{2\beta}|k(\mu - \nu) - k(\mu)| \leqslant C$$

for all ν in U_β, uniformly in β. Let $k_{\alpha,\gamma}$ be the function k on $U'_\alpha \cap U_\gamma$ and zero elsewhere. Define

$$K_{\alpha,\gamma}u(x) = k_{\alpha,\gamma} * u(x) = \Sigma^\gamma_\alpha k(\mu)u(x - \mu).$$

Then for $1 < p < \infty$ *and* u *in* l^p,

$$\|K_{a,\gamma} u\|_{l^p} \leqslant C_p \|u\|_{l^p}$$

and $C_p \leqslant C[p+1/(p-1)]$ *where* C *depends only on the uniform bounds in the definition of regular Vitali family, in* (3.1), (3.2), *and in* (3.3).

THEOREM 3.2. *Let* k *be a function on* $\boldsymbol{R}^n$ *which is* C^1 *except at the origin, positively homogeneous of degree* $-n$, *and has mean value* 0 *over* Σ. *Define* $k(0) = 0$ *and consider* k *to be a function defined on* $\boldsymbol{Z}^n$. *Then* k *is a singular kernel for the regular Vitali family* $\{U_a, 2a\}$. *Also,* (3.3) *is satisfied. Thus with* $K_{a,\gamma}$ *as defined in Theorem* 3.1, *we have for* $1 < p < \infty$,

$$\|K_{a,\gamma} u\|_{l^p} \leqslant C_p \|u\|_{l^p}, \qquad u \in l^p.$$

Also

$$C_p \leqslant C[p+1/(p-1)]\Big(\|k\|_{2,\Sigma} + \sum_{j=1}^{n} \|D_j k\|_{\infty,\Sigma}\Big). \tag{3.4}$$

Proof. In order to show that (3.1), (3.2), and (3.3) are satisfied, it suffices to consider U_a with $a = m+\frac{1}{2}$, m a non-negative integer. Define

$$V_a = U_a + Q = \{x = \mu + y\colon \mu \in U_a,\ y \in Q\}$$

and $V'_a = \boldsymbol{R}^n \setminus V_a$. Define the operator E by $Eu(x) = u(\mu)$ for x in $\mu+Q$. Let $\int_a^\gamma f(x)\,dx$ denote the integral of f over $V'_a \cap V_\gamma$. Set $V_\infty = \boldsymbol{R}^n$. Then

$$\Sigma_a^\gamma k(\mu) = \int_a^\gamma [Ek(x) - k(x)]\,dx + \int_a^\gamma k(x)\,dx. \tag{3.5}$$

The assumptions on k imply that

$$\Big|\int_a^\gamma k(x)\,dx\Big| \leqslant C\|k\|_{2,\Sigma} \tag{3.6}$$

with C independent of a and γ. Using Taylor's formula, we write

$$k(x) = Ek(x) + Rk(\mu, x)$$

for x in $\mu+Q$. Obviously,

$$\|Rk(\mu,\cdot)\|_{1,\mu+Q} \leqslant C|\mu|^{-n-1} \sum_{j=1}^{n} \|D_j k\|_{\infty,\Sigma}. \tag{3.7}$$

(3.1) now follows from (3.5), (3.6), and (3.7). The same method shows that (3.2) is satisfied. Let $\nu \in U_\beta$ and write

$$\Sigma_{2\beta}^\infty |k(\mu-\nu) - k(\mu)| \leqslant I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{2\beta}^\infty |Ek(x-\nu) - k(x-\nu)|\,dx,$$

$$I_2 = \int_{2\beta}^\infty |Ek(x) - k(x)|\,dx,$$

$$I_3 = \int_{2\beta}^\infty |k(x-\nu) - k(x)|\,dx.$$

(3.3) follows from trivial estimates for each of I_1, I_2, and I_3. The result now is an immediate consequence of Theorem 3.1.

Remark 3.3. Let k be as in Theorem 3.2 and for u in C_0^∞ and $h > 0$, define

$$K_{a,\gamma,h} u(x) = \Sigma_a^\gamma k(\mu) u(x - \mu h).$$

In view of Theorem 3.2, we have for $1 < p < \infty$, $h > 0$, and for all u in C_0^∞,

$$\|K_{a,\gamma,h} u\|_p \leqslant C_p \|u\|_p$$

with C_p as in (3.4).

Note that the series $K_{a,\gamma,h} u$ is absolutely convergent almost everywhere with bound independent of γ. Thus the limit as $\gamma \to \infty$ exists, and we write

$$K_{a,\infty,h} u(x) = \Sigma_a^\infty k(\mu) u(x - \mu h).$$

We immediately obtain

COROLLARY 3.4. *Let* k *be as in Theorem* 3.2. *Then for* $1 < p < \infty$, $h > 0$, *and for all* u *in* C_0^∞, *we have*

$$\|K_{a,\infty,h} u\|_p \leqslant C_p \|u\|_p$$

with C_p *as in* (3.4).

4. Discrete smoothing operators. We shall prove that the commutator of two SIO's of type β satisfies a discrete smoothing property similar to the smoothing property for operators with continuous kernels. The same result holds for discrete adjoints, and we prove boundedness of the operator $A_h \Lambda_h - \Lambda_h A_h$. We begin by summarizing the notation and properties of spherical harmonics which we need. The proofs may be found in [1] and [5]. Let $\{Y_{lm}\}$ be a complete orthonormal system of real spherical harmonics for $L^2(\Sigma)$. The positive integer m is the degree of Y_{lm} and the number of harmonics of degree m is $O(m^{n-2})$. Y_{lm} is C^∞ except at the origin and is positively homogeneous of degree 0. For each index α there is a constant C_α independent of the family $\{Y_{lm}\}$ such that

$$|D^\alpha Y_{lm}(\xi)| \leqslant C_\alpha m^{|\alpha|+(n-2)/2} |\xi|^{-|\alpha|}, \tag{4.1}$$

$$|D^\alpha [Y_{lm}(z)|z|^{-n}]| \leqslant C_\alpha m^{|\alpha|+(n-2)/2} |z|^{-n-|\alpha|}. \tag{4.2}$$

Also if p.v. denotes principal value, then

$$[\text{p.v. } Y_{lm}(\cdot)|\cdot|^{-n}]^{\wedge}(\xi) = \gamma_m Y_{lm}(\xi) \tag{4.3}$$

where

$$\gamma_m = i^m c_\pi \Gamma(m/2)\,[\Gamma((n+m)/2)]^{-1},$$

c_π is a real constant depending on π and n, and Γ is the gamma function. Thus

$$|\gamma_m| \leqslant Cm^{-n/2}, \qquad |\gamma_m^{-1}| \leqslant Cm^{n/2}. \tag{4.4}$$

We turn now to functions in B_β and operators of type β. Let $\beta > 0$ We say that f is in B_β provided

$$\sup |D^\alpha f(x)| < \infty$$

and

$$\sup |y|^{[\beta]-\beta} |D^\alpha f(x+y) - D^\alpha f(x)| < \infty,$$

where the first sup is over x in $\boldsymbol{R}^n$, $|\alpha| \leqslant [\beta]$, and the second sup is over $y \neq 0$, x in $\boldsymbol{R}^n$, and $|\alpha| = [\beta]$. We denote by $\|f\|_\beta$ the maximum of these suprema.

Consider an operator A defined by

$$Au(x) = a(x)u(x) + \text{p.v.} \int k(x, x-y)u(y)\,dy$$

for u in C_0^∞, a in B_β, and k having the following properties. For each x in $\boldsymbol{R}^n$, $k(x, z)$ is C^∞ in z except at the origin, positively homogeneous of degree $-n$ in z, and has mean value 0 over Σ. Let $T^{\wedge}(x, \xi)$ denote the Fourier transform in the second variable of p.v. $k(x, z)$. We write

$$\sigma(A)(x, \xi) = a(x) + (2\pi)^{n/2} T^{\wedge}(x, \xi)$$

and call $\sigma(A)$ the *symbol* of A.

Let N be the least even integer greater than $5n/2$. We say that A is an *operator of type* β provided that for each index α with $|\alpha| \leqslant N$ and for each ξ in Σ, the function $D_\xi^\alpha \sigma(A)$ is in B_β in x. We define

$$\|A\|_\beta = \sup \|D_\xi^\alpha \sigma(A)(\cdot, \xi)\|_\beta,$$

where the sup is over ξ in Σ and $|\alpha| \leqslant N$. Calderón used $N = 2n$ to obtain results on the boundedness of A on H_m^p and to study pseudoproducts and pseudoadjoints of operators of type β. Our method of proof of the boundedness of

$$\partial_{h,j}[(A_h B_h - A_h \circ B_h)u]$$

and

$$\partial_{h,j}[(A_h^* - A_h^{\#})u]$$

seems to require the choice of N above. However, we shall see that for boundedness of

$$A_h A_h - A_h A_h,$$

it is sufficient to choose N such that $N+1 > 3n/2$.

The series representations of Au and $\sigma(A)(x, \xi)$ with respect to $\{Y_{lm}\}$ will be written as

$$\begin{aligned} Au(x) &= \sum_{l;m\geqslant 0} a_{lm}(x) R_{lm} u(x), \\ \sigma(A)(x, \xi) &= a(x) + \sum_{l;m\geqslant 1} b_{lm}(x)\sigma(R_{lm})(\xi), \end{aligned} \tag{4.5}$$

where R_{lm} has kernel $Y_{lm}(z)|z|^{-n}$,

$$\sigma(R_{lm})(\xi) = (2\pi)^{n/2}\gamma_m Y_{lm}(\xi), \tag{4.6}$$

and the coefficients in (4.5) satisfy $a_{00} = a$, $\|a\|_\beta \leqslant C\|A\|_\beta$, and for $m \geqslant 1$,

$$\|a_{lm}\|_\beta \leqslant Cm^{-N+n/2}\|A\|_\beta, \qquad a_{lm} = \gamma_m^{-1} b_{lm}. \tag{4.7}$$

Also we have

$$\|R_{lm}\| \leqslant C_p, \tag{4.8}$$

where $\|R_{lm}\|$ denotes the operator norm of R_{lm} on L^p.

The approximation A_h is constructed as in [8] using the smooth function η. Thus

$$\begin{aligned} A_h u(x) &= [\sigma(A_h)(x, \cdot)u^{\wedge}]^{\vee}(x), \\ \sigma(A_h)(x, \xi) &= a(x) + (2\pi)^{n/2} T_h^{\wedge}(x, \xi), \end{aligned}$$

and $T_h^{\wedge}(x, \xi)$ is periodic in ξ with period $2\pi/h$, and for $h\xi$ in Q',

$$T_h^{\wedge}(x, \xi) = \eta(h\xi) T^{\wedge}(x, \xi).$$

It is easy to see that

$$A_h u(x) = \sum_{l;m\geqslant 0} a_{lm}(x) R_{lmh} u(x) \tag{4.9}$$

and

$$\sigma(A_h)(x, \xi) = a(x) + \sum_{l;m\geqslant 1} b_{lm}(x)\sigma(R_{lmh})(\xi).$$

Also the operators R_{lmh} and A_h have coefficients independent of h. By Theorem 3.1 of [8], we have

$$\|R_{lmh}\| \leqslant C_p. \tag{4.10}$$

We prepare for the discrete smoothing theorems by proving some technical lemmas. For simplicity in proofs we shall write R for R_{lm} and R_h for R_{lmh}. C denotes a constant which does not depend on l, m, h, or any function in B_β.

LEMMA 4.1. *Let K be an integer greater than n. There is a constant C_K such that for μ in $\mathbf{Z}^n$ and $\mu \neq 0$,*

$$\left|\text{p.v.}\int [\eta(\xi)-1]\sigma(R_{lm})(\xi)e^{i\langle\mu,\xi\rangle}d\xi\right| \leqslant C_K m^{K-1}|\mu|^{-K}. \tag{4.11}$$

Proof. Let φ be a C^∞ function such that

$$\operatorname{supp}(\varphi) \subset \{\xi \in \mathbf{R}^n:\ 2^{-1} < |\xi| < 2\}$$

and

$$\sum_{k=-\infty}^{\infty} \varphi_k(\xi) = 1, \qquad \xi \neq 0,$$

where $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Clearly,

$$|D^\alpha \varphi_k(\xi)| \leqslant C_\alpha 2^{-k|\alpha|}.$$

If $\eta(\xi) \neq 1$, then

$$\sum_{k=0}^{\infty} \varphi_k(\xi) = 1.$$

Let α be an index with $|\alpha| = K$ and consider

$$F_k(\mu) = \int \varphi_k(\xi)[\eta(\xi)-1]\sigma(R)(\xi)D^\alpha e^{i\langle\mu,\xi\rangle}d\xi.$$

Using integration by parts, (4.6), (4.4), and (4.1), we see that

$$|F_k(\mu)| \leqslant C_K m^{K-1}2^{k(n-K)}.$$

Since

$$\mu^\alpha \text{p.v.}\int [\eta(\xi)-1]\sigma(R)(\xi)e^{i\langle\mu,\xi\rangle}d\xi = \sum_{k=0}^{\infty} F_k(\mu),$$

the result follows immediately.

The next lemmas and those in Section 5 deal with an arbitrary function a in B_β. We assume $\beta > 1$. Then

$$|a(x)-a(x-y)| \leqslant n\|a\|_\beta |y|, \tag{4.12}$$

$$a(x)-a(x-y) = i\sum_{k=1}^{n} y_k D_k a(x) + b(x,y), \tag{4.13}$$

where with $\delta = \min(2, \beta)$,

$$|b(x,y)| \leqslant C\|a\|_\beta |y|^\delta. \tag{4.14}$$

$\partial_{h,j}$ will represent either the forward difference operator,

$$\partial_{h,j}u(x) = (ih)^{-1}[u(x+he_j)-u(x)]$$

or the symmetric difference operator,

$$\partial_{h,j}u(x) = (2ih)^{-1}[u(x+he_j)-u(x-he_j)],$$

where δ_{ij} is the Kronecker delta and $e_j = (\delta_{1j}, \ldots, \delta_{nj})$. $\tau_{h,j}$ represents the translation operator appearing in Leibniz's formula for $\partial_{h,j}(uv)$.

LEMMA 4.2. *There is a constant C such that for all u in C_0^∞ we have*

$$\|\partial_{h,j}[(aR_{lmh}-R_{lmh}a)u]\|_p \leqslant C\|a\|_\beta m^{n+1}\|u\|_p.$$

Proof. We shall consider only the forward difference. Let

$$Y(z) = Y_{lm}(z)|z|^{-n}$$

be the kernel of R. Since R_h has coefficients independent of h, we have

$$R_h u(x) = \sum_{\mu\in\mathbf{Z}^n} a_\mu u(x-\mu h)$$

where

$$a_\mu = (2\pi)^{-n/2}\int \eta(\xi)Y^\wedge(\xi)e^{i\langle\mu,\xi\rangle}d\xi.$$

In view of Lemma 4.1, we may write

$$a_\mu = Y(\mu)+F(\mu)$$

where $Y(0) = 0$, $F(0) = a_0$, and for $\mu \neq 0$,

$$F(\mu) = (2\pi)^{-n/2}\text{p.v.}\int [\eta(\xi)-1]Y^\wedge(\xi)e^{i\langle\mu,\xi\rangle}d\xi.$$

Then $R_h u = R_{hd}u + F_h u$, where

$$R_{hd}u(x) = \sum_\mu Y(\mu)u(x-\mu h) \quad \text{and} \quad F_h u(x) = \sum_\mu F(\mu)u(x-\mu h).$$

It is obvious that

$$\partial_{h,j}[(aR_h-R_h a)u] = (\partial_{h,j}a)R_h\tau_{h,j}u + J_1 + J_2,$$

where

$$J_1 = aR_{hd}\partial_{h,j}u - R_{hd}\partial_{h,j}(au),$$
$$J_2 = aF_h\partial_{h,j}u - F_h\partial_{h,j}(au).$$

Using (4.10), we see that

$$\|(\partial_{h,j}a)R_h\tau_{h,j}u\|_p \leqslant C\|a\|_\beta\|u\|_p.$$

It remains to estimate J_1 and J_2. Write

$$\partial_j F(\mu) = -i[F(\mu+e_j)-F(\mu)].$$

Then it is clear that

$$J_2(x) = h^{-1}\sum_\mu \partial_j F(\mu)[a(x)-a(x-\mu h)]u(x-\mu h).$$

Applying (4.6), (4.4), (4.1), and Lemma 4.1 with $K = n+2$, we obtain

$$\|J_2\|_p \leqslant C\|a\|_\beta m^{n+1}\|u\|_p.$$

We turn now to the more technical estimate for J_1. Write

$$\partial_j Y(\mu) = -i[Y(\mu+e_j)-Y(\mu)].$$

Then

$$J_1(x) = h^{-1}\sum_{\mu} \partial_j Y(\mu)\,[a(x)-a(x-\mu h)]u(x-\mu h). \tag{4.15}$$

We shall split J_1 into several pieces, one of which will involve a discrete singular operator. Recall the definition of Σ_α^γ from Section 3. Writing $t = h^{-1}$, we have

$$J_1 = J_{11}+J_{12}+J_{13}$$

where

$$J_{11}(x) = h^{-1}\Sigma^1\,\partial_j Y(\mu)\,[a(x)-a(x-\mu h)]u(x-\mu h),$$
$$J_{12}(x) = h^{-1}\Sigma_1^t\,\partial_j Y(\mu)\,[a(x)-a(x-\mu h)]u(x-\mu h),$$
$$J_{13}(x) = h^{-1}\Sigma_t^\infty\,\partial_j Y(\mu)\,[a(x)-a(x-\mu h)]u(x-\mu h).$$

J_{11} and J_{13} are easily estimated using (4.2) and (4.12). It remains only to estimate J_{12}. Using (4.13), we write

$$J_{12} = J_{121}+J_{122}+J_{123}$$

where

$$J_{121}(x) = i\sum_{k=1}^{n} D_k a(x)\,\Sigma_1^t\,\mu_k D_j Y(\mu)u(x-\mu h),$$
$$J_{122}(x) = i\sum_{k=1}^{n} D_k a(x)\,\Sigma_1^t\,\mu_k[\partial_j Y(\mu)-D_j Y(\mu)]u(x-\mu h),$$
$$J_{123}(x) = h^{-1}\Sigma_1^t\,\partial_j Y(\mu)b(x,\mu h)u(x-\mu h).$$

J_{122} and J_{123} are easily estimated using (4.2) and (4.14). Since $z_k D_j Y(z)$ is C^∞ except at the origin, positively homogeneous of degree $-n$, and has mean value 0 over Σ, we may apply Remark 3.3 and (4.2), and we obtain

$$\|J_{121}\|_p \leqslant C\|a\|_\beta m^{1+n/2}\|u\|_p.$$

This completes the proof of the lemma.

LEMMA 4.3. *There is a constant C such that for $0\leqslant M\leqslant[\beta-1]$, $1\leqslant j\leqslant n$, for $h>0$, and for all u in C_0^∞, we have*

$$\|\partial_{h,j}[(aR_{lmh}-R_{lmh}a)u]\|_{p,M} \leqslant C\|a\|_\beta m^{n+1}\|u\|_{p,M}.$$

Proof. The case $M=0$ is proved in Lemma 4.2. The general result is easily proved by induction.

We assume hereafter that A and B are operators of type β. We use the standard definitions of the adjoint A^* and pseudoadjoint $A^\#$ of A, the product AB and the pseudoproduct $A\circ B$ of A and B, the Riesz transform R_j, and the operator

$$\Lambda = \sum_{j=1}^{n} R_j D_j.$$

See [1] or [5] for these definitions. The discrete approximations A_h^*, $A_h^\#$, and $A_h\circ B_h$ are defined in [8]. We define

$$\Lambda_h = \sum_{j=1}^{n} R_{jh}\partial_{h,j}.$$

For future reference we list the series expansions of some of these operators. u^- denotes the complex conjugate of u.

$$A = \sum_{l,m} a_{lm}R_{lm},\qquad B = \sum_{l,m} b_{lm}R_{lm},$$
$$A^* = \sum(-1)^m R_{lm}a_{lm}^-,\qquad A^\# = \sum(-1)^m a_{lm}^- R_{lm},$$
$$AB = \sum a_{lm}R_{lm}b_{st}R_{st},\qquad A\circ B = \sum a_{lm}b_{st}R_{lm}R_{st}.$$

Similar expansions are valid for the discrete analogs of all these operators.

The discrete smoothing properties of the approximation operators are given in the next theorem.

THEOREM 4.4. *There is a constant C depending only on p, n, and β such that*

$$\|\partial_{h,j}(A_h^*u-A_h^\#u)\|_{p,M} \leqslant C\|A\|_\beta\|u\|_{p,M}, \tag{4.16}$$
$$\|\partial_{h,j}(A_hB_hu-A_h\circ B_hu)\|_{p,M} \leqslant C\|A\|_\beta\|B\|_\beta\|u\|_{p,M}, \tag{4.17}$$
$$\|A_h\Lambda_hu-\Lambda_hA_hu\|_{p,M} \leqslant C\|A\|_\beta\|u\|_{p,M}, \tag{4.18}$$

for $0\leqslant M\leqslant[\beta-1]$, $1\leqslant j\leqslant n$, $h>0$, and for all u in C_0^∞.

Proof. In order to prove (4.16), we write

$$A_h^*u-A_h^\#u = \sum_{l,m}(-1)^m(R_{lmh}a_{lmh}^- - a_{lm}^- R_{lmh})u.$$

Applying Lemma 4.3, we obtain

$$\|\partial_{h,j}(A_h^*u-A_h^\#u)\|_{p,M} \leqslant C\sum_{l,m} m^{n+1}\|a_{lm}\|_\beta\|u\|_{p,M}.$$

Since the number of harmonics of degree m is $O(m^{n-2})$, (4.16) follows from (4.7) and the choice of N. The proof of (4.17) is similar and will be omitted.

In order to prove (4.18), we write

$$A_h\Lambda_hu-\Lambda_hA_hu = \sum_{j=1}^{n}\sum_{l,m}\partial_{h,j}[(a_{lm}R_{jh}-R_{jh}a_{lm})R_{lmh}u] - \sum_{j=1}^{n}\sum_{l,m}(\partial_{h,j}a_{lm})R_{jh}R_{lmh}\tau_{h,j}u.$$

The second term on the right may be estimated using (4.7) and (4.10). In order to estimate the first term, we note that R_j has kernel $z_j|z|^{-n-1}$

and $z_j|z|^{-1}$ is a spherical harmonic of degree one. Applying Lemma 4.3 and (4.10), we see that the first sum is dominated by

$$\sum_{l,m} \|a_{lm}\|_\beta \|u\|_{p,M}.$$

It follows from (4.7) that the first sum converges if N is the least even integer such that $N+1 > 3n/2$.

5. Extended rates of convergence. In this section we shall investigate approximation of the derivatives of smoothing operators associated with operators of type β, and also the operator $AA-AA$. It follows easily from the approximation theorems in [8] that

$$\|D^\alpha[(A^*-A^\#)u-(A_h^*-A_h^\#)u]\|_p = O(1)$$

for $|\alpha| = M \leqslant [\beta]$, $h>0$, and u in H_M^p. In other words, there is no rate of convergence for highest derivatives of the approximation of the smoothing operator $A^*-A^\#$. A similar statement holds for $AB-A\circ B$.

We write $\beta = \beta_z+\varkappa$, where $0<\varkappa\leqslant 1$ and β_z is a positive integer. Loosely speaking, we shall show that

$$\|D_j(A^*-A^\#)u-\partial_{h,j}(A_h^*-A_h^\#)u\|_{p,M-1} = O(h^\varkappa)$$

as $h\to 0$. A similar result will hold for $AB-A\circ B$. In each case there will be slight restrictions on A and B. This result is suggested by the smoothing properties of the SIO's and the discrete smoothing properties established in Section 4.

We assume that if $0<\varkappa<1$ then $\partial_{h,j}$ represents the forward difference and the integer N in the definition of operator of type β is the least even integer greater than $5n/2$. If $\varkappa = 1$, then $\partial_{h,j}$ represents the symmetric difference and N is the least even integer greater than $1+5n/2$. These technical changes are necessary at only two points in the proofs. We shall assume that $\varkappa<1$ and only indicate at the appropriate places the reasons for the changes. As in Section 4, we first work with a single term in the series expansions of the operators, and use this to obtain the general results.

We shall use some estimates for the remainder term $b(x,y)$ in the expansion (4.13) of the function a. Recall that δ is the minimum of 2 and β. For μ in Z^n and y in Q we have

$$b(x,\mu h+hy)-b(x,\mu h) = a(x-\mu h)-a(x-\mu h-hy)-ih\sum_{k=1}^n y_k D_k a(x).$$

It follows that

$$b(x,\mu h+hy)-b(x,\mu h) = ih\sum_{k=1}^n y_k[D_k a(x-\mu h)-D_k a(x)]+b(x-\mu h, hy).$$

Thus

$$|b(x,\mu h+hy)-b(x,\mu h)| \leqslant C\|a\|_\beta h, \tag{5.1}$$

$$|b(x,\mu h+hy)-b(x,\mu h)| \leqslant C\|a\|_\beta h^\delta|\mu|^{\delta-1}, \tag{5.2}$$

and

$$\Big|\int_Q [b(x,\mu h+hy)-b(x,\mu h)]\,dy\Big| \leqslant C\|a\|_\beta h^\delta. \tag{5.3}$$

Lemma 5.1. *There is a constant C such that if $m\neq 2$ then for all u in C_0^∞ and for $0<h\leqslant 1$, we have*

$$\|D_j[(aR_{lm}-R_{lm}a)u]-\partial_{h,j}[(aR_{lmh}-R_{lmh}a)u]\|_p \leqslant C\|a\|_\beta m^{[\delta]+n}h^{\delta-1}\|u\|_{p,1}. \tag{5.4}$$

A necessary and sufficient condition that (5.4) be valid with $m = 2$ is that for all harmonics Y_{l2},

$$\sum_{k=1}^n D_k a(x)\int_\Sigma Y_{l2}(\xi)\,\xi_k\xi_j\,d\sigma = 0. \tag{5.5}$$

Proof. We write

$$D_j[(aR-Ra)u]-\partial_{h,j}[(aR_h-R_h a)u] = E+H,$$

where

$$E = (D_j a)Ru-(\partial_{h,j}a)R_h\tau_{h,j}u,$$

$$H = [aRD_ju-RD_j(au)]-[aR_h\partial_{h,j}u-R_h\partial_{h,j}(au)].$$

We can estimate E easily. We have

$$E = (D_ja)(R-R_h)u+(D_ja)R(u-\tau_{h,j}u)+(D_ja-\partial_{h,j}a)R_h\tau_{h,j}u.$$

Using (4.8), (4.10), and the method of Theorem 4.1 of [8], we see that

$$\|E\|_p \leqslant C\|a\|_\beta h^{\delta-1}\|u\|_{p,1}.$$

It remains to show that we can dominate H in L^p by the right-hand side in (5.4) if and only if (5.5) holds. This process is rather long and we shall divide it into several steps.

First step. We let R^ε have kernel, the kernel of R truncated at distance ε from the origin, and write

$$H^\varepsilon = [aR^\varepsilon D_ju-R^\varepsilon D_j(au)]-[aR_h\partial_{h,j}u-R_h\partial_{h,j}(au)]. \tag{5.6}$$

Let K_{jk} be the SIO with kernel $y_kD_jY(y)$ and K_{jk}^ε the SIO with the same kernel truncated at distance ε from the origin. It follows from ([5], Part I, Chapter IV, Theorem 7) that K_{jk} has operator norm on L^p

$$\|K_{jk}\| \leqslant C\|y_kD_jY\|_{2,\Sigma} \leqslant Cm^{n/2}. \tag{5.7}$$

Also

$$\sigma(K_{jk}) = -D_k[\xi_j\sigma(R)]. \tag{5.8}$$

We consider the continuous terms on the right in (5.6). Using integration by parts, we see that

$$(5.9)\quad [aR^\varepsilon D_j u - R^\varepsilon D_j(au)](x)$$

$$= i\sum_{k=1}^{n} D_k a(x) K_{jk}^\varepsilon u(x) + \int_{|y|>\varepsilon} D_j Y(y) b(x,y) u(x-y)\,dy - iB_j^\varepsilon u(x),$$

where $B_j^\varepsilon u$ is the boundary term,

$$B_j^\varepsilon u(x) = \varepsilon^{-1}\int_\Sigma y_j Y(y)\,[a(x)-a(x-\varepsilon y)]u(x-\varepsilon y)\,d\sigma.$$

Using (4.13) and Taylor's formula, we see that in L^p,

$$B_j^\varepsilon u(x) = u(x)\sum_{k=1}^{n} D_k a(x)\int_\Sigma y_k y_j Y(y)\,d\sigma + O(\varepsilon).$$

It follows from the decomposition theorem for homogeneous polynomials in terms of spherical harmonics ([5], Part II, Chapter III, Theorem 3.3) that if $m \neq 2$ then

$$\int_\Sigma y_j y_k Y(y)\,d\sigma = 0$$

and thus $B_j^\varepsilon u \to 0$ in L^p as $\varepsilon \to 0$. We shall show that the limits (as $\varepsilon\to 0$) of the other continuous terms on the right in (5.9) are approximated by the discrete terms in (5.6) for all m. Thus for (5.4) to hold, it is necessary that $B_j^\varepsilon u \to 0$ in L^p as $\varepsilon \to 0$, and hence (5.5) is necessary if $m = 2$. Since $H^\varepsilon \to H$ in L^p, we have shown that

$$(5.10)\quad H = i\sum_{k=1}^{n} D_k a(x) K_{jk} u(x) - [aR_h\partial_{h,j}u - R_h\partial_{h,j}(au)](x) + \\ + \int D_j Y(y) b(x,y) u(x-y)\,dy.$$

Note that it follows from standard theorems on SIO's that the last integral exists in L^p. This completes the first step of the proof.

Second step. We shall consider the discrete terms in (5.10) and focus our attention on approximation of each K_{jk}. It is easy to see that

$$(5.11)\quad [aR_h\partial_{h,j}u - R_h\partial_{h,j}(au)](x)$$

$$= i\sum_{k=1}^{n} D_k a(x)\sum_\mu \mu_k\partial_j a_\mu u(x-\mu h) + h^{-1}\sum_\mu \partial_j a_\mu b(x,\mu h)u(x-\mu h),$$

where $\partial_j a_\mu = -i(a_{\mu+e_j} - a_\mu)$ and the a_μ are the coefficients in R_h. In the third step we shall see that the last sum in (5.11) is an approximation to the last integral in (5.10).

An easy calculation shows that

$$(5.12)\quad \mu_k\partial_j a_\mu = a_{jk\mu} + z_{jk\mu},$$

where

$$a_{jk\mu} = (2\pi)^{-n/2}\int \eta(\xi) D_k[i(e^{i\xi_j}-1)\sigma(R)](\xi)e^{i\langle\mu,\xi\rangle}\,d\xi,$$

$$z_{jk\mu} = (2\pi)^{-n/2}\int D_k\eta(\xi)\,i(e^{i\xi_j}-1)\sigma(R)(\xi)e^{i\langle\mu,\xi\rangle}\,d\xi.$$

Consider the functions ω_{jk} and z_{jk} defined by

$$\omega_{jk}(\xi) = \eta(\xi)D_k[i(e^{i\xi_j}-1)\sigma(R)](\xi),$$

$$z_{jk}(\xi) = D_k\eta(\xi)\,i(e^{i\xi_j}-1)\sigma(R)(\xi).$$

Since η is in C_0^∞, it follows from (4.8) and (5.7) that ω_{jk} and z_{jk} are Fourier multipliers with norms bounded by $C_p m^{n/2}$ and C_p, respectively, $1 < p < \infty$. Define K_{jkh} and Z_{jkh} to be the discrete operators whose symbols $\sigma(K_{jkh})$ and $\sigma(Z_{jkh})$ are periodic with period $2\pi/h$ and for $h\xi$ in Q',

$$\sigma(K_{jkh})(\xi) = \omega_{jk}(h\xi),$$

$$\sigma(Z_{jkh})(\xi) = z_{jk}(h\xi).$$

It follows from Theorems 2.1 and 2.3 of [4] that $\sigma(K_{jkh})$ and $\sigma(Z_{jkh})$ are Fourier multipliers with norms bounded by $C_p m^{n/2}$ and C_p, respectively, $1 < p < \infty$. Using (5.11) and (5.12), we see that

$$[aR_h\partial_{h,j}u - R_h\partial_{h,j}(au)](x)$$

$$= i\sum_{k=1}^{n}[D_k a(K_{jkh}u + Z_{jkh}u)](x) + h^{-1}\sum_\mu \partial_j a_\mu b(x,\mu h)u(x-\mu h).$$

Inserting this into (5.10), we obtain

$$(5.13)\quad H = i\sum_{k=1}^{n}[D_k a(K_{jk}u - K_{jkh}u - Z_{jkh}u)](x) + \\ + \int D_j Y(y)b(x,y)u(x-y)\,dy - h^{-1}\sum_\mu \partial_j a_\mu b(x,\mu h)u(x-\mu h).$$

We conclude this step by estimating $\|K_{jk}u - K_{jkh}u\|$ and $\|Z_{jkh}u\|$. First, we write

$$(5.14)\quad [(K_{jk} - K_{jkh})u]^\wedge \\ = h\Psi(h\xi)(D_j u)^\wedge + [1-\eta(h\xi)][\sigma(K_{jk}) - \sigma(K_{jkh})](h\xi)u^\wedge,$$

where

$$\Psi(\xi) = \eta(\xi)\{-[i(e^{i\xi_j}-1)+\xi_j]\xi_j^{-2}\xi_j D_k\sigma(R)(\xi) + \\ + i\delta_{jk}(e^{i\xi_j}-1)\xi_j^{-1}\sigma(R)(\xi)\}.$$

Since $\eta(\xi)[i(e^{i\xi_j}-1)+\xi_j]\xi_j^{-2}$ and $\eta(\xi)(e^{i\xi_j}-1)\xi_j^{-1}$ are both in C_0^∞, it follows from (5.7) and (4.8) that Ψ is a Fourier multiplier with norm bounded by $C_p m^{n/2}$, $1 < p < \infty$. Using this fact, and applying the method of proof of Theorem 4.1 of [8] to estimate the inverse transform of the remaining term in (5.14), we see that

$$\|K_{jk}u - K_{jkh}u\|_p \leqslant Cm^{n/2} h\, |u|_{p,1}. \tag{5.15}$$

Since η is one near the origin, we may estimate $Z_{jkh}u$ by using the method of proof of Theorem 4.1 of [8] again. We obtain

$$\|Z_{jkh}u\|_p \leqslant Ch\, |u|_{p,1}. \tag{5.16}$$

This completes the second step of the proof.

Third step. It remains to estimate the expression

$$H_1 = \int D_j Y(y) b(x, y) u(x-y)\, dy - h^{-1} \sum_\mu \partial_j a_\mu b(x, \mu h) u(x-\mu h)$$

from (5.13). Write $a_\mu = Y(\mu) + F(\mu)$ as in the proof of Lemma 4.2 and let Ω denote the cube

$$\Omega = \{y \in \mathbf{R}^n \colon 2|y_j| \leqslant 3h,\ j = 1, \ldots, n\}.$$

Then

(5.17)

$$H_1 = \int_\Omega D_j Y(y) b(x, y) u(x-y)\, dy - h^{-1} \Sigma^1 \partial_j Y(y) b(x, \mu h) - u(x-\mu h) - \\ - h^{-1} \sum_\mu \partial_j F(\mu) b(x, \mu h) u(x-\mu h) + Su(x)$$

where

$$Su(x) = h^{-1} \Sigma_1^\infty \int_Q [D_j Y(\mu+y) b(x, \mu h + hy) u(x-\mu h - hy) - \\ - \partial_j Y(y) b(x, \mu h) u(x-\mu h)]\, dy.$$

Using (4.2) and (4.14), we see that

$$\Big\|\int_\Omega D_j Y(y) b(\cdot, y) u(\cdot - y)\, dy\Big\|_p \leqslant C\|a\|_\beta m^{n/2} h^{\delta-1} \|u\|_p.$$

The second term on the right in (5.17) may be estimated similarly. Applying Lemma 4.1 and (4.14) to estimate the next sum, we obtain

$$\Big\|h^{-1} \sum_\mu \partial_j F(\mu) b(\cdot, \mu h) u(\cdot - \mu h)\Big\|_p \leqslant C\|a\|_\beta m^{[\delta]+n} h^{\delta-1} \|u\|_p.$$

In order to estimate Su we write it in the form

$$Su(x) = h^{-1} \Sigma_1^\infty \sum_{r=1}^{7} \int_Q N_r(x, \mu, y)\, dy,$$

where

$$N_1(x, \mu, y) = [D_j Y(\mu+y) - \partial_j Y(\mu)]\, [b(x, \mu h + hy) - b(x, \mu h)] \times \\ \times [u(x-\mu h - hy) - u(x-\mu h)],$$

$$N_2(x, \mu, y) = \partial_j Y(\mu)\, [b(x, \mu h + hy) - b(x, \mu h)]\, [u(x-\mu h - hy) - \\ - u(x-\mu h)],$$

$$N_3(x, \mu, y) = [D_j Y(\mu) - \partial_j Y(\mu)] b(x, \mu h)\, [u(x-\mu h - hy) - u(x-\mu h)],$$

$$N_4(x, \mu, y) = [D_j Y(\mu+y) - \partial_j Y(\mu)]\, [b(x, \mu h + hy) - b(x, \mu h)] u(x-\mu h),$$

$$N_5(x, \mu, y) = \partial_j Y(\mu) b(x, \mu h)\, [u(x-\mu h - hy) - u(x-\mu h)],$$

$$N_6(x, \mu, y) = \partial_j Y(\mu)\, [b(x, \mu h + hy) - b(x, \mu h)] u(x-\mu h),$$

$$N_7(x, \mu, y) = [D_j Y(\mu+y) - \partial_j Y(\mu)] b(x, \mu h) u(x-\mu h).$$

Using (4.2), (5.1), and (4.14), we see that for $r = 1, 2$, and 3, we have

$$\Big\|\int_Q N_r(\cdot, \mu, y)\, dy\Big\|_p \leqslant C\|a\|_\beta m^{1+n/2} h^2 |\mu|^{-n-1} |u|_{p,1}.$$

Using (4.2) and (5.2), we see that

$$\Big\|\int_Q N_4(\cdot, \mu, y)\, dy\Big\|_p \leqslant C\|a\|_\beta m^{1+n/2} h^\delta |\mu|^{\delta-n-3} \|u\|_p.$$

Turning to N_6 and applying (4.2) and (5.3), we obtain

$$\Big\|\int_Q N_6(\cdot, \mu, y)\, dy\Big\|_p \leqslant C\|a\|_\beta m^{n/2} h^\delta |\mu|^{-n-1} \|u\|_p.$$

In order to estimate N_7 we note that

$$\Big|\int_Q [D_j Y(\mu+y) - \partial_j Y(\mu)\, dy\Big| \leqslant Cm^{[\delta]+n/2} |\mu|^{-n-1-[\delta]},$$

since $\partial_{h,j}$ represents the symmetric difference operator when $\beta = 2$. Now it follows from (4.14) that

$$\Big\|\int_Q N_7(\cdot, \mu, y)\, dy\Big\|_p \leqslant C\|a\|_\beta m^{[\delta]+n/2} h^\delta |\mu|^{-n-1+\delta-[\delta]} \|u\|_p.$$

Finally, we turn to N_5 which we split into four parts, one of which involves discrete singular operators.

$$\Sigma_1^\infty \int_Q N_5(x, \mu, y)\, dy = N_{51} + \ldots + N_{54},$$

where with $t = h^{-1}$,

$$N_{51} = \Sigma_1^t \partial_j Y(\mu) b(x, \mu h) \int_Q [u(x-\mu h - hy) - u(x-\mu h)] dy,$$

$$N_{52} = \Sigma_t^\infty \partial_j Y(\mu) \, [a(x) - a(x-\mu h)] \int_Q [u(x-\mu h - hy) - u(x-\mu h)] dy,$$

$$N_{53} = ih \sum_{k=1}^n D_k a(x) \Sigma_t^\infty \mu_k [D_j Y(\mu) - \partial_j Y(\mu)] \int_Q [u(x-\mu h - hy) - \\ - u(x-\mu h)] dy,$$

$$N_{54} = -ih \sum_{k=1}^n D_k a(x) \Sigma_t^\infty \mu_k D_j Y(\mu) \int_Q [u(x-\mu h - hy) - u(x-\mu h)] dy.$$

Elementary estimates show that for $s = 1, 2$, and 3, we have

$$\|N_{5s}\|_p \leqslant C \|a\|_\beta m^{1+n/2} h^2 |u|_{p,1}.$$

Applying Corollary 3.4, we obtain the same bound for N_{54}. Combining the estimates for all the N_r, we obtain

$$\|Su\|_p \leqslant C \|a\|_\beta m^{[\delta]+n/2} h^{\delta-1} \|u\|_{p,1}.$$

summarizing the third step, we have shown that

$$\|H_1\|_p \leqslant C \|a\|_\beta m^{[\delta]+n} h^{\delta-1} \|u\|_{p,1}.$$

Inserting this estimate, (5.15), and (5.16) into (5.13), we obtain the desired estimate for H, and thus we have completed the proof of the lemma. In the next lemma we extend the estimate to higher derivatives,

LEMMA 5.2. *There is a constant C such that if $m \neq 2$ and u is in C_0^∞. then*

$$\|D_j[(aR_{lm} - R_{lm} a) u] - \partial_{h,j}[(aR_{lmh} - R_{lmh} a) u]\|_{p,\beta_z - 1} \\ \leqslant C \|a\|_\beta m^{[\varkappa]+n+1} h^\varkappa \|u\|_{p,\beta_z}.$$

Proof. The case $1 < \beta \leqslant 2$ is exactly Lemma 5.1. The general case is easily proved using Leibniz's rule and then Lemma 5.1.

We are now ready to prove estimates for approximation of smoothing operators. Recall that N is the least even integer such that $N > 5n/2$ if $\varkappa < 1$ and $N > 1 + 5n/2$ if $\varkappa = 1$.

THEOREM 5.3. *Let A be an operator of type β such that for all spherical harmonics Y_{l2},*

$$\int_\Sigma \sigma(A)(x, \xi) Y_{l2}(\xi) d\sigma(\xi) \text{ is independent of } x.$$

Then there is a constant C such that for $0 < h \leqslant 1$ and for all u in C_0^∞, we have

$$\|D_j(A^* u - A^\# u) - \partial_{h,j}(A_h^* u - A_h^\# u)\|_{p,\beta_z - 1} \leqslant C \|A\|_\beta h^\varkappa \|u\|_{p,\beta_z}. \tag{5.18}$$

Proof. Using series expansions we write

$$D_j(A^* u - A^\# u) - \partial_{h,j}(A_h^* u - A_h^\# u)$$
$$= \sum_{l,m} (-1)^m \{D_j[(R_{lm} a_{\bar{lm}} - a_{\bar{lm}} R_{lm}) u] - \partial_{h,j}[(R_{lmh} a_{\bar{lm}} - a_{\bar{lm}} R_{lmh}) u]\}.$$

Since $a_{\bar{l2}}$ commutes with R_{l2}, it follows from Lemma 5.2 that the left-hand side is dominated by

$$h^\varkappa \sum_{l,m} \|a_{lm}\|_\beta m^{[\varkappa]+n+1} \|u\|_{p,\beta_z}.$$

Since the number of harmonics of degree m is $O(m^{n-2})$, (5.18) now follows from (4.7) and the choice of N. Clearly, the second choice of N is necessary if $\varkappa = 1$.

THEOREM 5.4. *Let A and B be operators of type β such that either $\sigma(B)$ is independent of x or for all spherical harmonics Y_{l2},*

$$\int_\Sigma \sigma(A)(x, \xi) Y_{l2}(\xi) d\sigma = 0.$$

There is a constant C such that for $0 < h \leqslant 1$ and for all u in C_0^∞, we have

$$\|D_j(ABu - A \circ Bu) - \partial_{h,j}(A_h B_h u - A_h \circ B_h u)\|_{p,\beta_z - 1} \leqslant C \|A\|_\beta \|B\|_\beta h^\varkappa \|u\|_{p,\beta_z}.$$

Proof. We write

$$ABu - A \circ Bu = \sum_{l,m} \sum_{s,t} a_{lm}(R_{lm} b_{st} - b_{st} R_{lm}) R_{st} u$$

with a similar expression for the discrete operators. By assumption, either $a_{l2} = 0$ or b_{st} commutes with R_{l2}. Using Lemma 5.2, we proceed as in the previous theorem to complete the proof.

The approximation of $AA - AA$ does not involve any restrictions on A, and we may allow N to be the least even integer greater than $3n/2 - 1$.

THEOREM 5.5. *Let A be an operator of type β. There is a constant C such that for $0 < h \leqslant 1$ and for all u in C_0^∞, we have*

$$\|(AA - AA) u - (A_h A_h - A_h A_h) u\|_{p,\beta_z - 1} \leqslant C \|A\|_\beta h^\varkappa \|u\|_{p,\beta_z}.$$

Proof. We write

$$(AA - AA) u = \sum_{j=1}^n \sum_{l,m} D_j[(a_{lm} R_j - R_j a_{lm}) R_{lm} u] - \\ - \sum_{j=1}^n \sum_{l,m} (D_j a_{lm}) R_j R_{lm} u$$

with a similar expression for the discrete operators. Since R_j has kernel $z_j|z|^{-n-1}$ and $z_j|z|^{-1}$ is a spherical harmonic of degree one, we may apply Lemma 5.2 in order to estimate

$$D_j[(a_{lm}R_j - R_j a_{lm})R_{lm}u] - \partial_{h,j}[(a_{lm}R_{jh} - R_{jh}a_{lm})R_{lm}u].$$

The remaining terms may be estimated easily using Theorem 4.1 of [8] and Lemma 4.3.

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Discontinuous translation invariant linear functionals on $L^1(G)$

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Abstract. The main result of this article is the proof of the existence of discontinuous translation invariant linear functionals on the group algebra $L^1(G)$, for all compactly generated locally compact abelian groups.

0. One of the well-known results in an abelian harmonic analysis on locally compact groups is that any *continuous* translation invariant linear functional on $L^1(G)$, the space of all Haar integrable functions on G, is just a complex multiple of the Haar integral I,

$$I(f) = \int f\,d\lambda, \quad f \in L^1(G),\ \lambda \text{ is a Haar measure on } G.$$

We may ask whether this result remains valid if we omit the word 'continuous' in the hypothesis.

The purpose of this note is to prove the following.

THEOREM. *Let G be any compactly generated locally compact abelian group. Then $L^1(G)$ admits discontinuous translation invariant linear functionals.*

The proof will be deferred till the end of Section 3.

The motivation has been provided to a great extent in [6], [7], where related problems have been solved for various function spaces associated with locally compact groups.

1. Let G be a locally compact abelian group (LCAG); we will denote by $M(G)$ the algebra of all bounded Borel measures on G, with convolution as multiplication, i.e., we define

$$(1) \qquad \mu * \nu(E) = \int \mu(E-g)\,d\nu(g) \qquad \forall E,$$

where $\mu, \nu \in M(G)$, E is a Borel set in G and $E-g = \{h: h+g \in E\}$, with the total variation norm on $M(G)$.

It is well known that $M(G)$ can be identified with the strong dual of $C_0(G)$, the space of all continuous complex valued functions vanishing at infinity.

Also, $L^1(G)$ can be canonically embedded in $M(G)$ and will be a norm