Proof. We assume that the unitary ball is the support of K; then

$$(2.13)\qquad |K_\lambda(f)(x)| \leqslant \int_{|y|\leqslant 1} |K(y)||f(x-\lambda^{-1}y)|\,dy \leqslant \|K\|_{q'}\Big(\int_{|y|\leqslant 1} |f(x-\lambda^{-1}y)|^q dy\Big)^{1/q},$$

where $p' \leqslant q < \infty$, $1/q' + 1/q = 1$.

If $\tilde{f}_q$ denotes the Hardy–Littlewood maximal function of $|f|^q$, we can rewrite (2.13) as

$$(2.14)\qquad \bar{f}(x) \leqslant C\big(\tilde{f}_q(x)\big)^{1/q}.$$

Now the fact that $\tilde{f}$, i.e., the Hardy–Littlewood maximal operator of f, is weak-type (1, 1) (see [2]) implies (2.12). If $q = \infty$, we split K as follows: ${}_NK = KX_{\{|x|\leqslant N\}}$, ${}^NK = K - {}_NK$. Then

$$K_\lambda * f(x) - f(x)\int K(x)\,dx = \int {}_NK(y)\{f(x-\lambda^{-1}y) - f(x)\}\,dy + \int {}^NK(y)\{f(x-\lambda^{-1}y) - f(x)\}\,dy = {}_NI_\lambda(x) + {}^NI_\lambda(x).$$

It is not difficult to establish the following two inequalities:

$$(2.15)\qquad |{}^NI_\lambda(x)| \leqslant \big(\|f\|_\infty + |f(x)|\big)\int_{|x|\geqslant N} |K(x)|\,dx,$$

$$(2.16)\qquad |{}_NI_\lambda(x)| \leqslant \big(|f(x)| + \|f\|_\infty\big)\int_{\{x:\,|{}_NK(x)|\geqslant T\}} |{}_NK(x)|\,dx + T\lambda \int_{|y|\leqslant \lambda^{-1}N} |f(x-y) - f(x)|\,dy.$$

Now, considering $\overline{\lim}_{N\to\infty}\overline{\lim}_{T\to\infty}\overline{\lim}_{\lambda\to\infty}$ on (2.15) and (2.16), we have

$$\overline{\lim_{\lambda\to\infty}}\Big|K_\lambda * f(x) - f(x)\int K(x)\,dx\Big| = 0 \text{ a.e.}$$

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The l_1^n problem and degrees of non-reflexivity

by

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1. Introduction. The material presented in this paper originated from an attempt to solve the "l_1^n problem" first raised by R. C. James [6]. Let X be a non-reflexive Banach space. Is it true that for every $\varepsilon > 0$ and integer n there is a subspace B of X such that $d(B, l_1^n) < 1+\varepsilon$?

Before continuing let us recall the definition of the notions appearing in the statement of this problem and some closely related notions.

We consider here Banach spaces over the reals and l_1^n denotes the n-dimensional L_1 space, i.e. the space of n-tuples $x = (x_1, x_2, \ldots, x_n)$ of reals with

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

By l_1 we denote as usual the space of all sequences $x = (x_1, x_2, \ldots)$ with $\|x\| = \sum_{i=1}^{\infty}|x_i| < \infty$. We say that $d(B, C) \leqslant \lambda$ for some Banach spaces B and C and a real $\lambda \geqslant 1$ if there is an (always linear here) operator T from B onto C such that $\|T\|\|T^{-1}\| \leqslant \lambda$. A Banach space Y is said to be *finitely represented* in a Banach space X if for every finite dimensional subspace B of Y and every $\varepsilon > 0$ there is a subspace C of X such that $d(B, C) \leqslant 1+\varepsilon$. If P is a property which is meaningful for general Banach spaces we say that a Banach space X is "super P" if every Banach space Y finitely represented in X has property P. Of particular importance is the property *super reflexive* introduced by James. Thus, according to the general rule, a Banach space X is super reflexive if every Banach space Y finitely represented in X is reflexive.

In paper [6] in which James posed the l_1^n problem he proved that the

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answer to it is positive for $n = 2$. We tried to solve James' problem by combining the basic idea of his proof for $n = 2$ with a careful study of the "degree" of non-reflexivity of X and first of all with the properties of the quotient space $R(X) = X^{**}/X$ (here and later on we assume, unless stated otherwise, that X is embedded canonically in X^{**}). By extending the arguments of James we showed that if $R(X)$ is "big" then the answer to James' question is positive for $n = 3$ and that similar (but more complicated) statements hold for arbitrary n. Together with this result we found two constructions which produce for every non-reflexive space X a space Y which is finitely represented in X and for which $R(Y)$ is "big". However, it turned out that the bigness of $R(Y)$ obtained in these constructions is not of the type which ensures the existence of copies of l_1^3 in Y (and thus in X).

Thus our approach did not lead to a solution of James' problem. There was a good reason for this failure. James [9] recently constructed an ingenious example of a non-reflexive Banach space X which does not contain almost isometric copies of l_1^3. In view of this negative solution to the l_1^n problem in general there is, however, some interest in the partial results obtained in our approach, in particular, since it singles out a certain "degree" of non-reflexivity which is reflected by the local structure of Banach spaces.

In this paper we introduce a notion of local k-dimensional structure (local k-structure in short). This notion by definition depends only on the local structure of a Banach space X (i.e. the finite dimensional subspaces of X) and is invariant under isomorphism. Also, by definition, local $k+1$ structure implies local k-structure. Moreover, it is trivially seen to be a self-dual property (i.e. X admits a local k-structure if and only if X^* admits such a structure). For $k = 1$ this notion agrees with the negation of super-reflexivity (i.e. X admits a local 1-structure if and only if X is not super-reflexive). We show that if $R(X) = X^{**}/X$ admits a local k-structure for some integer k then X admits a local $(k+1)$-structure. There is also a partial converse to this statement: If X admits a local $(k+1)$-structure, then there is a space Y, finitely represented in X, such that $R(Y)$ admits a local k-structure. Thus, for example, X admits a local 2-structure if and only if there is a Banach space Y which is finitely represented in X with $R(Y)$ not super-reflexive. More generally, if we define the (quite complicated) object $R^k(X)$ in an obvious way (i.e. $R^1(X) = R(X)$, $R^{k+1}(X) = R(R^k(X))$), we show that X admits a local k-structure if and only if there is a space Y finitely represented in X so that $R^k(Y) \neq \{0\}$. The connection between the notion of local k-structure and the l_1^n problem is provided by showing that if X admits a local k-structure then for every $\varepsilon > 0$ there is a subspace B of X with $d(B, l_1^{k+1}) \leqslant 1+\varepsilon$, and actually a somewhat stronger statement holds. The proof is a "k-dimensional" version of the proof of James of the main result in [6]. In view of all these facts the recent example of James [9] shows therefore that there is a space X which admits a local 1-structure but not a local 2-structure. This example provides the main justification for the introduction of the notion of local k-structures.

In Section 2 of this paper we make only a beginning of the study of k-structures. It is not clear to us whether they play a role also in other problems in Banach space theory and thus deserve much more careful study. Two questions are, however, of quite obvious interest and remain unanswered here. (1) Does there exist for every k a space admitting a local k-structure but not a local $(k+1)$-structure? James' example does not seem to be easily generalized to the case where $k > 1$. (2) Is it true that a Banach space X admits a local k-structure if and only if for every equivalent norm $|||\ |||$ in X and every $\varepsilon > 0$ there is a subspace B of $(X, |||\ |||)$ with $d(B, l_1^{k+1}) \leqslant 1+\varepsilon$? (Of course, it is the "if" part which is open.) For $k = 1$ the answer is known to be yes. This is the content of the beautiful result of Enflo [4] and James [7], which shows that a space is super-reflexive if and only if it is uniformly convexifiable or if and only if it has a uniformly non-square norm. The connection between local 1-structure and uniform convexity indicates perhaps the existence of a notion of "k-uniform convexity".

In Section 3 we continue the discussion of Section 2 but with a somewhat different aim in mind. Our interest in Section 3 is in enlarging $R^k(X)$ for spaces admitting a local k-structure without "enlarging" the local structure itself. Our first construction is that of taking iterated duals. For every non-reflexive Banach space X we have the natural and canonical chain of inclusions

$$X \subset X^{**} \subset X^{(4)} \subset X^{(6)} \ldots \subset X^{(2n)} \subset \ldots$$

We can therefore define also a space $X^{(\omega)}$ as the completion of $\bigcup_{n=1}^{\infty} X^{(2n)}$ and continue to define by an obvious transfinite induction the spaces $X^{(\alpha)}$ for every even ordinal number α. It follows immediately from the principle of local reflexivity [10] that for every α the space $X^{(\alpha)}$ is finitely represented in X. We show in Section 3 that if X is a Banach space such that $R^k(X) \neq \{0\}$ for some integer k, then if α is large enough ($\alpha \geqslant \omega^2$ to be precise) $R^k(X^{(\alpha)})$ is infinite dimensional. Thus, taking e.g. the case $k = 1$, it follows that there is no meaningful notion of "super quasi-reflexivity". If X is a space such that for every Z finitely represented in X we have $\dim R(Z) = \dim Z^{**}/Z < \infty$, then X is already super reflexive. This fact should be compared with the result of Section 2 which shows that "super Z^{**}/Z reflexive" is a meaningful notion (i.e. that of admitting a local 2-structure).

Another construction presented in Section 3 emphasizes in a stronger form the non existence of "super quasi-reflexivity". Even the formally stronger notion "super density character of $Z^{**}/Z \leqslant$ density character of Z'' already implies super reflexivity. If X is any Banach space with $R^k(X) \neq \{0\}$, we construct a separable function space Y on the k-dimensional cube $[0,1]^k$ so that Y is finitely represented in X and $R^k(Y)$ is non-separable.

2. Local k-structure and the l_1^n-problem. We start with the definition of the main concept appearing in this paper.

Definition 1. A Banach space X is said to admit a *local k-dimensional structure* (*local k-structure* in short) if there is a constant M having the following property. *For every integer n there exist n^k elements $\{x_{i_1,i_2,\ldots,i_k}\}$, $1 \leqslant i_1, i_2, \ldots, i_k \leqslant n$ in X and n^k elements $\{f_{j_1,j_2,\ldots,j_k}\}$, $1 \leqslant j_1, j_2, \ldots, j_k \leqslant n$ in X^* such that*

$$\|x_{i_1,i_2,\ldots,i_k}\| \leqslant M, \quad \|f_{j_1,j_2,\ldots,j_k}\| \leqslant M, \tag{2.1}$$

and

$$f_{j_1,j_2,\ldots,j_k}(x_{i_1,i_2,\ldots,i_k}) = \begin{cases} 1 & \text{if} \quad j_p \leqslant i_p,\ p = 1, \ldots, n, \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

It is worthwhile to read this definition with the following geometric picture in mind. The $x_{i_1,i_2,\ldots,i_k}$ are functions on the k-dimensional lattice of length n (i.e. consisting of n^k points). They are the characteristic functions of the set $\{(j_1, j_2, \ldots, j_k);\ j_p \leqslant i_p \text{ for all } p\}$. The $f_{j_1,j_2,\ldots,j_k}$ are the (extensions to X of the) evaluation functionals at the points of the lattice. To illustrate this definition assume that X contains c_0. Let $f_j \in X^*$ be extensions of norm 1, say, of the evaluation functional of the jth coordinate in c_0. Take any n^k different elements of these $\{f_j\}_{j=1}^{\infty}$ and arrange them into a k-dimensional lattice of length n. Then for any subset of this lattice there is an element of norm 1 in c_0 and thus in X which is the characteristic function of this set. Thus a space X which contains c_0 (or, more generally, in which c_0 can be finitely represented) admits a local k-structure for every k (here M can be taken even as 1). The same is easily seen to be true if c_0 is replaced by l_1 (see Proposition 1 below). Another simple fact which is worthwhile to keep in mind is the following. If X admits a local k-structure and Y admits a local h-structure, then $X \otimes Y$ with any tensorial norm admits a local $(k+h)$-structure. (Let $x_{i_1,\ldots,i_k}$, $f_{j_1,\ldots,j_k}$ and $y_{\sigma_1,\sigma_2,\ldots,\sigma_h}$, $g_{\tau_1,\tau_2,\ldots,\tau_h}$ be the elements ensured by the existence of local structures in X and Y, respectively. Then $x_{i_1,\ldots,i_k} \otimes y_{\sigma_1,\ldots,\sigma_h}$ and $f_{j_1,\ldots,j_k} \otimes g_{\tau_1,\ldots,\tau_h}$ can be used to verify that $X \otimes Y$ admits an $(h+k)$-structure.) In order to simplify somewhat the notation we shall introduce a special symbol to denote the right-hand side of (2.2); we shall denote it by $\Gamma_n\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix}$.

Proposition 1. *A Banach space X admits a local k-structure if and only if X^* admits such a structure.*

Proof. Assume that X has a local k-structure and let $x_{i_1,\ldots,i_k} \in X$ and $f_{j_1,\ldots,j_k} \in X^*$ satisfy (2.1) and (2.2). Put $x^*_{i_1,\ldots,i_k} = f_{n-i_1,n-i_2,\ldots,n-i_k} \in X^*$ and $F_{j_1,\ldots,j_k} = Jx_{n-j_1,n-j_2,\ldots,n-j_k} \in X^{**}$ where $J\colon X \to X^{**}$ denotes the canonical embedding. Then clearly all elements are of norm $\leqslant M$ and

$$F_{j_1,\ldots,j_k}(x^*_{i_1,\ldots,i_k}) = \Gamma_n\begin{pmatrix} n-j_1, \ldots, n-j_k \\ n-i_1, \ldots, n-i_k \end{pmatrix} = \Gamma_n\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix}.$$

Hence X^* has a local k-structure. Assume conversely that X^* has a local k-structure. By the first part of the proof the same is true for X^{**}. Using the w^*-density of the unit ball of X in that of X^{**} or, more precisely, the theorem Day [3] calls *Helly's theorem*, it follows that X itself also admits a local k-structure (the constant M has only to be replaced by $M+\varepsilon$ for any $\varepsilon > 0$).

Proposition 2. *Let X be a Banach space such that $R(X) = X^{**}/X$ admits a local k-structure. Then X admits a local $(k+1)$-structure.*

Proof. There is no loss of generality to assume that X is complemented in X^{**}. Indeed, it is trivial and well known that $X^{***} = X^* \oplus X^{\perp}$ where X^* is canonically embedded in X^{***} and $X^{\perp}$ denotes those elements in X^{***} which vanish on the canonical embedding of X in X^{**}, i.e. it can be identified with $R(X)^*$. Hence $R(X^*)$ is isomorphic to $R(X)^*$ and by Proposition 1 it is enough to prove the present result for X^* which is, as remarked above, complemented in its second dual.

Let therefore P be a projection from X^{**} onto X and let $Y = (I-P)X^{**}$. By our assumption there is a constant M such that for any integer n there are elements $\{y_{i_1,i_2,\ldots,i_k};\ 1 \leqslant i_1, i_2, \ldots, i_k \leqslant n\}$ in Y and elements $\{F_{j_1,j_2,\ldots,j_k};\ 1 < j_1, j_2, \ldots, j_k \leqslant n\}$ in $X^{\perp} \subset X^{***}$ all of norm $\leqslant M$ so that

$$F_{j_1,j_2,\ldots,j_k}(y_{i_1,i_2,\ldots,i_k}) = \Gamma_n\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix}.$$

By Helly's theorem there are elements of norm $< M+1$ in X^* which we denote by $\{f_{j_1,j_2,\ldots,j_k,1};\ 1 \leqslant j_1, j_2, \ldots, j_k \leqslant n\}$ so that

$$y_{i_1,i_2,\ldots,i_k}(f_{j_1,j_2,\ldots,j_k,1}) = \Gamma_n\begin{pmatrix} i_1, i_2, \ldots, i_k \\ j_1, j_2, \ldots, j_k \end{pmatrix}.$$

Again by Helly's theorem there are $\{x_{i_1,i_2,\ldots,i_k,1};\ 1 \leqslant i_1, i_2, \ldots, i_k \leqslant n\}$, in X so that

$$f_{j_1,j_2,\ldots,j_k,1}(x_{i_1,i_2,\ldots,i_k,1}) = \Gamma_n\begin{pmatrix} i_1, i_2, \ldots, i_k \\ j_1, j_2, \ldots, j_k \end{pmatrix}.$$

Observe that since $Y \subset X^{\perp}$, we have that $F_{j_1,j_2,\ldots,j_k}(x_{i_1,i_2,\ldots,i_k,1}) = 0$ for all $\{j_p\}_{p=1}^k$ and $\{i_p\}_{p=1}^k$ (we remind the reader that we assume that X is identified with its canonical image in X^{**}). By using again Helly's theorem we find elements $\{f_{j_1,j_2,\ldots,j_k,2};\ 1 \leqslant j_1, j_2, \ldots, j_k \leqslant n\}$ in X^* of norm $< M+1$ which take the same values as those of $F_{j_1,j_2,\ldots,j_k}$ on those finitely many elements of X^{**} (and for that matter of X) which have already been singled out. In other words,

$$y_{i_1,i_2,\ldots,i_k}(f_{j_1,j_2,\ldots,j_k,2}) = \Gamma_n\begin{pmatrix} i_1, & \ldots, & i_k \\ j_1, & \ldots, & j_k \end{pmatrix}, \quad f_{j_1,j_2,\ldots,j_k,2}(x_{i_1,i_2,\ldots,i_k,1}) = 0.$$

Next we apply Helly's theorem to find elements $\{x_{i_1,i_2,\ldots,i_k,2}\}$ in X which take the same value as $y_{i_1,i_2,\ldots,i_k}$ on those elements of X^* which we already singled out. Continuing in this manner n times we construct for every $p \leqslant n$ elements $f_{j_1,j_2,\ldots,j_k,p}$ in X^* and $x_{i_1,i_2,\ldots,i_k,p}$ in X all of norm $\leqslant M+1$ so that

$$f_{j_1,j_2,\ldots,j_k,p}(x_{i_1,i_2,\ldots,i_k,s}) = \begin{cases} \Gamma_n\begin{pmatrix} i_1, & \ldots, & i_k \\ j_1, & \ldots, & j_k \end{pmatrix} & \text{if} \quad p \leqslant s, \\ 0 & \text{if} \quad p > s. \end{cases}$$

In other words, we have

$$f_{j_1,j_2,\ldots,j_k,j_{k+1}}(x_{i_1,i_2,\ldots,i_k,i_{k+1}}) = \Gamma_n\begin{pmatrix} i_1, & \ldots, & i_k, & i_{k+1} \\ j_1, & \ldots, & j_k, & j_{k+1} \end{pmatrix}$$

and this concludes the proof.

The converse to Proposition 2 is obviously false. There are even reflexive spaces X (for which thus $X^{**}/X = \{0\}$) which admit a local k-structure for every k. However, if we take into consideration also spaces Y which can be finitely represented in X we get a valid converse to Proposition 2. In order to do this we introduce first another definition.

DEFINITION 2. A Banach space X is said to admit *a global k-dimensional structure* if there is a bounded set $\{x_{i_1,i_2,\ldots,i_k};\ 1 \leqslant i_1, i_2, \ldots, i_k < \infty\}$, in X and a bounded set $\{f_{j_1,j_2,\ldots,j_k};\ 1 \leqslant j_1, j_2, \ldots, j_k < \infty\}$ in X^* so that $f_{j_1,j_2,\ldots,j_k}(x_{i_1,i_2,\ldots,i_k}) = \Gamma_\infty\begin{pmatrix} i_1, & \ldots, & i_k \\ j_1, & \ldots, & j_k \end{pmatrix}$.

(The symbol $\Gamma_\infty(\ )$ is the obvious extension of $\Gamma_n(\ )$ to an infinite k-dimensional lattice.) It is easily seen and well known (cf. e.g. [6]) that a Banach space X admits a global 1-structure if and only if it is non-reflexive. The connection between global k-structure and local k-structure is clarified in the next proposition.

PROPOSITION 3. *A Banach space X admits a local k-structure if and only if there is a Banach space Y with a global k-structure which is finitely represented in X.*

Proof. The "if" part is obvious. To verify the "only if" part, let M be the constant ensured by the assumption that X has a local k-structure. Choose for every integer n elements $\{x_{i_1,\ldots,i_k}(n):\ 1 \leqslant i_1, i_2, \ldots, i_k \leqslant n\}$ in X of norm $\leqslant M$ for which there exist functionals $f_{j_1,\ldots,j_k}(n) \in X^*$ so that (2.1) and (2.2) hold. By a diagonal process we find a sequence $\{n_m\}_{m=1}^\infty$ of integers so that

$$(2.3) \qquad \Big|\Big|\Big| \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}\hat{x}_{i_1,\ldots,i_k} \Big|\Big|\Big| = \lim_{m\to\infty} \Big|\Big|\Big| \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} x_{i_1,\ldots,i_k}(n_m) \Big|\Big|\Big|$$

exists for every finite sum $\sum$ and all choices of rational $a_{i_1,\ldots,i_k}$ (by $\hat{x}_{i_1,\ldots,i_k}$ we denote elements which form an algebraic basis for an abstract linear space Y_0). It is clear that the limit in (2.3) exists automatically also for arbitrary real a's and that $|||\ |||$ defines a seminorm on Y_0. The existence of the functionals $f_{j_1,j_2,\ldots,j_k}(n)$ in X^* implies also easily that the equation $f_{j_1,j_2,\ldots,j_k}(\hat{x}_{i_1,i_2,\ldots,i_k}) = \Gamma_\infty\begin{pmatrix} i_1, & \ldots, & i_k \\ j_1, & \ldots, & j_k \end{pmatrix}$ defines for every $j_1, \ldots, j_k$ (a unique) functional of norm $\leqslant M$ on $(Y_0, |||\ |||)$. From this we get also that $|||\ |||$ is actually a norm (not only a seminorm) on Y_0. The completion Y of Y_0 is a space having a global k-structure which is finitely represented in X.

PROPOSITION 4. *Let X be a Banach space admitting a global k-structure* ($k > 1$). *Then $R(X) = X^{**}/X$ admits a global* ($k-1$)*-structure.*

Proof. Let $\{x_{i_1,i_2,\ldots,i_k}\} \subset X$ and $\{f_{j_1,j_2,\ldots,j_k}\} \in X^*$ be the sets ensured by the existence of the global k-structure in X. Since, as easily verified, $R(Y)$ is isomorphic to a subspace of $R(Z)$ whenever Y is isomorphic to a subspace of Z, there is no loss of generality to assume that the $\{x_{i_1,i_2,\ldots,i_k}\}$ span all of X. It follows from this assumption that for every fixed $j_1, j_2, \ldots, j_{k-1}$ the sequence $\{f_{j_1,j_2,\ldots,j_{k-1},n}\}_{n=1}^\infty$ tends w^* to 0. Let now, for each $i_1, \ldots, i_{k-1}$, $x^{**}_{i_1,\ldots,i_{k-1}}$ be a limit point in the w^*-topology of X^{**} of the sequence $\{x_{i_1,i_2,\ldots,i_{k-1},n}\}_{n=1}^\infty$. Then, clearly,

$$x^{**}_{i_1,\ldots,i_{k-1}}(f_{j_1,j_2,\ldots,j_{k-1},j_k}) = \Gamma_\infty\begin{pmatrix} i_1, & \ldots, & i_{k-1} \\ j_1, & \ldots, & j_{k-1} \end{pmatrix}.$$

For every $j_1, \ldots, j_{k-1}$ let $F_{j_1,\ldots,j_{k-1}} \in X^{***}$ be a w^*-limit point (i.e. in the weak topology determined by X^{**}) of $\{f_{j_1,j_2,\ldots,j_{k-1},n}\}_{n=1}^\infty$. Then $F_{j_1,\ldots,j_{k-1}} \in X^\perp$ and $F_{j_1,\ldots,j_{k-1}}(x^{**}_{i_1,\ldots,i_{k-1}}) = \Gamma_\infty\begin{pmatrix} i_1, & \ldots, & i_{k-1} \\ j_1, & \ldots, & j_{k-1} \end{pmatrix}$ and this proves the existence of a global $(k-1)$-structure in X^{**}/X.

Summing up the contents of Propositions 1–6 we get

THEOREM 1. *Let X be a Banach space, and $k \geqslant 1$ an integer. The following three assertions are equivalent.*

(i) *X admits a local k-structure,*

(ii) *X^* admits a local k-structure,*

(iii) *There is a Banach space Y finitely represented in X such that $R^k(Y) \neq \{0\}$.*

Proof. If $R^k(Y) \neq \{0\}$, then $R^{k-1}(Y)$ is non-reflexive and thus admits (even a global) 1-structure. By Proposition 2, Y admits a local

k-structure. The same will be true for X if Y is finitely represented in X. If, conversely, we assume (i) (or (ii) which is equivalent to (i) by Proposition 1), then by Proposition 3 there is Y which has a global k-structure which is finitely represented in X. By Proposition 4, this Y satisfies $R^k(Y) \neq \{0\}$.

The relation between local k-structure and the l_1^n problem is exhibited in the next theorem.

THEOREM 2. *Let X admit a local k-structure. Then for every $\varepsilon > 0$ there is a subspace B of X with $d(B, l_1^{k+1}) < \varepsilon$. More generally, for every integer n and every $\varepsilon > 0$ there are n vectors $\{x_i\}_{i=1}^n$ in X all of norm 1 so that*

$$\|\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n\| \geqslant n - \varepsilon \tag{2.4}$$

for every choice of signs $\{\theta_i\}_{i=1}^n$ in which there are at most k changes (i.e. each θ_i is either $+1$ or -1 and $\theta_i \theta_{i+1} = -1$ for at most k indices i).

It is clear that for $n = k+1$ the second statement of Theorem 2 reduces to the first one. For $n = 2$, $k = 1$, Theorem 2 is exactly the main result of [6]. James ([6], [7]) and others proved and used this result in the case $k = 1$ and n arbitrary. The proof we give here is the natural generalization of James' argument to the k-dimensional setting. Since the notation becomes quite involved for large n and k, we present the proof in detail only for $n = 3$, $k = 2$. At the end we indicate briefly what has to be done for arbitrary n and k.

Proof. By Proposition 3, we may assume without loss of generality that X has a global 2-structure determined by elements $\{x_{i_1,i_2}\}$, $1 \leqslant i_1, i_2 < \infty$ in X and $\{f_{j_1,j_2}\}$, $1 \leqslant j_1, j_2 < \infty$ in X^*. Let M be an upper bound for the norm of all these elements.

Let $m \geqslant 1$, $p_1 < p_2 < p_3 < \ldots < p_{2m}$, $\pi_1 < \pi_2 < \pi_3 < \ldots < \pi_{2m}$ be integers and put

$$S(m, (p_i), (\pi_i)) = \{x \in X;\ f_{i_1,i_2}(x) = (-1)^{h+l}, \text{ for } p_{2h-1} < i_1 \leqslant p_{2h} \text{ and } \pi_{2l-1} < i_2 \leqslant \pi_{2l},\ 1 \leqslant h, l \leqslant m\}. \tag{2.5}$$

Let also

$$K(m, (p_i), (\pi_i)) = \inf\{\|x\|;\ x \in S(m, (p_i), (\pi_i))\} \tag{2.6}$$

and

$$K(m) = \lim_{p_1\to\infty} \lim_{p_2\to\infty} \ldots \lim_{p_{2m}\to\infty} \lim_{\pi_1\to\infty} \ldots \lim_{\pi_{2m}\to\infty} K(m, (p_i), (\pi_i)). \tag{2.7}$$

The fact that the iterated limits exist is easy to verify. Observe that if

$$w_{h,l} = x_{p_{2h},\pi_{2l}} - x_{p_{2h-1},\pi_{2l}} - x_{p_{2h},\pi_{2l-1}} + x_{p_{2h-1},\pi_{2l-1}}$$

then $w_{h,l}$ is (regarded as a function on the two-dimensional "lattice" f_{i_1,i_2}) the characteristic function of the rectangle $p_{2h-1} < i_1 \leqslant p_{2h}$, $\pi_{2l-1} < i_2 \leqslant \pi_{2l}$. Hence the vector $\sum_{h=1}^m \sum_{l=1}^m (-1)^{h+l} w_{h,l}$ is a member of the set $S(m, (p_i), (\pi_i))$. It follows that every $K(m)$ is finite, more precisely,

$$K(m) \leqslant 4m^2 M, \quad m = 1, 2, \ldots \tag{2.8}$$

Further it follows from the definition that for every m,

$$M^{-1} \leqslant K(m) \leqslant K(m+1).$$

It follows from this and (2.8) that

$$\varliminf_{m\to\infty} K(m)/K(m-1) = 1. \tag{2.9}$$

Let $\varepsilon > 0$ be given. By (2.9) we can find an integer m and a $\delta > 0$ so that

$$(K(m-1) - \delta)/(K(m) + \delta) > 1 - \tfrac{1}{3}\varepsilon. \tag{2.10}$$

By the definition of $K(m)$ and $K(m-1)$ there are sequences (p_i), (r_i), (s_i), (π_i), (ϱ_i), (σ_i) all of length $2m$ so that

$$p_1 < r_1 < s_1 < p_2 < p_3 < r_2 < r_3 < s_2 < s_3 < p_4 < \ldots < s_{2m},$$
$$\pi_1 < \varrho_1 < \sigma_1 < \pi_2 < \pi_3 < \varrho_2 < \varrho_3 < \sigma_2 < \sigma_3 < \varrho_n < \ldots < \sigma_{2m},$$

and satisfying

$$\left.\begin{array}{l} K(m, (p_i), (\pi_i)) \\ K(m, (r_i), (\varrho_i)) \\ K(m, (s_i), (\sigma_i)) \end{array}\right\} < K(m) + \delta \tag{2.11}$$

and

(2.12)

$$\left.\begin{array}{l} K(m-1, (s_1, p_2, s_3, p_n, \ldots, s_{2m-3}, p_{2m-2}), (\sigma_1, \pi_2, \sigma_3, \ldots, \pi_{2m-2})) \\ K(m-1, (p_3, r_2, p_5, r_4, \ldots, p_{2m-1}, r_{2m-2}), (\sigma_1, \pi_2, \sigma_3, \ldots, \pi_{2m-2})) \\ K(m-1, (r_3, s_2, r_5, s_4, \ldots, r_{2m-1}, s_{2m-2}), (\sigma_1, \pi_2, \sigma_3, \ldots, \pi_{2m-2})) \\ K(m-1, (r_3, s_2, r_5, s_4, \ldots, r_{2m-1}, s_{2m-2}), (\pi_3, \varrho_2, \pi_5, \varrho_4, \ldots, \varrho_{2m-2})) \end{array}\right\} > K(m-1) - \delta.$$

Choose now u, v, and w to be vectors of norm $< K(m) + \delta$ in $S(m, (p_i), (\pi_i))$, $S(m, (r_i), (\varrho_i))$ and $S(m, (s_i), (\sigma_i))$, respectively. With these choices of u, v and w we have

$$\tfrac{1}{3}(u + v + w) \in S(m-1, (s_1, \ldots, p_{2m-2}), (\sigma_1, \ldots, \pi_{2m-2})),$$
$$\tfrac{1}{3}(-u + v + w) \in S(m-1, (p_3, \ldots, r_{2k-2}), (\sigma_1, \ldots, \pi_{2m-2})),$$
$$\tfrac{1}{3}(-u - v + w) \in S(m-1, (r_3, \ldots, s_{2k-2}), (\sigma_1, \ldots, \pi_{2m-2})),$$
$$\tfrac{1}{3}(u - v + w) \in S(m-1, (r_3, \ldots, s_{2k-2}), (\pi_3, \ldots, \varrho_{2m-2})).$$

It follows from (2.12) and (2.13) that for every choice of signs $\|u\pm v\pm w\| \geqslant 3(K(m-1)-\delta)$. Thus in view of (2.10) we get that if $x=u/\|u\|$, $y=v/\|v\|$ and $z=w/\|w\|$, then $\|x\pm y\pm z\|\geqslant 3-\varepsilon$ for every choice of signs and this concludes the proof of the theorem (for $k=2$ and $n=3$).

In order to prove the theorem for $k=2$ and $n>3$ we work instead of with three sets of the form $S(m,(p_i),(\pi_i))$ with n such sets. The requirement (2.10) on m must be replaced by $(K(m-n+2)-\delta)/(K(m)+\delta) > 1-\varepsilon/n$ and also (2.12) has to be generalized in a suitable form. The procedure is the same as the extension of the theorem in the case $k=1$ to $n>2$ (see [6] for $n=3$ and [7] for general n). In order to prove the theorem for $k>2$ we again repeat the same procedure but in the k-dimensional setting.

We state now the simplest case in which our results give new information.

Corollary 1. *Assume that X is a Banach space with X^{**}/X non-reflexive. Then X contains arbitrarily close copies of l_1^3.*

Proof. Use Theorems 1 and 2.

Corollary 2. *A Banach space X admits a local k-structure for every k if and only if l_1 is finitely represented in X.*

Proof. For the "if" part use the remark following Definition 1. For the "only if" part use Theorem 2.

The example of James [9] of a non-reflexive space which does not contain arbitrarily close copies of l_1^3 is in view of Corollary 1 an example of a space admitting a local 1-structure but not a local 2-structure.

As for a possible converse to Theorem 2, for $k=1$ the converse of the second statement of Theorem 2 (i.e. for $n>1$ arbitrary) is trivially valid. This is no longer the case if $k>1$. Indeed, let X be a space having a global 1-structure with $\{x_i\}_{i=1}^\infty\subset X$ all of norm 1 and $\{f_j\}_{j=1}^\infty \in X^*$ bounded (this is the case in every non-reflexive space). Introduce in X an equivalent norm $|||\ |||$ by putting

$$|||x||| = \sup\big(\|x\|,\ \sup_{j_1<j_2<j_3} |f_{j_3}(x)-2f_{j_2}(x)+2f_{j_1}(x)|\big).$$

Then $|||x_i|||=1$ for all i and for all $j_1<j_2<j_3$ we have

$$|||x_1+x_2+\ldots+x_{j_1}-x_{j_1+1}-x_{j_1+2}-\ldots-x_{j_2}+x_{j_2+1}+\ldots+x_{j_3}||| = j_3$$

(apply $f_1-2f_{j_1+1}+2f_{j_2+1}$ to this vector). In $(X, |||\ |||)$ the conclusion of Theorem 2 for $k=2$ holds for every $n\geqslant 2$ and even with $\varepsilon=0$. However, James' example shows that X may fail to have a local 2-structure. We do not know however whether the validity of the conclusion of Theorem 2 for X *in every equivalent norm* already implies that X admits a local k-structure.

3. Methods of enlarging $R^k(X)$. In this section we present two methods which enable under certain conditions the enlargement of $R^k(X)$ without "enlarging" the local structure. Before we do this, we would like to recall two simple observations which were used already in the previous section and will be used repeatedly below.

(i) For every Banach space X, $R(X)^*$ is isomorphic to $R(X^*)$.

(ii) If Y is isomorphic to a subspace of X, then $R(Y)$ is isomorphic to a subspace of $R(X)$.

(To verify (ii), let $T\colon Y\to X$ be an isomorphism into. Then $T^{**}\colon Y^{**}\to X^{**}$ extends T and induces in an obvious way an operator $\tilde{T}\colon Y^{**}/Y\to X^{**}/X$ which is, as easily verified, also an isomorphism into.)

The first method we present for enlarging $R^k(X)$ is that of taking transfinite duals, which were already defined in the introduction.

Proposition 5. *For every Banach space X and every integer k*

$$\dim R^k(X^{(\omega)})\geqslant 2\dim R^k(X).$$

Proof. The only case which requires a proof is of course that in which the dimension of $R^k(X)$ is finite and positive. In that case the dimension of $R^k(X)$ can be characterized by the existence of "triangular" k-dimensional structures in X which is a notion between the local and global k-dimensional structures studied in Section 2. The characterization is as follows. The dimension of $R^k(X)$ is $\geqslant n$ if and only if there are at least n independent triangular k-dimensional structures in X, i.e. vectors $\{x^m_{i_1,i_2,\ldots,i_k}\}$ in X where $1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_k<\infty$ and $m=1,\ldots,n$ and vectors $\{f^l_{j_1,j_2,\ldots,j_k}\}$ in X^* where $1\leqslant j_1\leqslant j_2\leqslant\ldots\leqslant j_k<\infty$ and $l=1,\ldots,n$ so that these vectors form bounded sets and

$$f^l_{j_1,j_2,\ldots,j_k}(x^m_{i_1,i_2,\ldots,i_k}) = \begin{cases} 1 & \text{if } j_p\leqslant i_p,\ 1\leqslant p\leqslant k, \text{ and } l=m,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

The proof of this fact is very similar to the arguments presented in Section 2. More precisely, the proof of the "only if" part resembles that of Proposition 2. In that proof we stopped the construction at the nth step. This was of course not necessary. What was important in the proof of Proposition 2 was the fact that at each step of the construction we had to consider only a finite number of vectors constructed in the previous stages. This is the case in the triangular structure because given any integer j there are only finitely many $f_{j_1,\ldots,j_k}$ for which $j_k=j$. The proof of the "if" part is similar to that of Proposition 3. We omit the details of the verification of this criterion.

Using this criterion we proceed to the proof of the proposition and for simplicity of notation we assume that $k=2$. Put $n=\dim R^2(X)$.

By the above-mentioned criterion, there are bounded sets $\{x^m_{i_1,i_2}\}_{1\leqslant i_1\leqslant i_2<\infty}$, $m=1,\ldots,n$, in X and bounded sets $\{f^l_{j_1,j_2}\}_{1\leqslant j_1\leqslant j_2<\infty}$, $m=1,\ldots,n$, which satisfy (3.1).

We consider the elements of X also as elements of $X^{(2r)}$ for every integer r (without changing their notation) and elements of X^* as elements in $X^{(2s+1)}$ for every integer s. More precisely, if $J_Y\colon Y\to Y^{**}$ denotes the canonical embedding we consider the following chains of isometries

$$X\xrightarrow{J_X}X^{**}\xrightarrow{J_{X^{**}}}X^{(4)}\xrightarrow{J_{X^{(4)}}}X^{(6)}\to\ldots,$$

$$X^*\xrightarrow{J_{X^*}}X^{***}\xrightarrow{J_{X^{***}}}X^{(5)}\to\ldots$$

Thus e.g. an element $x\in X$ is identified with the element $J_{X^{(4)}}J_{X^{**}}J_Xx$ of $X^{(6)}$, and similarly, an element $u\in X^{(5)}$ is identified with $J_{X^{(7)}}J_{X^{(5)}}u$ in $X^{(9)}$. With these identifications the bilinear form (u,v) with $u\in X^{(2r)}$ for some r and $v\in X^{(2s+1)}$ for some s is well defined (in particular, independent from the choice of r and s).

With these identifications in mind we choose for every $1\leqslant i,\ r<\infty$ and $1\leqslant m\leqslant n$ an element $y^m_{i,r}\in X^{(2r)}$ which is a limit point in the weak topology determined by $X^{(2r-1)}$ of the sequence $\{x^m_{i,p}\}^\infty_{p=i}$. Similarly, we choose for $1\leqslant j,\ s<\infty$ and $1\leqslant l\leqslant n$ an element $g^l_{j,s}\in X^{(2s+1)}$ which is a limit point of the sequence $\{f^l_{j,p}\}^\infty_{p=j}$ in the weak topology determined by $X^{(2s)}$.

By passing to the appropriate limits in (3.1) we get

$$(x^m_{i_1,i_2},g^l_{j,s})=0\quad\text{all } i_1,i_2,j,s,m,l. \tag{3.2}$$

$$(y^m_{i,r},f^l_{j_1,j_2})=\begin{cases}1 & \text{if } j_1\leqslant i \text{ and } m=l,\\ 0 & \text{otherwise.}\end{cases} \tag{3.3}$$

By passing in (3.2) to the limit (with respect to i_2) we get

$$(y^m_{i,r},g^l_{j,s})=0\quad\text{all } i,j,m,l \text{ if } r>s. \tag{3.4}$$

Similarly, we get from (3.3) for $s\geqslant r$

$$(y^m_{i,r},g^l_{j,s})=\begin{cases}1 & \text{if } j\leqslant i \text{ and } m=l,\\ 0 & \text{otherwise.}\end{cases} \tag{3.5}$$

Put now $z^m_{i,r}=y^m_{i,1}-y^m_{i,r+1}$; then by (3.3)

$$(z^m_{i,r},f^l_{j_1,j_2})=0\quad\text{all } i,r,j_1,j_2,m,l, \tag{3.6}$$

and by (3.4), and (3.5)

$$(z^m_{i,r},g^l_{j,s})=\begin{cases}1 & \text{if } j\leqslant i,\ s\leqslant r \text{ and } m=l,\\ 0 & \text{otherwise.}\end{cases} \tag{3.7}$$

The relations (3.1), (3.2), (3.6) and (3.7) show that in $X^{(\omega)}$ we have at least $2n$ independent triangular structures, i.e. $\{x^m_{i_1,i_2}\}$ and $\{z^m_{i,r}\}$ with the corresponding functionals being $\{f^l_{j_1,j_2}\}$ and $\{g^l_{j,s}\}$. (Actually the $\{z^m_{i,r}\}$ form a full rectangular structure or global 2-structure in the sense of Definition 2 since we do not have to require that $i\leqslant r$, but this does not matter of course.) This concludes the proof of the proposition.

Remark. For a finite integer n we have that $R^k(X^{(n)})$ is isomorphic to $\big(R^k(X)\big)^{(n)}$. Proposition 5 shows that this is no longer the case if we replace n by an infinite ordinal α.

The objects $R^k(X^{(\alpha)})$ are obviously very complicated even for relatively simple spaces X (unless X is reflexive). In general, they cannot be described explicitly. It is worthwhile, however, to consider at least one example which is relatively simple and which illustrates Proposition 5. Let J be the classical example of James [5] of a quasi-reflexive space. The space J consists of all the sequences $x=(x_1,x_2,\ldots)$ of reals such that

$$\|x\|=\sup\Big[\sum_{i=1}^n(x_{p_i}-x_{p_{i+1}})^2+(x_{p_{n+1}}-x_{p_1})^2\Big]^{1/2}<\infty, \tag{3.8}$$

where the sup is taken over all $p_1\leqslant p_2\leqslant\ldots\leqslant p_{n+1}$, and for which in addition $\lim_{n\to\infty}x_n=0$. James showed in [5] that J^{**} can be identified with the space of all the sequences $x=(x_n)$ for which (3.8) holds. Since (3.8) easily implies that $x_\infty=\lim_{n\to\infty}x_n$ exists, J^{**} is obtained by adding the constant sequence to J. James observed also in [5] that the map $T\colon J^{**}\to J$ defined by

$$T(x_1,x_2,\ldots)=(-x_\infty,x_1-x_\infty,x_2-x_\infty,\ldots)$$

is an isometry onto. Consider now the space Y of two-sided sequences of reals

$$x=(\ldots x_{-2},x_{-1},x_0,x_1,x_2,\ldots)$$

for which (3.8) holds and $\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{-n}=0$. Let Y_n be the subspace of Y consisting of all $x=(x_i)^\infty_{i=-\infty}$ in Y for which $x_i=0$ if $i<-n$. It is clear that each Y_n is isometric to J. We identify J with Y_0. The isometry T defined above can therefore be considered as an isometry T_1 from J^{**} onto Y_1, for which $T_{1|J}$ is the identity. Similarly, the same T defines for every k an isometry T_k from $J^{(2k)}$ onto Y_k such that $T_{k|J^{(2k-2)}}=T_{k-1}$. It follows that $J^{(\omega)}$ can be identified with the subspace $\overline{\bigcup_{k=0}^\infty Y_k}$

of Y which is easily seen to be equal to Y itself. In particular, $J^{(\omega)}$ is isomorphic to $J\oplus J$ and $\dim R(J^{(\omega)}) = 2$. For ordinals $\alpha > \omega$ we do not have such a neat representation of $J^{(\alpha)}$. It follows however that $J^{(\omega\cdot 2)}$ is isomorphic to $J^{(\omega)}\oplus J^{(\omega)}$, i.e. to $J\oplus J\oplus J\oplus J$ and, in general, that $J^{(\omega\cdot k)}$ is isomorphic to $(J\oplus J\oplus \ldots \oplus J)$, with 2^k terms, for every integer k. We have not examined $J^{(\omega^2)}$ in detail. (We know that $R(J^{(\omega^2)})$ is infinite dimensional by the preceding remarks but we did not check e.g. whether $R^2(J^{(\omega^2)}) \neq \{0\}$.)

We state now formally an immediate consequence of Proposition 5.

THEOREM 3. *Let X be a Banach space and k an integer so that $R^k(X) \neq \{0\}$. Then $R^k(X^{(\omega^2)})$ is infinite dimensional.*

Of course, $\dim R^k(X^{(\alpha)}) = \infty$ also for all $\alpha > \omega^2$. On the other hand, the example considered above shows that ω^2 cannot be replaced by a smaller ordinal. If X is the space of James [9], then the results of Section 2 show that in spite of the fact that $R(X^{(\alpha)})$ grows as α is allowed to grow this space remains reflexive for every ordinal α. We would like also to mention here the remark we made already in the introduction, namely that by the principle of local reflexivity [10], for every Banach space X and every ordinal α, $X^{(\alpha)}$ is finitely represented in X.

We pass now to the second construction which enables the enlargement of $R^k(X)$. We give the details only for $k = 1$ since here we can apply a result of Brunel and Sucheston [1]. For $k > 1$ we have first to generalize the result of Brunel and Sucheston to this setting. We shall later on indicate briefly the form this generalization takes for arbitrary k.

THEOREM 4. *Let X be a Banach space admitting a local 1-structure. Then there is a separable Banach space Y which is finitely represented in X such that Y^{**}/Y is non-separable.*

Proof. Brunel and Sucheston [1] proved that there is a Banach space Z which is finitely represented in X and which has an ESA (equal signs additive) normalized basis $\{e_i\}_{i=1}^{\infty}$. A basis $\{e_i\}_{i=1}^{\infty}$ is said to be an ESA *basis* if whenever $p_0 = 0 < p_1 < \ldots < p_m$ and $\operatorname{sgn} a_i = \operatorname{sgn} a_j$ for $p_r < i$, $j \leqslant p_{r+1}$ $(r = 0, \ldots, m-1)$ then

$$\Big\|\sum_{i=1}^{p_m} a_i e_i\Big\| = \Big\|\sum_{r=1}\Big(\sum_{i=p_{r-1}+1}^{p_r} a_i\Big) e_r\Big\|. \tag{3.9}$$

In general, Z^{**} is "small" and actually $\dim Z^{**}/Z = 1$, if Z contains no copy of c_0 or l_1, however starting with such a Z it is easy to construct the desired space Y. Let Y_0 be the linear space of real-valued functions on the unit interval $[0, 1]$ which is spanned by characteristic functions of intervals. For every $f \in Y_0$ there is therefore a partition $t_0 = 0 < t_1 < t_2 < \ldots < t_n = 1$ such that f is constant on $t_i < x < t_{i+1}$, $i = 0, \ldots, n-1$.

Put now for such an f

$$\|f\| = \Big\|\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i} f(t)\,dt\Big) e_i\Big\|. \tag{3.10}$$

If we replace the partition $\{t_i\}_{i=0}^n$ by a finer partition $\{s_j\}_{j=0}^m$ say (i.e. we add new partition points), then property (3.9) ensures that in (3.10) we get the same value for $\|f\|$ if we use in it the partition $\{s_j\}_{j=0}^m$. Hence $\|f\|$ is a well-defined quantity and it is easily verified that it defines a norm on Y_0. The space Y_0 is finitely represented in Z (and thus in X) since for every given $\{f_j\}_{j=1}^l$ in Y_0 there is a partition $\{t_i\}_{i=1}^n$ of $[0, 1]$ such that all the f_j and thus all their linear combinations are constant on each of the intervals (t_i, t_{i+1}) of the partition. By the definition (3.10) of the norm in Y_0 it follows therefore that $\operatorname{span}\{f_j\}_{j=1}^l$ is isometric to a subspace of $\operatorname{span}\{e_i\}_{i=1}^n$ in Z.

Let Y be the completion of Y_0 (normed by (3.10)). Clearly, Y is a Banach space which is finitely represented in X and which is separable (the characteristic functions of intervals with rational end points span a dense subspace of Y).

For every $t \in [0, 1]$ the functional $\varphi_t(f) = \int_0^t f(s)\,ds$ is a functional of norm $\leqslant 1$ on Y. This follows from (3.10) and the trivial observation that for normalized ESA basis $\{e_i\}_{i=1}^{\infty}$ the relation $x^*\big(\sum_i \lambda_i e_i\big) = \sum_i \lambda_i$ defines a functional of norm 1. Let now $0 \leqslant t < \tau \leqslant 1$ and let $g_{t,\tau}$ denote the characteristic function of the interval $[t, \tau]$. Then, by (3.10), $\|g_{t,\tau}\| = \tau - t$. From the definition of φ_t we get that $\varphi_t(g_{t,\tau}) = 0$ and $\varphi_\tau(g_{t,\tau}) = \tau - t$. Hence $\|\varphi_t - \varphi_\tau\| \geqslant 1$ for $t \neq \tau$ and this proves that Y^* and hence also Y^{**} is not separable.

Remark 1. If X is the space constructed by James [9], then by Theorem 1 the space Y constructed in Theorem 4 has the additional property that Y^{**}/Y is reflexive. Several other examples of spaces Y with Y separable and Y^{**}/Y non-separable but reflexive were constructed recently (see [8], [11] and [2]).

Remark 2. The statement of the generalization of Theorem 4 to general k is as follows: Let X admit a local k-structure. Then there is a separable space Y finitely represented in X such that $R^k(Y)$ is non-separable. The space Y is defined as a completion of a space of functions on the k-dimensional unit cube $[0, 1]^k$. The first step of the proof is to generalize the Brunel–Sucheston result to a k-dimensional setting. This can be done by using essentially the same arguments as in [1]. For $k = 2$ the result reads as follows: There is a space Z finitely represented in X and unit vectors $\{e_{i,j}\}_{1\leqslant i,j<\infty}$ in Z satisfying for every choice of scalars $a_{i,j}$ (with only finitely many $\neq 0$) every integer p and every $\lambda \geqslant 0$

(3.11) $$\Big\|\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{p}a_{i,j}e_{i,j}+\lambda a_{p,j}e_{p+1,j}+\sum_{i=p+2}^{\infty}a_{i-1,j}e_{i,j}\Big)\Big\| = \Big\|\sum_{j=1}^{\infty}\Big(\sum_{\substack{i=1\\ i\neq p}}^{\infty}a_{i,j}e_{i,j}+(1+\lambda)\,a_{p,j}e_{p,j}\Big)\Big\|,$$

and a similar relation holding if we interchange the roles of i and j.

The space Y (for $k=2$) is constructed as a completion of the space Y_0 consisting of those functions f on $[0,1]^2$ such that for some $0\leqslant t_0<t_1<\ldots<t_n=1$ and $0=s_0<s_1<\ldots<s_m=1$ the function f is constant on $(t_i,t_{i+1})\times(s_j,s_{j+1})$ for each i and j. The norm of f is now defined by

(3.12) $$\|f\|=\Big\|\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j}f(s,t)\,dsdt\cdot e_{i,j}\Big\|.$$

As in the case $k=1$ the relation (3.11) shows that $\|f\|$ is well defined by (3.12), i.e. does not change by passing to a finer partition. The generalizations to $k>2$ should now be obvious. Let us only remark that the proof that $R^k(Y)$ is non-separable is done by induction on k.

Remark 3. Brunel and Sucheston [1] were led to the notion of the ESA basis and all the questions related to the l_1^n problem by considering ergodic properties of isometries. The notion of k-structure (local or global) will arise in their approach if we study instead of only one isometry a commuting family of k isometries. In view of James' example [9] and the results of this paper it seems that the gap between J convexity and B convexity in the terminology of [1] can be filled from the ergodic point of view only by taking into consideration such families of commuting isometries.

Remark 4. There is a realization of $R^k(X)$ which may be of use in studying its properties. The natural embeddings of $X^{(2k-2)}$ into $X^{(2k)}$ are $J_X^{(2k-2)}, J_{X^{**}}^{(2k-4)}, J_{X^{(4)}}^{(2k-6)}, \ldots, J_{X^{(2k-2)}}$. $R^k(X)$ is simply $X^{2k}/[J_X^{(2k-2)}X^{(2k-2)}, \ldots, J_{X^{(2k-2)}}X^{(2k-2)}]$.

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