A note on approximation of the identity

by

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Abstract. Let $K : L \ln L [0, 1]$ be a non-decreasing function on $[0, 1]$ and $|E_n| = \int_{E_{n+1}} (1, x)$, then it is proved the existence of the pointwise limit of $K_{x-n}(f)(x)$ when $E_n$ is a lacunary sequence and $f(x)$ for lacunary sequences it is shown that there exits $x \in L^1$ such that $K_{x-n}(f)(x)$ diverges a.e. whenever $K$ is unbounded outside the origin. Results of similar nature are also discussed for $f \in L^q$, $1 < p < \infty$.

0. Introduction. In the present note we consider the pointwise convergence of an approximation of the identity, a problem which is usually stated as follows: consider an integrable function $K : R \rightarrow L$ for every function $1 \leq p < \infty$, $K_i(f)(x) = \int_{-\infty}^{\infty} K_i(x)dx$, and let $K_i(f)(x)$ be the convolution of $K_i$ with $f$ at the point $x$; then the problem concerning us will be: Does the $\lim K_i(f)(x)$ exist almost everywhere when $i \rightarrow \infty$?

While the existence of the limit of $K_i(f)$ in norm $p, 1 \leq p < \infty$, is an easily obtainable fact, the mere assumption that $K$ be an $L^p$ function is by no means enough to obtain almost everywhere convergence. It is known that if $K$ is majorized by a radial $L^p$ function the pointwise limit exists a.e. for any $f \in L^p$. A nice treatment can be found in [1].

Using a decomposition lemma for an $L^p$ function, the author establishes inequalities for the maximal operator $\sup_{i} |K_i(f)(x)|$, which will imply pointwise convergence under conditions for $K$ slightly more general than those stated. The goodness of the method is more appreciated when we see that, using it, we can rescue convergence where $K$ has singularities, outside the origin, $K$ in the class $L \ln L$ at its singularities, and a lacunary sequence $\{E_n\}$ is used instead of the continuous parameter $e$. Non-lacunary sequences will be enough to obtain divergence almost everywhere on $L^p$, when the function $K$ is unbounded outside the origin regardless of its type of singularity. We also consider divergence almost everywhere on $L^p$ for $1 < p < \infty$.

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L. We shall present a theorem which is a useful tool in the study of pointwise convergence for a family of convolution operators. Its proof rests principally on the following lemma due to Calderón and Zygmund.

**Lemma 1.** Let $f$ be a non-negative integrable function on $\mathbb{R}^d$, and let $t$ be a number greater than zero. Then there exist sets $G$ and $B$ on $\mathbb{R}^d$ such that

1. $G \cup B = \mathbb{R}^d$, $G \cap B = \emptyset$,
2. for almost every $x \in G$, $f(x) \leq t$,
3. $B$ is the union of cubes $Q_j$, whose interiors are disjoint, and for each cube $Q_j$ we have

\[
\int_{Q_j} f(x) \, dx \leq c t,
\]

where $c$ is a constant depending only on the dimension, $|Q|$ denotes the Lebesgue measure of $Q$.

For a proof of this lemma see for example [1].

**Theorem 1.** Let $\{K_j\}_{j \in \mathbb{N}}$ be a family of measurable functions on $\mathbb{R}^d$ which verifies the two following conditions:

1. The integrals $\int f(x) \, dx$ are uniformly bounded in $a$.
2. If $\phi(x, y)$ denotes the expression $\sup_{a \in \mathbb{R}^d} |K_a(x-y) - K_a(x)|$, then $\int \phi(x, y) \, dx$ is uniformly bounded in $y$ for some constant $0 < q < 1$.

Then the operator $\tilde{f}(x) = \sup_{a \in \mathbb{R}^d} |K_a(x-y)| f(x)$ verifies

(h) For all $f$ belonging to $L^p(\mathbb{R}^d)$ we have

\[
|\phi(x, f(x)) > 0| \leq c_{p} t^{-1} \int |f(y)| \, dy,
\]

where $c_{p}$ depends on the constants arising from (j), (jj) and on the dimension.

(hh) For all $f$ belonging to $L^p(\mathbb{R}^d)$, $1 < p < \infty$, $\|f\|_p \leq \epsilon \|f\|_p$ holds, $c_{p}$ depends on (j), (jj), and $p$ the dimension.

As a reference for the future an operator $\tilde{f}$ will be called weak-type $(1, 1)$ if (h) holds for some constant $c_{1}$, and strong-type $(p, p)$ if for all $p$-integrable functions $f$, $\|f\|_p \leq \epsilon \|f\|_p$ where the constant $c$ is independent of $f$.

**Proof of Theorem 1.** Taking in account (j), we see that $\tilde{f}$ is strong-type $(\infty, \infty)$. If we know in addition that $\tilde{f}$ is weak-type $(1, 1)$, from the Marcinkiewicz interpolation theorem we obtain that our operator is strong-type $(p, p)$ for any $p$, $1 < p < \infty$. For a proof of the Marcinkiewicz theorem see [2]. Therefore we focus our attention on proving that $\tilde{f}$ is weak-type $(1, 1)$.

Consider $f$ an integrable function and $t > 0$, fixed. Now we apply Lemma 1 to $|f|$, $t$ and set

\[
g = fX_0 + \sum_{j} q_j X_j,
\]

where $q_j$ is the average of $f$ over the cube $Q_j$. Let $b$ be the function such that $b + g = f$. From (1.1) and (j), there exists a constant $\mu$ depending on the dimension and the family $\{K_j\}$ such that $\|b\|_1 \leq \mu$. Since $f \leq \tilde{f}$, we have

\[
|\{x \in \mathbb{R}^d, \tilde{f}(x) < \varepsilon\}| \leq 2 \|f\|_1 + \mu|\{x \in \mathbb{R}^d, \tilde{f}(x) < \varepsilon\}|.
\]

Let us denote by $Q_j$ the cube $Q_j$ expanded by a fixed quantity, depending on the dimension and $q_j$ in such a way that if $x \in Q_j$ and $y \in Q_j$, then $|x - y| > r$ where $y$ is the center of the cube $Q_j$. Call $B^c$ the union of the $Q_j$ and the complement in $\mathbb{R}^d$ of $B^c$, $\mathbb{R}^d$. Our election of $Q_j$ and (1.1) allow us to write $\|B\| \leq \mu t^{-1} \|f\|_1$, $\mu$ depending on $q$ and the dimension. Hence

\[
|\{x \in \mathbb{R}^d, \tilde{f}(x) > \mu\}| \leq \mu t^{-1} \|f\|_1 + |\{x \in \mathbb{R}^d, \tilde{f}(x) > \mu\}|.
\]

In order to conclude the estimate for the last term in (1.3), we observe that the integral of $bX_0$ is zero for every $j$. Therefore for any $K_j$

\[
\tilde{b}(x) = \sum_{j} \int_{Q_j} \phi(x, y) b(y) \, dy.
\]

Writing $\tilde{b}(x, y) = \sup_{p} |K_p(x-y) - K_p(x-y)| b(y)$, it follows from (1.4)

\[
\tilde{b}(y) = \sum_{j} \int_{Q_j} \phi(x, y) b(y) \, dy.
\]

Integrating (1.5) over $G^c$, we have

\[
\tilde{b}(x) dx \leq \sum_{j} \int_{Q_j} b(y) \left( \int_{Q_j} \phi(x, y) dx \right) dy
\]

\[
\leq \sum_{j} \int_{Q_j} b(y) \left( \int_{Q_j} \phi(x, y) dx \right) dy \leq \mu_{2} \sum_{j} \int_{Q_j} b(y) \, dy.
\]

Here $\mu_{2}$ is the constant arising from (jj). The fact $\|b\|_2 \leq 2 \|f\|_1$ gives

\[
|\{x \in \mathbb{R}^d, \tilde{b}(x) > \mu\}| \leq \varepsilon t^{-1} \|f(x) dx\|_1.
\]

Collecting (1.2), (1.3) and (1.6), we obtain (h) and the theorem follows.
We shall denote by $L$ a function on $R^M$ such that:

(i) $L$ is a positive integrable function.

(ii) For any $x \in R^M \setminus \{0\}$ there exists the gradient of $L$ and moreover

$$|F_L(x)| \leq C|\|\|^{-M-1}.$$ 

Given a function $K$, write $K_\lambda(x) = \lambda^M K(\lambda x)$ and $K_\lambda(f) = K_\lambda * f$.

**Theorem 2.** Let $L$ be under the conditions (i), (ii) and $K$ be a measurable function such that $|K(x)| \leq L(x)$ for $x \geq 1$. Then the operator $f(x) = \sup_{\lambda > 0} |K_\lambda(f)(x)|$ is weak-type $(1, 1)$, strong-type $(p, p)$, $1 < p < \infty$. Moreover,

$$\lim_{\lambda \to \infty} K_\lambda(f)(x) = f(x) \int K(y) \, dy \quad \text{a.e.}$$

(1.7)

for any $f \in L^p, \ 1 < p < \infty$.

Proof. First observe that

$$f(x) \leq \sup_{\lambda > 0} L_\lambda ||f|| = \sup_{\lambda > 0} L_\lambda * f(x)$$

if our function $f$ is non-negative, but it is clear that it is enough to consider only this case. Hence the statements for the operator $f$ follow if (1.1) holds for the family $(L_\lambda)_{\lambda > 1}$; indeed, we have:

$$\int_0^1 \varphi(x, y) \, dy = \int_0^1 \varphi \left( x + \frac{y}{|y|} \right) \, dy.$$ 

By the mean value theorem and (ii),

$$\lambda^M \left| \int_0^1 \left( x + \frac{y}{|y|} \right) \, dy - L(\lambda x) \right| \leq C \frac{e}{|\lambda x|^{-M-1}}.$$ 

Using (1.8) and (1.9), we have

$$\sup_{\lambda > 1} \int_0^1 \varphi(x, y) \, dy < \infty.$$ 

In order to obtain (1.7) observe first that it is true for any function $f \in C_0(R^M)$, i.e. for $f$ continuous with compact support. Now define a.e. the following operator

$$\tilde{f}(x) = \lim_{\lambda \to \infty} f(x) \int K(\lambda y) \, dy - K_\lambda(f)(x).$$

It will be enough to prove that $\tilde{f}(x) = 0$ a.e. Take a sequence $(f_n)$ in $C_0(R^M)$ such that $|f - f_n|_{L^p} \to 0$, $1 < p < \infty$. Since

$$\tilde{f}(x) = \|\tilde{f}(x)\| + \tilde{f}(x), \tilde{f}(x) = f - f_n(x),$$

we have

$$(1.10) \quad \|f - f_n\| \leq C_\lambda t^{\frac{\alpha}{\beta}} \int |f - f_n| \, ds.$$ 

Then (1.10) and the fact that $f_n$ tends in norm to $f$ establishes for all

$$(1.11) \quad \|f - f_n\| \to 0,$$ 

i.e. $f_n \to f$ a.e.

Hence (1.7) for any $f \in L^p$ can be obtained by reducing this to the case already done. In fact, take any natural number $N$, and write $f_N = f \chi_{|x| < N}$, $f_N = f - f_1$. We know that $\lim_{\lambda \to \infty} K_\lambda(f_N)(x)$ exists a.e., besides

$$\int_{|x| > N} |K_\lambda(f_N)(x)| \leq \int_{|x| > N} |K_\lambda| \, dx.$$ 

if $|x| \leq N/2$, that is, $K_\lambda(f_N)(x) \to 0$ as $\lambda \to \infty$. This completes the theorem.

The condition given in (1.1) to obtain the pointwise convergence is the following:

$$I(|x|) \leq 1,$$

(1.11)

where $I(|x|)$ is finite and $I(r)$ non-increasing. But if we are under condition (1.11) for $I$ it is easy to construct $L$ which satisfies (I) and (II) and still $|K(x)| \leq L(x)$. A natural process that allows us to smooth $I(|x|)$ as much as we need outside the origin is the following

$$L(x) = \frac{2^M}{|x|^{M+1}} \int_{|y| \leq 1} I(r)^{M-1} \, dr,$$

then we have

$$I(|x|) \leq L(x) \leq \frac{2^M}{|x|^{M+1}} \int_{|y| \leq 1} \frac{1}{|y|^{M+1}} \, dr$$

and

$$|V_L(x)| \leq \frac{C}{|x|^{M+1}} \quad \text{for} \quad x \neq 0, \text{a.e.}$$

The next goal will be to use Theorem 1 to obtain pointwise convergence for an approximation of the identity where we translate our majorant function $L$ of Theorem 2. We shall restrict our attention to the one-dimensional case, this will simplify the computations. A sequence $(\lambda_n)$ is called a lacunary sequence if there exists a number $q$, $0 < q < 1$, such that for any $n$ we have $0 < \lambda_n < q^{2n+1}$. If for a function $f$ defined on a measurable set $B \subseteq R$ we have that

$$\int_B |f| \chi_{\{f \geq 0\}} \, dx = \int_B |f| \chi_{\{f \leq 0\}} \, dx$$

is finite, we say that $f$ belongs to the class $L^1(B)$. 

\[ \text{An approximation of the identity} \]

\[ \text{115} \]
Theorem 3. Let $L$ be a positive non-decreasing function with support in $[0, 1]$ belonging to $L^\infty([0, 1])$, let $(\lambda_n)$ be a lacunary sequence, and let $K$ be a measurable function such that $|K(x)| \leq L(x)$; then the operator $f(x) = \sup_n |E_n f(x)|$ is weak-type $(1,1)$. Moreover, for any function $f$ locally integrable on $R$ we have

\[
\lim_{n \to \infty} E_n f(x) = f(x) \frac{1}{g(y)} \int_0^1 K(y) dy \text{ a.e.} \tag{1.12}
\]

Proof. We see first that $f$ is weak-type $(1,1)$. It is enough to verify condition $(jj)$ in Theorem 1 for the family $(\lambda_n L(\lambda_n x))$. Let $q$ be the number attached to the lacunary sequence $(\lambda_n)$; choose any integer $s$ such that $q - 2q < 1$. Without loss of generality we may assume that $L(x) = 0$ if $0 \leq x < q - q$. For the remaining part of $L$ we apply Theorem 2. Call, as before,

\[
\varphi(x, y) = \sup_n |L(\lambda_n (x - y)) - L(\lambda_n x)|
\]

and observe that

\[
\sup_{x \in [0, 1]} \int_{x + q}^{x + q} \varphi(x, y) dy \leq C \int_0^1 L(y) dy.
\]

So it suffices to consider the case $0 < y < q$. Negative $y$'s have analogous calculation. Let us consider $y \in (\lambda_n^{-1}, \lambda_n^{-1})$ and let us split the integral $\int_0^1 \varphi(x, y) dx$ as follows

\[
\sum_{n=0}^{s-1} \int_{\lambda_{n+1}}^{\lambda_n} + \int_{\lambda_n}^{\lambda_{n+1}} + \int_{\lambda_{n+1}}^{1} + \int_{1}^{x + q} + \int_{x + q}^{x + q}.
\]

We assume that $\lambda_n = 1$. Denote the integrals in (1.13) by $I_1, I_2, A, B$, respectively. It is not hard to see that $A + B \leq C \int_0^1 L(y) dy$, where the constant $C$ depends only on $q$. While for the $I_2$'s we have the estimate

\[
I_2 = \int_{\lambda_n}^{\lambda_{n+1}} L(\lambda_{n+1}(x - y)) dy + \int_{\lambda_n}^{\lambda_{n+1}} L(\lambda_n x) dx \leq \int_{\lambda_n}^{\lambda_{n+1}} \frac{1}{y} \int_0^1 L(1 - x) dx + \frac{1}{y} \int_0^1 L(1 - x) dx \leq 1 \int_{\lambda_n}^{\lambda_{n+1}} \frac{1}{y} \int_0^1 L(1 - x) dx.
\]

It is on $J_1$, where the fact that we have a difference is used

\[
J_1 \leq \int_{\lambda_n}^{\lambda_{n+1}} L(\lambda_{n+1}(x - y)) dy + \int_{\lambda_n}^{\lambda_{n+1}} L(\lambda_n x) dx \leq 1 \int_{\lambda_n}^{\lambda_{n+1}} \frac{1}{y} \int_0^1 L(1 - x) dx.
\]

But

\[
\sum_{n=0}^{s-1} \int_{\lambda_n}^{\lambda_{n+1}} L(1 - x) dx \leq D \int_0^1 L(x) dx
\]

This concludes the proof that $f$ is weak-type $(1,1)$. Since the existence of the limit (1.12) is a local problem, it suffices to consider $f$ in $L^1(R)$ and the proof follows the lines of that Theorem 2.

We shall see that the character of lacunarity for the sequence $(\lambda_n)$ is an essential condition in order to assume existence of the limit (1.12) if we allow for the function $K$ singularities outside the origin. We write $B + a = (y + x, y + x) R$.

Lemma 2. Let $(E_n)$ be a sequence of measurable sets in $[0, 1]$ such that $\sum |E_n| = \infty$; then there exists a sequence $(a_n)$ in $[0, 1]$ and an interval $A \subseteq [0, 1]$ such that for almost every $x \in A$ there are infinitely many $E_n + a_n$ which contain $x$.

For a proof of this lemma with a slightly different statement see [2], p. 116, vol. II.

Theorem 4. Let $L$ be a non-decreasing positive unbounded function on $[0, 1]$, let $(\lambda_n)$ be a sequence increasing to infinity but non-lacunary, then there exists a function $f$ in $L^1(R)$ for which

\[
\limsup_{n \to \infty} L_n f(x) = + \infty \text{ a.e. } x \in R.
\]

Proof. Call $L(x) = \sup_{n \in \mathbb{N}} \lambda_n L(\lambda_n x)$, where $\lambda_n = q_n$, and set

\[
\lambda(t) = \mathbb{1}_{(0, \frac{1}{2})} \mathbb{1}_{(L(x) \geq t)}.
\]

A computation shows that

\[
\lambda(t) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \mathbb{1}_{(0, \frac{1}{2})} \mathbb{1}_{(L(x) \geq t)}.
\]

Taking in account the nature of our problem (1.14), and the non-lacunarity of the sequence $(\lambda_n)$ we can assume that $\lim_{n \to \infty} \lambda_n \lambda_n^{-1} = 1$. Hence there
exists a sequence \( \{S_n\} \) increasing to 1 in such a way that \( L(s) \geq n \) for \( S_n \leq s \leq 1 \) and a sequence of integers \( \{m(n)\} \) for which we have \( \lambda_m \geq \lambda_{m+1}, \) where \( m \geq m(n). \) Let us set \( t_n = n^{-1} s_m(n) \), then

\[
(1.15) \quad t_n \rightarrow \infty. \]

Therefore it is possible to find a sequence \( \{f_n\} \) of positive functions in \( L^1(0,1) \) whose integrals are 1, a sequence \( \{a_n(\theta)\} \) tending to infinity and a sequence \( \{e_n\} \) such that

\[
(1.16) \quad |\{0 < s < 4^{-1}; f_n(s) > e_n\}| \geq c_n e_n^{-1}. \]

Indeed, if the construction of (1.16) were not possible we would have for the sequence \( f_n = n \chi_{[0,1/m]}, \) for example, that the following inequality is true

\[
|\{0 < s < 4^{-1}; f_n(s) > t\}| \leq e/\|t\| \]

for some constant \( e. \) This implies, after a limit process, that

\[
\sup_{t \geq 0} \|t\| < \infty \]

which is in contradiction with (1.15). That is, (1.16) is true. Now, choosing a subsequence, with repetitions if it were necessary, we can assume for the elements in (1.16) the additional property

\[
\sum c_n e_n^{-1} = \infty, \quad \sum s^2 < \infty. \]

We apply Lemma 1 to the sets in (1.16) and we find a sequence \( \{a_n\} \) in \( [0,1] \) with the properties established there for an interval \( A \subseteq [0,1], \) Consider the sequence \( a_n \) tending to infinity but such that \( \sum m_n e_n^{-1} < \infty \) and define the function

\[
f(s) = \sum m_n e_n^{-1} f_n(s - a_n); \]

\( f \) is an integrable function and \( \int f(s) = +\infty \) a.e. on \( A. \) Hence we have

\[
\limsup_{n \to \infty} E_n(f(s)) = +\infty \quad \text{a.e. on } A, \]

and now (1.14) follows easily.

2. After Theorems 3 and 4 it is reasonable to ask ourselves whether or not there is divergence a.e. for \( f \in L^p(R) \) where \( p > 1. \) Naturally, the answer will depend on how strong is the singularity of the approximating kernel \( K, \) outside the origin. We shall see that it is possible to have an \( L^1 \) kernel \( K \) for which, given \( p, 1 < p < \infty, \) there always exists a function \( f \in L^p \) such that

\[
\limsup_{n \to \infty} K_n f(s) = +\infty \quad \text{a.e.} \]

Also there are kernels whose singularity is so weak that only for \( p = 1 \) we can select a function in \( L^1 \) with the above property, while for the remaining \( 1 < p < \infty, \) we obtain pointwise convergence a.e.

For the next theorem, \( \{\lambda_n\} \) will denote any sequence tending to infinity such that the quotients \( Q_n = \lambda_{n+1} e_n^{-1} \) are non-increasing, and for some \( \alpha \geq 1 \) we have \( 1 < Q_n < 1 + \alpha e_n^{-1} \), or any sequence for which we can find a subsequence with those properties. If we call \( t_m \) the minimum \( i \) such that the following is true

\[
(2.1) \quad \lambda_m \leq \lambda_i, \quad Q_i + \alpha e_i^{-1} \leq 1 + \alpha e_i^{-1}; \]

then it follows for our type of sequences that

\[
(2.2) \quad \text{If } \lambda_i \geq \lambda_m, \text{ then (2.1) holds for } \lambda_i, \]

\[
(2.3) \quad \lambda_i \leq \alpha e_i^{-1}. \]

We also use the following notation. If \( L \) is a function in \( L^1(0,1) \) and \( p \) is the number, \( 1 < p < \infty, \) given by

\[
\sup_{g \geq 1} \|g \geq 1; L \in L^p([0,1])\|
\]

we denote by \( p_L \) the conjugate of \( p, \) i.e. \( p_L^{-1} + p^{-1} = 1 \) for \( 1 < p < \infty, \)

\[
p_L = 1 \quad \text{if } p = \infty \quad \text{if } p = 1. \]

THEOREM 5. Let \( L \) be a positive function, non-decreasing, in \( L^1(0,1) \) but unbounded. Then

(k) If \( 1 < q < p \) there exists \( f \in L^p(R) \) such that

\[
(2.4) \quad \liminf_{n \to \infty} E_n(f(s)) = +\infty \quad \text{a.e.} \]

if \( p_L = 1, (2.4) \) holds with \( q = 1. \)

(kk) If \( p_L < q < \infty \) for all \( f \in L^p(R), \)

\[
(2.5) \quad \lim_{n \to \infty} L_n(f(s)) = f(s) \int L(s) \, ds \quad \text{a.e.}; \]

if \( p_L = +\infty, (2.5) \) holds with \( q = +\infty. \)

Proof of (k). We write

\[
L_n(f(s)) = \lambda_n L\lambda_n(s), \quad f_m(s) = \lambda_m e_n^{-1} \chi_{[n,m]}(\lambda_m e_n^{-1} s),
\]

\[
(2.6) \quad L_n(f_m(s)) = L_n f_m(s) = \lambda_n e_n^{-1} \int_{\min(t_m,N_{t_m})}^{\max(t_m,N_{t_m})} L(y) \, dy,
\]

\[
f_m(s) = \sup_n L_n(f_m(s)). \]
Observing that if $1 < q < p_L$, then $\sup_m C_m = \infty$. Indeed, if this were not true, we would have
\[
\mathcal{L}^{1/q}(\mathbb{R}^d) \left( 1 - \frac{1}{\lambda_m} \right) \leq \mathcal{L}^{1/q}(\mathbb{R}^d) \int_{1 - \frac{1}{\lambda_m}}^1 L(y) dy \leq \varepsilon;
\]
therefore, for other constant $\delta$,
\[
L^r(x) \leq \delta \left( \frac{1}{1-m^s} \right)^{\alpha}, \quad 0 < s < 1.
\]
Then, if we choose $p < a < q$, we would have $L \in L^r[0,1]$, which contradicts our definition of $p$. Hence, taking a subsequence with repetitions, if it were necessary, we can assume
\[
\sum_m |I_m| = \infty, \quad \sum_m S_m^{-r} < \infty;
\]
finally, take a sequence $R_m$ tending to infinity, but

\[
\sum_m R_m^r S_m^{-r} < \infty.
\]

Let us set
\[
f(x) = \sum_m R_m S_m^{-1} h_{\alpha_m}(x - a_m),
\]
where $\{a_m\}$ is a sequence chosen in such a way that $\max(I_m + a_m) = \min(I_{m+1} + a_{m+1})$. The fact that $|h_{\alpha_m}|_p = 1$, (2.8) and (2.10) imply $f \in L^r_R([R])$. Moreover, the function $f$ has the property that given $\lambda > 0$ we can select $\alpha > 0$ such that

\[
\{a_m\}, \quad (\pi_1 + \infty) \subseteq \{a_m\} : f(x) > \lambda).
\]

(2.11) follows from: $f(x) \geq R_m S_m^{-1} h_{\alpha_m}(x - a_m)$ for all $m$, (2.8), (2.9), the fact that $\{I_m + a_m\}$ are non-overlapping and $\lim R_m = \infty$.

Now the function
\[
F(x) = \sum_m u^{-1} f(x + \pi_m)
\]
belongs to $L^r_R([R])$ but $F(x) = +\infty$ if $x > 0$.

The second result, (kk), is contained in the next lemma.

**Lemma 3.** [K < L^p(R^d), K of bounded support. Suppose f \in L^p(R^d)]

where $p' \leq q \leq \infty, p'$ the conjugate of $p$. Then

\[
\lim_{\lambda \to \infty} \lambda f(x) - f(x) \int K(x) dx a.e.
\]

Moreover, if $q = \infty$, nothing but the integrability of $K$ is necessary.
Proof. We assume that the unitary ball is the support of \( K \); then

\[
|K_k(f)(x)| \leqslant \int_{|y| \leqslant 1} |K(y)| |f(x - y)| dy \\
\leqslant |K|_\infty \left( \int_{|y| \leqslant 1} |f(x - y)| dy \right)^{1/q},
\]

where \( p' = q < \infty, \quad 1/q' + 1/q = 1. \)

If \( f \) denotes the Hardy-Littlewood maximal function of \(|f|^q\), we can rewrite (2.13) as

\[
\tilde{f}(x) \leqslant C \left( \tilde{f}(x) \right)^{1/q}.
\]

Now the fact that \( f \), i.e., the Hardy-Littlewood maximal operator of \( f \), is weak-type \((1, 1)\) (see [2]) implies (2.12). If \( q = \infty \), we split \( K \) as follows:

\[
\tilde{K}(x) = K(Y \cap K) + K(Y \setminus K).
\]

Then

\[
\|K_k f - f\|_1 \leqslant \int_{|y| \leqslant 1} |K(y)| \|f(x - y) - f(x)\| dy + \\
\int |K(y)| |f(x - y) - f(x)| dy = \|K_k f - f\|_1.
\]

It is not difficult to establish the following two inequalities:

\[
\|K_k f - f\|_1 \leqslant \left( \int \frac{K(y)}{|y|^N} \right)^{1/q} \|f\|_1,
\]

\[
\|K_k f - f\|_1 \leqslant \left( \int \frac{|K(y)|}{|y|^N} \right)^{1/q} \|f\|_1 + \\
\int |K(y)| |f(x - y) - f(x)| dy + T\lambda \int_{|y| < 1 - \lambda} |f(x - y) - f(x)| dy.
\]

By \( l_1 \) we denote as usual the space of all sequences \( x = (x_1, x_2, \ldots) \) with \( \|x\| = \sum |x_i| < \infty. \) We say that \( d(B, C) \leqslant \lambda \) for some Banach spaces \( B \) and \( C \) and a real \( \lambda \) if there is an (always linear here) operator \( T \) from \( B \) onto \( C \) such that \( \|T\| \leqslant \lambda. \) A Banach space \( X \) is said to be \textit{finitey represented} in a Banach space \( X \) if for every finite dimensional subspace \( B \) of \( Y \) and every \( \epsilon > 0 \) there is a subspace \( C \) of \( X \) such that \( d(B, C) \leqslant 1 + \epsilon. \) If \( F \) is a property which is meaningful for general Banach spaces we say that a Banach space \( X \) is \textit{super} \( P \) if every Banach space \( Y \) \textit{finite}ly represented in \( X \) has property \( P. \) Of particular importance is the property \textit{super reflexive} introduced by James. Thus, according to the general rule, a Banach space \( X \) is super reflexive if every Banach space \( Y \) \textit{finite}ly represented in \( X \) is reflexive.

In paper [6] in which James posed the \( l_1^n \) problem he proved that the

\[\text{References}\]


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