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A note on approximation of the identity

by

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Abstract. Let $K \in L \ln L[0,1]$ be a non-decreasing function on $[0,1]$ and $|K_{\lambda_n}(x)| = \lambda_n K(\lambda_m x)$, then it is proved the existence of the pointwise limit of $K_{\lambda_n} * f(x)$ when $\{\lambda_n\}$ is a lacunary sequence and $f \in L^1_{\text{loc}}$. For non-lacunary sequences it is shown that there exists $f \in L^1$ such that $K_{\lambda_n} * f(x)$ diverges a.e. whenever K is unbounded outside the origin. Results of similar nature are also discussed for $f \in L^p$, $1 < p < \infty$.

0. Introduction. In the present note we consider the pointwise convergence of an approximation of the identity, a problem which is usually stated as follows: consider an integrable function K on $\boldsymbol{R}^M$; let f be an L^p function, $1 \leqslant p \leqslant \infty$, $K_\lambda(x) = \lambda^M K(\lambda x)$, and let $K_\lambda(f)(x)$ be the convolution of K_λ with f at the point x; then the problem concerning us will be: *Does the* $\lim K_\lambda(f)(x)$ *exist almost everywhere when* λ *tends to infinity?*

While the existence of the limit of $K_\lambda(f)$ in norm p, $1 \leqslant p < \infty$, is an easily obtainable fact, the mere assumption that K be an L^1 function is by no means enough to obtain almost everywhere convergence. It is known that if K is majorized by a radial L^1 function the pointwise limit exists a.e. for any $f \in L^p$, $1 \leqslant p \leqslant \infty$. A nice treatment can be found in [1].

Using a decomposition lemma for an L^1 function, we shall establish inequalities for the "maximal operator" $\sup\limits_{\lambda>0} |K_\lambda(f)(x)|$, which will imply pointwise convergence under conditions for K slightly more general than those stated. The goodness of the method is more appreciable when we see that, using it, we can rescue convergence where K has singularities, outside the origin, K in the class $L \ln^+ L$ at its singularities, and a lacunary sequence $\{\lambda_n\}$ is used instead of the continuous parameter λ. Non-lacunary sequences will be enough to obtain divergence almost everywhere on L^1, when the function K is unbounded outside the origin regardless of its type of singularity. We also consider divergence almost everywhere on L^p for $1 < p < \infty$.

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1. We shall present a theorem which is a useful tool in the study of pointwise convergence for a family of convolution operators. Its proof rests principally on the following lemma due to Calderón and Zygmund.

LEMMA 1. *Let f be a non-negative integrable function on $\boldsymbol{R}^M$, and let t be a number greater than zero. Then there exist sets G and B on $\boldsymbol{R}^M$ such that*

(i) $G\cup B = \boldsymbol{R}^M$, $G\cap B = \emptyset$,

(ii) *for almost every $x\in G$, $f(x)\leqslant t$,*

(iii) *B is the union of cubes Q_j, whose interiors are disjoint, and for each cube Q_j we have*

$$t\leqslant \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx\leqslant ct, \tag{1.1}$$

where c is a constant depending only on the dimension, $|Q|$ denotes the Lebesgue measure of Q.

For a proof of this lemma see for example [1].

THEOREM 1. *Let $\{K_\alpha\}_{\alpha\in\Delta}$ be a family of measurable functions on $\boldsymbol{R}^M$ which verifies the two following conditions:*

(j) *The integrals $\int|K_\alpha(x)|\,dx$ are uniformly bounded in α.*

(jj) *If $\varphi(x,y)$ denotes the expression $\sup_{\alpha\in\Delta}|K_\alpha(x-y)-K_\alpha(x)|$, then $\int_{q|x|\geqslant|y|}\varphi(x,y)\,dx$ is uniformly bounded in y for some constant $0<q<1$.*

Then the operator

$$\bar f(x) = \sup_{\alpha\in\Delta}|K_\alpha * f|(x)$$

verifies

(h) *For all f belonging to $L^1(\boldsymbol{R}^M)$ we have*

$$|\{x, \bar f(x)>t\}|\leqslant c_1 t^{-1}\int|f(x)|\,dx,$$

c_1 depends on the constants arising from (j), (jj) *and on the dimension.*

(hh) *For all f belonging to $L^p(\boldsymbol{R}^M)$, $1<p\leqslant\infty$, $\|\bar f\|_p\leqslant c_p\|f\|_p$ holds, c_p depends on* (j), (jj), *p and the dimension.*

As a reference for the future an operator $\bar f$ will be called *weak-type* $(1,1)$ if (h) holds for some constant c_1, and *strong-type* (p,p) if for all p-integrable functions f, $\|\bar f\|_p\leqslant c\|f\|_p$ where the constant c is independent of f.

Proof of Theorem 1. Taking in account (j), we see that $\bar f$ is strong-type (∞,∞). If we know in addition that $\bar f$ is weak-type $(1,1)$, from the Marcinkiewicz interpolation theorem we obtain that our operator is strong-type (p,p) for any p, $1<p\leqslant\infty$. For a proof of the Marcinkiewicz theorem see [2]. Therefore we focus our attention on proving that $\bar f$ is weak-type $(1,1)$. Consider f an integrable function and $t>0$, fixed. Now we apply Lemma 1 to $|f|$, t and set

$$g = fX_G+\sum_j a_j X_0,$$

where a_j is the average of f over the cube Q_j. Let b be the function such that $b+g=f$. From (1.1) and (j), there exists a constant μ depending on the dimension and the family $\{K_\alpha\}$ such that $\|\bar g\|_\infty\leqslant\mu t$. Since $\bar f\leqslant\bar g+\bar b$, we have

$$|\{x\in\boldsymbol{R}^M, \bar f(x)>2\mu t\}|\leqslant|\{x\in\boldsymbol{R}^M, \bar b(x)>\mu t\}|. \tag{1.2}$$

Let us denote by $Q_j^\sim$ the cube Q_j expanded by a fixed quantity, depending on the dimension and q, in such a way that if $x\notin Q_j^\sim$ and $y\in Q_j$, then $|x-y_j|q\geqslant|y-y_j|$ where y_j is the center of the cube Q_j. Call $B^\sim$ the union of the $Q_j^\sim$ and the complement in $\boldsymbol{R}^M$ of $B^\sim$, $G^\sim$. Our election of $Q_j^\sim$ and (1.1) allow us to write $|B^\sim|\leqslant\mu_1 t^{-1}\|f\|_1$, μ_1 depending on q and the dimension. Hence

$$|\{x\in\boldsymbol{R}^M, \bar b(x)>\mu t\}|\leqslant\mu_1 t^{-1}\|f\|_1+|\{x\in G^\sim, \bar b(x)>\mu t\}|. \tag{1.3}$$

In order to conclude the estimate for the last term in (1.3) we observe that the integral of bX_{Q_j} is zero for every j. Therefore for any K_β

$$K_\beta * b(x) = \sum_j\int_{Q_j}\{K_\beta(x-y)-K_\beta(x-y_j)\}b(y)\,dy. \tag{1.4}$$

Writing $\varphi_j(x,y) = \sup_\beta|K_\beta(x-y)-K_\beta(x-y_j)|$, it follows from (1.4)

$$\bar b(x)\leqslant\sum_j\int_{Q_j}\varphi_j(x,y)|b(y)|\,dy. \tag{1.5}$$

Integrating (1.5) over $G^\sim$, we have

$$\int_{G^\sim}\bar b(x)\,dx\leqslant\sum_j\int_{Q_j}|b(y)|\Big(\int_{x\notin Q_j^\sim}\varphi_j(x,y)\,dx\Big)dy$$

$$\leqslant\sum_j\int_{Q_j}|b(y)|\Big(\int_{|x-y_j|q\geqslant|y-y_j|}\varphi_j(x,y)\,dx\Big)dy\leqslant\mu_2\sum_j\int_{Q_j}|b(y)|\,dy.$$

Here μ_2 is the constant arising from (jj). The fact $\|b\|_1\leqslant 2\|f\|_1$ gives

$$|\{x\in G^\sim, \bar b(x)>\mu t\}|\leqslant ct^{-1}\int|f(x)|\,dx. \tag{1.6}$$

Collecting (1.2), (1.3) and (1.6), we obtain (h) and the theorem follows.

As an easy consequence of Theorem 1 we have the next theorem about approximations of the identity.

We shall denote by L a function on $\boldsymbol{R}^M$ such that:

(l) L is a positive integrable function.

(ll) For any $x \in \boldsymbol{R}^M - \{0\}$ there exists the gradient of L and moreover

$$|\nabla L(x)| \leqslant C\,|x|^{-M-1}.$$

Given a function K, write $K_\lambda(x) = \lambda^M K(\lambda x)$ and $K_\lambda(f) = K_\lambda * f$.

THEOREM 2. *Let L be under the conditions* (l), (ll) *and K be a measurable function such that $|K(x)| \leqslant L(x)$ for a.e. x. Then the operator $\bar{f}(x) = \sup\limits_{\lambda>0} |K_\lambda(f)(x)|$ is weak-type* $(1, 1)$, *strong-type* (p, p), $1 < p \leqslant \infty$. *Moreover,*

$$\lim_{\lambda\to\infty} K_\lambda(f)(x) = f(x) \int K(y)\,dy \quad \text{a.e.} \tag{1.7}$$

for any $f \in L^p$, $1 \leqslant p \leqslant \infty$.

Proof. First observe that

$$f(x) \leqslant \sup_{\lambda>0} |L_\lambda| * |f|(x) = \sup_{\lambda>0} L_\lambda * f(x)$$

if our function f is non-negative, but it is clear that it is enough to consider only this case. Hence the statements for the operator $\bar{f}$ follow if (jj) holds for the family $\{L_\lambda\}_{\lambda>0}$; indeed, we have:

$$\int\limits_{|x|\geqslant 2|y|} \varphi(x, y)\,dx = \int\limits_{|x|\geqslant 2} \varphi\left(x, \frac{y}{|y|}\right) dx. \tag{1.8}$$

By the mean value theorem and (ll),

$$\lambda^M \left| L\left(\lambda\left(x - \frac{y}{|y|}\right)\right) - L(\lambda x)\right| \leqslant \frac{c}{(|x|-1)^{M+1}}. \tag{1.9}$$

Using (1.8) and (1.9), we have

$$\sup_y \int\limits_{|x|\geqslant 2|y|} \varphi(x, y)\,dx < \infty.$$

In order to obtain (1.7) observe first that it is true for any function $f \in C_0(\boldsymbol{R}^M)$, i.e. for f continuous with compact support. Now define a.e. the following operator

$$\tilde{f}(x) = \varlimsup_{\lambda\to\infty} \left| f(x) \int K(y)\,dy - K_\lambda(f)(x)\right|.$$

It will be enough to prove that $\tilde{f}(x) = 0$ a.e. Take a sequence $\{f_n\}$ in $C_0(\boldsymbol{R}^M)$ such that $\|f - f_n\|_p \to 0$, have $1 \leqslant p < \infty$. Since

$$\tilde{f}(x) \leqslant C\,|f|(x) + \bar{f}(x),\ \tilde{f}(x) = \overline{f - f_n}(x),$$

we have

$$|\{x, \tilde{f}(x) > t\}| \leqslant C_p t^{-p} \int |f - f_n|^p\,dx. \tag{1.10}$$

Then (1.10) and the fact that f_n tends in norm to f establishes for all $t > 0$

$$|\{x, \tilde{f}(x) > t\}| = 0, \quad \text{i.e.} \quad \tilde{f}(x) = 0 \text{ a.e.}$$

Hence (1.7) for any $f \in L^\infty$ can be obtained by reducing this to the case already done. In fact, take any natural number N, and write $f_1 = f X_{\{|x|\leqslant N\}}$, $f_2 = f - f_1$. We know that $\lim\limits_{\lambda\to\infty} K_\lambda(f_1)(x)$ exists a.e., besides

$$|K_\lambda f_2(x)| \leqslant |f|_\infty \int\limits_{|y|\geqslant N\lambda/2} |K(y)|\,dy$$

if $|x| \leqslant N/2$, that is, $K_\lambda(f_2)(x) \to 0$ as $\lambda \to \infty$. This completes the theorem.

The condition given in [1] to obtain the pointwise convergence is the following:

$$|K(x)| \leqslant l(|x|), \tag{1.11}$$

where $\int\limits_0^\infty r^{M-1} l(r)\,dr$ is finite and $l(r)$ non-increasing. But if we are under condition (1.11) for l it is easy to construct L which satisfies (l) and (ll) and still $|K(x)| \leqslant L(x)$. A natural process that allow us to smooth $l(|x|)$ as much as we need outside the origin is the following

$$L(x) = \frac{2^M}{|x|^M} \cdot \int\limits_{|x|/2}^{|x|} l(r) r^{M-1}\,dr;$$

then we have

$$l(|x|) \leqslant L(x) \leqslant 2^{M-1} l\left(\frac{|x|}{2}\right) \quad \text{and} \quad |\nabla L(x)| \leqslant \frac{c}{|x|^{M+1}} \quad \text{for } x \neq 0, \text{ a.e.}$$

The next goal will be to use Theorem 1 to obtain pointwise convergence for an approximation of the identity where we translate our majorant function L of Theorem 2. We shall restrict our attention to the one-dimensional case, this will simplify the computations. A sequence $\{\lambda_n\}$ is called a *lacunary sequence* if there exists a number q, $0 < q < 1$, such that for any n we have $0 < \lambda_n \leqslant q\lambda_{n+1}$. If for a function f defined on a measurable set $E \subseteq \boldsymbol{R}$ we have that

$$\int\limits_E |f| \ln[\max(1, |f|)]\,dx = \int\limits_E |f| \ln^+ |f|\,dx$$

is finite, we say that *f belongs to the class $L \ln^+ L(E)$.*

THEOREM 3. *Let L be a positive non-decreasing function with support in $[0,1]$ belonging to $L\ln^+L[0,1]$, let $\{\lambda_n\}$ be a lacunary sequence, and let K be a measurable function such that $|K(x)|\leqslant L(x)$; then the operator $\bar f(x) = \sup_n |K_{\lambda_n}*f|(x)$ is weak-type (1,1). Moreover, for any function f locally integrable on $\boldsymbol{R}$ we have*

$$\lim_{n\to\infty} K_{\lambda_n}*f(x) = f(x)\int_0^1 K(y)\,dy \text{ a.e.} \tag{1.12}$$

Proof. We see first that $\bar f$ is weak-type $(1,1)$. It is enough to verify condition (jj) in Theorem 1 for the family $\{\lambda_n L(\lambda_n x)\}$. Let q be the number attached to the lacunary sequence $\{\lambda_n\}$; choose any integer s such that $q+2q^s<1$. Without loss of generality we may assume that $L(x)=0$ if $0\leqslant x\leqslant q+q^s$. For the remaining part of L we apply Theorem 2. Call, as before,

$$\varphi(x,y) = \sup_n \lambda_n\big|L\big(\lambda_n(x-y)\big)-L(\lambda_n x)\big|$$

and observe that

$$\sup_{0<c\leqslant|y|}\int_{q|x|\geqslant|y|}\varphi(x,y)\,dx\leqslant C\int_0^1 L(x)\,dx.$$

So it suffices to consider the case $0<y<q^s$. Negative y's have analogous calculation. Let us consider $y\in[\lambda_{k+1}^{-1},\lambda_k^{-1}]$ and let us split the integral $\int_{q|x|\geqslant|y|}\varphi(x,y)\,dx$ as follows

$$\sum_{j=0}^{k-s}\int_{\lambda_{j+1}^{-1}}^{\lambda_{j+1}^{-1}+y}+\int_{\lambda_{j+1}^{-1}+y}^{\lambda_j^{-1}}+\int_{yq^{-1}}^{\lambda_{k-s+1}^{-1}}+\int_1^{1+y}. \tag{1.13}$$

We assume that $\lambda_0=1$. Denote the integrals in (1.13) by I_j, J_j, A, B, respectively. It is not hard to see that $A+B\leqslant C\int_0^1 L(x)\,dx$, where the constant C depends only on q. While for the I_j's we have the estimate

$$\lambda_{j+1}\int_{\lambda_{j+1}^{-1}}^{\lambda_{j+1}^{-1}+y} L\big(\lambda_{j+1}(x-y)\big)\,dx+\lambda_j\int_{\lambda_{j+1}^{-1}}^{\lambda_{j+1}^{-1}+y}L(\lambda_j x)\,dx$$

$$\leqslant\int_0^{\lambda_{j+1}y}L(1-x)\,dx+\int_0^{\lambda_j y}L(1-x)\,dx\leqslant 2\int_0^{\lambda_{j+1}y}L(1-x)\,dx.$$

It is on J_j where the fact that we have a difference is used

$$J_j\leqslant\int_{\lambda_{j+1}^{-1}+y}^{\lambda_j^{-1}}\lambda_j\big\{L(\lambda_j x)-L\big(\lambda_j(x-y)\big)\big\}\,dx$$

$$\leqslant\int_0^{\lambda_j y}L(1-x)\,dx\leqslant\int_0^{q^{k-j}}L(1-x)\,dx.$$

But

$$\sum_{j=0}^{k-s}\int_0^{q^{k-j}}L(1-x)\,dx\leqslant D\Big(1+\int_0^1 L(x)\ln^+L(x)\,dx\Big).$$

This concludes the proof that $\bar f$ is weak-type $(1,1)$. Since the existence of the limit (1.12) is a local problem, it suffices to consider f in $L^1(\boldsymbol{R})$ and the proof follows the lines of that Theorem 2.

We shall see that the character of lacunaricity for the sequence $\{\lambda_n\}$ is an essential condition in order to assume existence of the limit (1.7) if we allow for the function K singularities outside the origin. We write $E+x=\{y+x,\ y\in E\}$.

LEMMA 2. *Let $\{E_n\}$ be a sequence of measurable sets in $[0,\frac14]$ such that $\sum_n|E_n|=\infty$; then there exists a sequence $\{x_n\}$ in $[0,1]$ and an interval $A\subseteq[0,1]$ such that for almost every $x\in A$ there are infinite many E_n+x_n which contain x.*

For a proof of this lemma with a slightly different statement see [2], p. 116, vol. II.

THEOREM 4. *Let L be a non-decreasing positive unbounded function on $[0,1]$, let $\{\lambda_n\}$ be a sequence increasing to infinite but non-lacunary, then there exists a function f in $L^1(\boldsymbol{R})$ for which*

$$\limsup_{n\to\infty} L_{\lambda_n}(f)(x) = +\infty \quad \text{a.e. } x\in\boldsymbol{R}. \tag{1.14}$$

Proof. Call $\bar L(x)=\sup_{n\geqslant 0}\lambda_n L(\lambda_n x)$, where $\lambda_0=4$, and set

$$\lambda(t)=|\{0\leqslant x\leqslant\tfrac14,\ \bar L(x)\geqslant t\}|,$$

a computation shows that

$$\lambda(t)=\sum_{n=0}^{\infty}\frac{1}{\lambda_n}\Big|\Big\{\frac{\lambda_n}{\lambda_{n+1}}\leqslant x\leqslant 1,\ \lambda_n L(x)\geqslant t\Big\}\Big|.$$

Taking in account the nature of our problem, (1.14), and the non-lacunaricity of the sequence $\{\lambda_n\}$ we can assume that $\lim_{n\to\infty}\lambda_n\lambda_{n+1}^{-1}=1$. Hence there exists a sequence $\{S_n\}$ increasing to 1 in such a way that $L(x) \geqslant n$ for $S_n \leqslant x \leqslant 1$ and a sequence of integers $\{n_0(n)\}$ for which we have $\lambda_m \geqslant \lambda_{m+1} s_n$ where $m \geqslant n_0(n)$. Let us set $t_n = n\lambda_{n_0(n)}$; then

$$t_n \lambda(t_n) \geqslant n. \tag{1.15}$$

Therefore it is possible to find a sequence $\{f_n\}$ of positive functions in $L^1[0,1]$ whose integrals are 1, a sequence $\{c_n\}$ tending to infinity and a sequence $\{s_n\}$ such that

$$|\{0 \leqslant x \leqslant 4^{-1} \colon \bar{f}_n(x) > s_n\}| \geqslant c_n s_n^{-1}. \tag{1.16}$$

Indeed, if the construction of (1.16) were not possible we would have for the sequence $f_n = nX_{[0,1/n]}$, for example, that the following inequality is true

$$|\{0 \leqslant x \leqslant 4^{-1},\ \bar{f}_n(x) > t\}| \leqslant c/t$$

for some constant c. This implies, after a limit process, that

$$\sup_{t>0} t\lambda(t) < \infty$$

which is in contradiction with (1.15). That is, (1.16) is true. Now, choosing a subsequence, with repetitions if it were necessary, we can assume for the elements in (1.16) the additional property

$$\sum c_n s_n^{-1} = \infty, \qquad \sum s_n^{-1} < \infty.$$

We apply Lemma 1 to the sets in (1.16) and we find a sequence $\{x_n\}$ in $[0,1]$ with the properties established there for an interval $A \subseteq [0,1]$. Consider the sequence m_n tending to infinity but such that $\sum m_n s_n^{-1} < \infty$ and define the function

$$f(x) = \sum m_n s_n^{-1} f_n(x - x_n);$$

f is an integrable function and $\bar{f}(x) = +\infty$ a.e. $x \in A$. Hence we have

$$\limsup_{n\to\infty} L_{\lambda_n}(f)(x) = +\infty \quad \text{a.e. } x \in A$$

and now (1.14) follows easily.

2. After Theorems 3 and 4 it is reasonable to ask ourselves whether or not there is divergence a.e. for $f \in L^p[\boldsymbol{R}]$ where $p > 1$. Naturally, the answer will depend on how strong is the singularity of the approximating kernel K, outside the origin. We shall see that it is possible to have an L^1 kernel K for which, given p, $1 \leqslant p < \infty$, there always exists a function $f \in L^p$ such that

$$\limsup_{\lambda\to\infty} K_\lambda * f(x) = +\infty \text{ a.e.}$$

Also there are kernels whose singularity is so weak that only for $p = 1$ we can select a function in L^1 with the above property, while for the remaining p, $1 < p \leqslant \infty$, we obtain pointwise convergence a.e.

For the next theorem, $\{\lambda_n\}$ will denote any sequence tending to infinity such that the quotients $Q_n = \lambda_{n+1}\lambda_n^{-1}$ are non-increasing, and for some $\omega \geqslant 1$ we have $1 < Q_n \leqslant 1 + c\lambda_n^{-\omega}$, or any sequence for which we can find a subsequence with those properties. If we call i_m the minimum i such that the following is true

$$\lambda_m^\omega \leqslant \lambda_i, \quad Q_i + \lambda_m^{-1} \leqslant 1 + \lambda_{i+1}\lambda_m^{-1-\omega}, \tag{2.1}$$

then it follows for our type of sequences that

$$\textit{If } \lambda_i \geqslant \lambda_{i_m}, \textit{ then (2.1) holds for } \lambda_i, \tag{2.2}$$

$$\lambda_{i_m} \leqslant c\lambda_m^\omega. \tag{2.3}$$

We also use the following notation. If L is a function in $L^1[0,1]$ and p is the number, $1 \leqslant p \leqslant \infty$, given by

$$\sup\{q \geqslant 1;\ L \in L^q[0,1]\},$$

we denote by p_L the conjugate of p, i.e. $p_L^{-1} + p^{-1} = 1$ for $1 < p < \infty$, $p_L = 1$ if $p = \infty$ and $p_L = \infty$ if $p = 1$.

THEOREM 5. *Let L be a positive function, non-decreasing, in $L^1[0,1]$ but unbounded. Then*

(k) *If $1 \leqslant q < p_L$ there exists $f \in L^q[\boldsymbol{R}]$ such that*

$$\limsup L_{\lambda_n}(f)(x) = +\infty \text{ a.e.} \tag{2.4}$$

if $p_L = 1$, (2.4) holds with $q = 1$.

(kk) *If $p_L < q \leqslant \infty$ for all $f \in L^q[\boldsymbol{R}]$,*

$$\lim_{\lambda\to\infty} L_\lambda(f)(x) = f(x)\int_0^1 L(x)\,dx \text{ a.e.}; \tag{2.5}$$

if $p_L = +\infty$, (2.5) holds with $q = +\infty$.

Proof of (k). We write

$$L_n(x) = \lambda_n L(\lambda_n x), \qquad f_m(x) = \lambda_m^{\omega+1} X_{[0,1]}(\lambda_m^{\omega+1} x),$$

$$L_n(f_m)(x) = L_n * f_m(x) = \lambda_m^{1+\omega} \int\limits_{\max(0,\lambda_n(x-\lambda_m^{-1-\omega}))}^{\min(1,\lambda_n x)} L(y)\,dy, \tag{2.6}$$

$$\bar{f}_m(x) = \sup_n L_n(f_m)(x).$$

Our first aim is to prove

$$\bar{f}_m(x) \geq \lambda_m^{1+\omega} \int\limits_{1-\lambda_m^{-1}}^{1} L(y)\,dy\,X_{[0,\lambda_{i_m}^{-1}]}(x); \tag{2.7}$$

i_m as in (2.1).

Consider $i_m \leq i$ and $\lambda_{i+1} \leq \lambda_m^{\omega+1}$; we have

$$\bar{f}_m(x) \geq \lambda_m^{1+\omega} \int\limits_{\lambda_{i+1}\left(x-\frac{1}{\lambda_m^{\omega+1}}\right)}^{1} L(y)\,dy \quad \text{if} \quad \frac{1}{\lambda_{i+1}} \leq x \leq \frac{1}{\lambda_i}.$$

Using (2.1) and (2.2), we have

$$\lambda_m^{1+\omega} \int\limits_{\lambda_{i+1}\left(x-\frac{1}{\lambda_m^{\omega+1}}\right)}^{1} L(y)\,dy \geq \lambda_m^{1+\omega} \int\limits_{\lambda_{i+1}\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_m^{\omega+1}}\right)}^{1} L(y)\,dy$$

$$\geq \lambda_m^{1+\omega} \int\limits_{1-\lambda_m^{-1}}^{1} L(y)\,dy \quad \text{if} \quad \frac{1}{\lambda_{i+1}} \leq x \leq \frac{1}{\lambda_i},$$

and clearly,

$$\bar{f}_m(x) \geq \lambda_m^{1+\omega} \int\limits_0^1 L(y)\,dy\,X_{[0,\lambda^{-1-\omega}]}(x).$$

Let us write

$$I_m = [0, \lambda_{i_m}^{-1}], \qquad h_m = f_m \|f_m\|_q^{-1},$$

$$S_m = \lambda_m^{1+\omega} \int\limits_{1-\lambda_m^{-1}}^{1} L(y)\,dy\, \|f_m\|_q^{-1},$$

$$C_m = C\lambda_m \Big(\int\limits_{1-\lambda_m^{-1}}^{1} L(y)\,dy \Big)^q,$$

where C is a constant.

We translate (2.7) as follows:

$$\operatorname{supp} h_m \subseteq I_m \subseteq \{0 \leq x \leq 1;\ \bar{h}_m(x) \geq S_m\}. \tag{2.8}$$

By (2.3) we have

$$|I_m| \geq S_m^{-q} C_m.$$

Observe that if $1 \leq q < p_L$, then $\sup\limits_m C_m = \infty$. Indeed, if this were not true, we would have

$$\lambda_m^{1/(q-1)} L\left(1-\frac{1}{\lambda_m}\right) \leq \lambda_m^{1/q} \int\limits_{1-\lambda_m^{-1}}^{1} L(y)\,dy \leq c;$$

therefore, for other constant $\bar{c}$,

$$L^a(x) \leq \bar{c}\,\frac{1}{(1-x)^{a/q'}}, \qquad 0 < x < 1.$$

Then, if we choose $p < a < q'$, we would have $L \in L^a[0,1]$, which contradicts our definition of p. Hence, taking a subsequence with repetitions, if it were necessary, we can assume

$$\sum |I_m| = \infty, \qquad \sum S_m^{-q} < \infty; \tag{2.9}$$

finally, take a sequence R_m tending to infinity, but

$$\sum R_m^q S_m^{-q} < \infty. \tag{2.10}$$

Let us set

$$f(x) = \sum R_m S_m^{-1} h_m(x-x_m),$$

where $\{x_m\}$ is a sequence chosen in such a way that $\max(I_m+x_m) = \min(I_{m+1}+x_{m+1})$. The fact that $\|h_m\|_q = 1$, (2.8) and (2.10) imply $f \in L_+^q[\boldsymbol{R}]$. Moreover, the function f has the property that given $\lambda > 0$ we can select $\pi_\lambda > 0$ such that

$$(\pi_\lambda, +\infty) \subseteq \{x; \bar{f}(x) \geq \lambda\}. \tag{2.11}$$

(2.11) follows from: $\bar{f}(x) \geq R_m S_m^{-1} h_m(x-x_m)$ for all m, (2.8), (2.9), the fact that $\{I_m+x_m\}$ are non-overlaping and $\lim R_m = \infty$.

Now the function

$$F(x) = \sum n^{-2} f(x+\pi_{n^3})$$

belongs to $L_+^q[\boldsymbol{R}]$ but $\bar{F}(x) = +\infty$ if $x > 0$.

The second result, (kk), is contained in the next lemma.

LEMMA 3. *Let $K \in L^p(\boldsymbol{R}^M)$, K of bounded support. Suppose $f \in L^q(\boldsymbol{R}^M)$ where $p' \leq q \leq \infty$, p' the conjugate of p. Then*

$$\lim_{\lambda\to\infty} K_\lambda * f(x) = f(x) \int K(x)\,dx \text{ a.e.} \tag{2.12}$$

Moreover, if $q = \infty$, nothing but the integrability of K is necessary.

Proof. We assume that the unitary ball is the support of K; then

$$|K_\lambda(f)(x)| \leqslant \int_{|y|\leqslant 1} |K(y)|\,|f(x-\lambda^{-1}y)|\,dy \leqslant \|K\|_{q'}\Big(\int_{|y|\leqslant 1} |f(x-\lambda^{-1}y)|^q\,dy\Big)^{1/q}, \tag{2.13}$$

where $p' \leqslant q < \infty$, $1/q' + 1/q = 1$.

If $\tilde{f}_q$ denotes the Hardy–Littlewood maximal function of $|f|^q$, we can rewrite (2.13) as

$$\bar{f}(x) \leqslant C\big(\tilde{f}_q(x)\big)^{1/q}. \tag{2.14}$$

Now the fact that $\tilde{f}$, i.e., the Hardy–Littlewood maximal operator of f, is weak-type $(1, 1)$ (see [2]) implies (2.12). If $q = \infty$, we split K as follows: ${}_NK = K\chi_{\{|x|\leqslant N\}}$, ${}^NK = K - {}_NK$. Then

$$K_\lambda * f(x) - f(x)\int K(x)\,dx = \int {}_NK(y)\{f(x-\lambda^{-1}y) - f(x)\}\,dy + \int {}^NK(y)\{f(x-\lambda^{-1}y) - f(x)\}\,dy = {}_NI_\lambda(x) + {}^NI_\lambda(x).$$

It is not difficult to establish the following two inequalities:

$$|{}^NI_\lambda(x)| \leqslant \big(\|f\|_\infty + |f(x)|\big)\int_{|x|\geqslant N} |K(x)|\,dx, \tag{2.15}$$

$$|{}_NI_\lambda(x)| \leqslant \big(|f(x)| + \|f\|_\infty\big)\int_{\{x:\,|{}_NK(x)|\geqslant T\}} |{}_NK(x)|\,dx + T\lambda\int_{|y|\leqslant \lambda^{-1}N} |f(x-y) - f(x)|\,dy. \tag{2.16}$$

Now, considering $\overline{\lim}_{N\to\infty}\,\overline{\lim}_{T\to\infty}\,\overline{\lim}_{\lambda\to\infty}$ on (2.15) and (2.16), we have

$$\overline{\lim_{\lambda\to\infty}}\Big|K_\lambda * f(x) - f(x)\int K(x)\,dx\Big| = 0 \text{ a.e.}$$

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The l_1^n problem and degrees of non-reflexivity

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1. Introduction. The material presented in this paper originated from an attempt to solve the "l_1^n problem" first raised by R. C. James [6]. Let X be a non-reflexive Banach space. Is it true that for every $\varepsilon > 0$ and integer n there is a subspace B of X such that $d(B, l_1^n) < 1+\varepsilon$?

Before continuing let us recall the definition of the notions appearing in the statement of this problem and some closely related notions.

We consider here Banach spaces over the reals and l_1^n denotes the n-dimensional L_1 space, i.e. the space of n-tuples $x = (x_1, x_2, \ldots, x_n)$ of reals with

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

By l_1 we denote as usual the space of all sequences $x = (x_1, x_2, \ldots)$ with $\|x\| = \sum_{i=1}^{\infty}|x_i| < \infty$. We say that $d(B, C) \leqslant \lambda$ for some Banach spaces B and C and a real $\lambda \geqslant 1$ if there is an (always linear here) operator T from B onto C such that $\|T\|\,\|T^{-1}\| \leqslant \lambda$. A Banach space Y is said to be *finitely represented* in a Banach space X if for every finite dimensional subspace B of Y and every $\varepsilon > 0$ there is a subspace C of X such that $d(B, C) \leqslant 1+\varepsilon$. If P is a property which is meaningful for general Banach spaces we say that a Banach space X is "super P" if every Banach space Y finitely represented in X has property P. Of particular importance is the property *super reflexive* introduced by James. Thus, according to the general rule, a Banach space X is super reflexive if every Banach space Y finitely represented in X is reflexive.

In paper [6] in which James posed the l_1^n problem he proved that the

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