On the summability of Fourier series of functions of $A$-bounded variation

by

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Abstract. For $f$ in $ABV$ with $A = (a^{2+1})$, it is shown that the Fourier series of $f$, $S[f]$, is everywhere $(G, \beta)$ bounded, $-1 < \beta < 0$, and everywhere $(G, \alpha)$ summable for $a > \beta$. If $f$ is continuous in $(a^{2+1})$-variation, $-1 < \beta < 0$, $S[f]$ is everywhere $(G, \beta)$ summable. These results hold uniformly on each closed interval of continuity of $f$. If $ABV = (a^{2+1})$-BV properly, there is a continuous function $f$ in $ABV$ such that $S[f]$ is not $(G, \beta)$ bounded at some point. A lemma on the continuity of the $A$-variation and a summability test similar to the Lebesgue convergence test are the principal tools. A new proof is indicated for the fact that functions of harmonic bounded variation satisfy the Lebesgue test.

It is well known that functions of various generalized bounded variation classes have Fourier series which converge everywhere and converge uniformly on each closed interval of continuity. Here we shall show that the Fourier series of functions of one such class, the definition of which depends on a parameter $\beta$, $-1 < \beta < 0$, are everywhere $(G, \beta)$ bounded, uniformly $(G, \beta)$ bounded on each closed interval of continuity and, for $a > \beta$, are everywhere $(G, \alpha)$ summable and uniformly $(G, \alpha)$ summable on each closed interval of continuity. With an additional restriction, we may choose $a = \beta$ for $\beta > -1$. We also show that, for $\beta > -1$, each larger class of $ABV$ functions contains a continuous function whose Fourier series is not $(G, \beta)$ bounded at some point.

In §1 we present the basic concepts and state the principal results. In §2 we prove the result which shows in what sense the main theorem is best possible. In §3 we establish a lemma on the continuity of the $A$-variation which is the principal tool in the proof of the main theorem. In §4 we present the proof of the main theorem and, finally, in §5 we show that the lemma of §3 can be employed to furnish a simpler demonstration of the convergence properties of functions of harmonic bounded variation.

1. Definitions and results. Let $f$ be a real function of period $2\pi$. Let $A$ denote a non-decreasing sequence of positive numbers $\lambda_n$ such that $\sum 1/\lambda_n$ diverges, $\{I_n\}$ a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset [0, 2\pi]$, and let $f(I_n) = f(b_n) - f(a_n)$. The function $f$ is said to be of
A-bounded variation (ABV) if the A-variation of $f$,$$
abla_A(f) = \sup \left\{ \sum |f(I_\alpha)|/|\lambda_\alpha|^A : (I_\alpha) \right\} < \infty.$$The collection of ABV functions is a Banach space with norm \( \|f\| = |f(0)| + \nabla_A(f) \). Let \( A^m = \{ \lambda_{A} \}_{m=0}^{\infty} \), \( m = 0, 1, 2, \ldots \). A function \( f \) of ABV is said to be continuous in A-variation if \( \nabla_A(f) \to 0 \) as \( m \to \infty \). When \( A = |n| \), the class ABV is referred to as the functions of harmonic bounded variation. We have studied the convergence behavior of the Fourier series of functions of this class in a previous paper [3].

Here we shall consider the summability properties of Fourier series of functions of class \( \{n^{\beta+1}\}BV \) with \( -1 \leq \beta < 0 \).

For \( \beta = -1 \) it is clear that our class is exactly the functions of bounded variation in the usual sense. A numerical series \( \sum a_n \) is said to be \( (C, -1) \) summable if it converges and \( a_n = o(n^{-1}) \). It is said to be \( (C, -1) \) bounded if its partial sums are bounded and \( a_n = O(n^{-1}) \). For a series of functions we may speak of \( (C, -1) \) summability or boundedness on a set if the convergence and the order condition are uniform on that set. Clearly, the Fourier series of a BV function is \( (C, -1) \) bounded, convergent everywhere, uniformly convergent on each closed interval of continuity, and will, therefore, by a theorem of Hardy and Littlewood ([1], p. 121), be \( (C, u) \) summable for every \( u > -1 \) and uniformly \( (C, u) \) summable on each closed interval of continuity.

It is known that functions of ABV have only simple discontinuities. We may assume, without loss of generality, that they are regulated, that is, \( f(x) = \frac{1}{2} [f(x+) + f(x-)] \) for each \( x \).

Our main result is the following:

**Theorem 1.** The Fourier series, \( S[f] \), of a function \( f \) of class \( \{n^{\beta+1}\}BV \), \( -1 \leq \beta < 0 \), is everywhere \( (C, \beta) \) bounded and is uniformly \( (C, \beta) \) bounded on each closed interval of continuity. For \( \alpha > \beta \), \( S[f] \) is everywhere \( (C, \alpha) \) summable to sum \( f(x) \) and summability is uniform on each closed interval of continuity.

If \( f \) is continuous in \( \{n^{\beta+1}\} \)-variation, \( -1 \leq \beta < 0 \), then \( S[f] \) is everywhere \( (C, \beta) \) summable to sum \( f(x) \) and summability is uniform on each closed interval of continuity.

This result is best possible in a certain sense, as is expressed in our second theorem.

**Theorem 2.** If \( ABV = \{n^{\beta+1}\}BV \) properly, \( -1 \leq \beta < 0 \), then there is a continuous function in \( ABV \) whose Fourier series is not \( (C, \beta) \) bounded at some point.

The proof of Theorem 2 is quite direct. Theorem 1 is based on a test for \( (C, \beta) \) summability obtained jointly with B.N. Sahney [2]. As is usual we denote the Fourier series of an integrable function \( f \) of period \( 2\pi \) by \( S[f] \) and set
\[
\varphi(t) = \varphi_s(t; f) = \frac{1}{2} \left[ f(x+t) + f(x-t) - 2f(x) \right],
\]
\[
\Phi(t) = \Phi_s(t; f) = \int_0^t \varphi(u) du.
\]

Our test is embodied in the following

**Theorem.** \( S[f] \) is \( (C, \beta) \) summable to \( f(x) \), \(-1 < \beta < 0 \), at every point \( x \) at which
\[
(i) \quad \Phi(\lambda) = o(\lambda),
\]
\[
(ii) \quad \Phi(\lambda) - \Phi(\lambda - h) = o(h^{-\beta}),
\]
\[
(iii) \quad \int_0^1 \frac{\varphi(t) - \varphi(\frac{t}{1 + \eta})}{\eta^{\beta+1}} d\eta = o(\eta^{1-1/2}) \text{ as } \eta \to 0,
\]
and summability is uniform over any closed interval of continuity where conditions (i) and (iii) are satisfied uniformly. If \( f \) is bounded, condition (iii) may be omitted; if \( f \) is regulated, conditions (i) and (ii) may be omitted.

If \( (i'), (ii'), (iii') \) denote (i), (ii), (iii) with "\( o^* \) replaced by "\( O \), \( S[f] \) will be \( (C, \beta) \) bounded at every point \( x \) at which \( (i'), (ii'), (iii') \) hold, and uniformly \( (C, \beta) \) bounded on every set on which the conditions are satisfied uniformly. If \( f \) is bounded, conditions \( (i') \) and \( (ii') \) may be omitted.

We turn next to the proof of Theorem 2.

**Proof of Theorem 2.** Since \( ABV = \{n^{\beta+1}\}BV \) properly, there is a sequence \( \{a_n\} \), such that \( a_n \to 0 \) and \( \sum a_n / \lambda_n < \infty \) and \( \sum a_n / \lambda_n = \infty \). Let \( a_n \) be the characteristic function of the interval \( I_n = (\tau(n), \tau(n+1)) \) and \( \Phi_n \) be of period \( 2\pi \) on the real line. Clearly, in \( ABV \),

\[
\|f_n\| \leq 2 \sum_n |a_n| / \lambda_n = b_n = O(1) \text{ as } n \to \infty.
\]

Letting \( c_n^2 \) denote the nth \( (C, \beta) \) mean of \( S[f] \) at \( x \), for some \( \beta(t) \), \( d(t) \leq 1 \), we have ([4], p. 85)

\[
c_n^2 = \int_0^{2\pi} \frac{1}{2\pi} \left[ f(t) + f(2\pi-t) - 2f(t) \right] dt
\]

and

\[
\sum_n \int_{I_n} c_n^2 dt = \ldots
\]
\[ \sum_{\ell=1}^{n-1} \frac{2}{2n+\beta+1} \frac{a_{\ell}}{(n+\beta+1)^{\beta+1}} \left( a_{\ell} \right)^{\beta+1} \frac{2}{2n+\beta+1} \sum_{\ell=1}^{n-1} \frac{a_{\ell}}{(n+\beta+1)^{\beta+1}} \left( a_{\ell} \right)^{\beta+1} \]

where \( C \) and \( C' \) are independent of \( n \). The last sum is \( O(1) \) as \( n \to \infty \); the first becomes unbounded. Thus

\[ ||V_n(0; f)|| \leq ||V_n(0; f; f)||_{B_{\infty}} \to \infty \]

as \( n \to \infty \), implying that there is an \( f \in ABV \) for which \( ||V_n(0; f)|| \) is unbounded.

Since the continuous functions of \( ABV \) themselves form a Banach space and the \( f_n \) can be modified so as to be continuous without any substantial change in the argument, we have the desired result.

3. Preliminaries to the proof of Theorem 1. We have already defined the \( A \)-variation of a function \( f \) with domain \([0, 2\pi]\). If \( I \) is any closed interval in \([0, 2\pi]\) and \( \{I_n\} \) denotes a collection of non-overlapping intervals, we may define the \( A \)-variation of \( f \) on \( I \) to be

\[ V_A(f; I) = \sup \left\{ \sum_{n} |f(I_n)| \mid \text{ for each } I_n \text{ such that } I_n \subset I \right\}. \]

The following lemma on the continuity of the \( A \)-variation will be of considerable use to us.

**Lemma.** Let \( f \) be of class \( ABV \) on \( I = [a, b] \). Then

(i) If \( f \) is right continuous at \( a \),
\[ V_A(f; [a, a]) \to 0 \quad \text{as} \quad a \to a, \]

(ii) If \( f \) is left continuous at \( b \),
\[ V_A(f; [b, b]) \to 0 \quad \text{as} \quad b \to b, \]

(iii) If \( [a, b] \subset \overline{I} \), then
\[ V_A(f; [a, b]) \to 0, \]

as \( a \) and \( b \) together approach either \( a \) or \( b \).

**Proof.** It is clear that (i) implies (ii), for if we suppose \( [a, b] \subset \overline{I} \) and set \( g(x) = f(x) \) for \( x \in (a, b) \) and \( g(x) = f(a+) \) and \( g(x) = g(b-) \) then \( g \) is right continuous at \( a \) and left continuous at \( b \) and

\[ 0 \leq V_A(f; [a, a]) = V_A(g; [a, a]) \leq V_A(g; [a, b]) \to 0 \quad \text{as} \quad a \to b. \]

Similarly,

\[ 0 \leq V_A(f; [a, b]) = V_A(g; [a, b]) \to 0 \quad \text{as} \quad a \to b. \]

Thus we need only prove (i). We consider only the case in which \( f \) is right continuous at \( a \).

There is a collection of non-overlapping intervals \( \{I_n\} \), such that

\[ f(I_n) \cap f(I) = \emptyset, \quad I_n \subset [a, b], \quad f(I_n) \neq \emptyset \quad \text{for each} \quad n, \quad \text{and} \]

\[ \sum_{I_n} |f(I_n)| \cap I_n > \frac{1}{2} V_A(f; [a, b]). \]

Since \( f \) is right continuous at \( a \), we may assume that \( I_n \subset [a, b] \) for each \( n \).

Choose \( y_{i+1}(a, b) \) such that

\[ I_n \cap \left\{ y_{i+1}(a, b) \right\} = \emptyset, \quad y_{i+1} < (a + b)/2, \]

and \( I = [a, b] \) implies

\[ |f(I)| \leq \min \{ |f(I_n)| : n = 1, \ldots, N \}. \]

Then there is a collection of non-overlapping intervals \( \{I_n\} \) such that

\[ f(I_n) \cap f(I) = \emptyset, \quad I_n \subset [a, b], \quad f(I_n) \neq \emptyset \quad \text{for each} \quad n, \quad \text{and} \]

\[ \sum_{I_n} |f(I_n)| \cap I_n > \frac{1}{2} V_A(f; [a, b]). \]

Continuing in this manner we can choose, for \( k = 1, 2, \ldots, y_{i+1} \), (by choosing \( y_{i+1} \subset (a + b)/2 \) and \( I_{N+1}, \ldots, I_{N+1} \), non-overlapping intervals containing \( [y_{i+1}, y_{i+2}] \) with

\[ \sum_{I_n} |f(I_n)| \cap I_n > \frac{1}{2} V_A(f; [a, b]). \]

and \( f(I) \) as.

Let \( |f(I_n)| = a_n, 1/a_n = b_n. \) Then \( a_n \to 0, b_n \to 0 \). Now \( V_A(f; [a, b]) \)

\[ \sum_{I_n} a_n b_n \text{ implies that given } \epsilon > 0, \text{ there is an } N \text{ such that} \]

\[ \sum_{I_n} a_n b_n < \epsilon \]

and, therefore, since for \( j > 0, a_{n+j} b_n < a_n b_n \),

\[ \sum_{I_n} a_n b_n < \epsilon. \]

Now there is a \( J(\epsilon) \) such that

\[ \sum_{J(\epsilon)} a_n b_n \leq a_{n+1} \sum_{I_n} b_n < \epsilon \quad \text{if} \quad j > J(\epsilon). \]
Writing \( f = N_x \), we see that
\[
\frac{1}{2} V_f(f; [a, y]) \leq \sum_{i=1}^m a_i b_i \delta < 2 \varepsilon
\]
if \( \delta \) is sufficiently large. Hence
\[
V_f(f; [a, y]) \to 0 \quad \text{as} \quad \delta \to \infty
\]
implies that
\[
V_f(f; [a, y]) \to 0 \quad \text{as} \quad y \to a
\]
since \( V_f(f; [a, y]) \) is a monotone function of \( y \).

4. Proof of Theorem 1. Clearly \( S[f] \) is \((\mathcal{C}, \mathcal{L})\) summable everywhere and is uniformly \((\mathcal{C}, \mathcal{L})\) summable on each closed interval of continuity. Hence if we show that \( S[f] \) is \((\mathcal{C}, \mathcal{L})\) bounded at a point, \(-1 < \beta < 0\), it follows from a well-known convexity theorem ([1], p. 277), that \( S[f] \) is \((\mathcal{C}, \mathcal{L})\) summable at that point for \( \alpha > \beta \). The result on uniform \((\mathcal{C}, \mathcal{L})\) summability follows from uniform \((\mathcal{C}, \mathcal{L})\) boundedness in the same way.

From the statement of the summability test we see that we need only verify condition (ii) or (iii). Let us consider, for \( \eta < \delta < \pi, \eta = \pi / (\alpha + \beta + 1) / 2 \),
\[
I(n, \alpha, \beta) = \int_0^{\pi} \frac{|f(t) - f(t + \eta)|}{|f(t + \eta)|^{1/2}} \sin \theta \sin \eta \, dt.
\]
We have
\[
2I(n, \alpha, \beta) \leq \int_0^{\pi} |f(t + \eta) - f(t)| (t^{1/2} \alpha + 1) \, dt + \int_0^{\pi} |f(t - \eta) - f(t - \eta)| (t^{1/2} \alpha - 1) \, dt = I_1 + I_2.
\]
Now if \( m = (\delta / \eta) \), letting \( V([a, b]) \) denote the \((n^{-1})\)-variation of \( f \) on \([a, b] \), we have
\[
n^{-\beta} I_2 \leq n^{-\beta} \sum_{i=1}^m \int_0^{\pi} \frac{|f(x + \eta) - f(x + \eta + \delta)|}{{|f(x + \eta)|}^{1/2}} \, dt
\]
\[
\leq n^{-\beta} \eta \sum_{i=1}^m \sec(f; [a + \eta, a + \eta + \delta]) \|f\|_{\delta, 1} \leq 2n^{-\beta} V([a + \eta, a + \eta + \delta])
\]
Applying the lemma of § 3 we see that
\[
n^{-\beta} I_1(n, \alpha, \beta) = o(1) \quad \text{as} \quad \delta \to 0, n \to \infty.
\]
If we suppose that \( a_n \to a_0 \), a point of continuity of \( f \), then
\[
n^{-\beta} I_1(n, a_n, \delta) \leq 2n^{-\beta} V([a_n + \eta, a_n + \delta + 2\eta])
\]
Adopting the convention that the variation of \( f \) on the empty interval is zero, we have
\[
n^{-\beta} I_1(n, a_n, \delta) \leq 2n^{-\beta} \left\{ V([a_n + \eta, a_n + \delta + 2\eta]) + V([a_n + \eta, a_n + \delta + 2\eta]) \right\}
\]
by the lemma. Similar results hold for \( n^{-\beta} J_f \). Thus we see that
\[
n^{-\beta} J_f(n, a_n, \delta) = o(1) \quad \text{as} \quad \delta \to 0, n \to \infty
\]
for each \( a_n \), and the convergence is uniform at each point of continuity. Let us now consider
\[
J_{\infty}(n, a, \delta) = \int_0^{\pi} \frac{|f(t) - f(t + \eta)|}{|f(t + \eta)|^{1/2}} \sin \theta \sin \eta \, dt \leq \frac{1}{2} (J_1 + J_2)
\]
where \( J_1 \) and \( J_2 \) are obtained in the same manner as \( I_1 \) and \( I_2 \). Then if \( m = (\delta / \eta) \),
\[
n^{-\beta} J_{\infty} \leq n^{-\beta} \sum_{i=1}^m \sec(f; [a + i\eta, a + (i + 2)\eta]) \|f\|_{\delta, 1} \leq 2n^{-\beta} V_m
\]
where \( V_m \) denotes the \((n^{-1})\)-variation of \( f \) on \([0, 2\pi] \). The same inequality holds for \( J_2 \).

Thus, if \( f \) is continuous in \((n^{-1})\)-variation,
\[
n^{-\beta} J_f(n, a, \delta) = o(1) \quad \text{as} \quad \delta \to \infty
\]
uniformly in \( a \). If our hypothesis is that \( f \) is of \((n^{-1})\)BV, then \( V_m \) is dominated by the \((n^{-1})\)-variation of \( f \) and so
\[
n^{-\beta} J_f(n, a, \delta) = O(1) \quad \text{as} \quad \delta \to \infty
\]
uniformly in \( a \).

Now choose \( \delta_k \to 0 \) so that \( n \delta_k \to \infty \). If \( f \) is of \((n^{-1})\)BV, then for each \( a \)
\[
w^{-\alpha} (I(n, a, \delta_k)^2 + J(n, a, \delta_k)) = o(1) + O(1) = O(1) \quad \text{as} \quad n \to \infty.
\]
If \( a_n \to a_0 \), a point of continuity of \( f \), we see that
\[
w^{-\alpha} (I(n, a_n, \delta_k)^2 + J(n, a_n, \delta_k)) = o(1) + O(1) = O(1) \quad \text{as} \quad n \to \infty.
\]
Thus $S[f]$ is everywhere $(C, \beta)$ bounded, uniformly $(\mathcal{C}, \beta)$ bounded at $a_0$ and, therefore, on each closed interval of continuity. If $f$ is continuous in $(n^{n+1})$-variation, then for each $x$

$$n^{-\beta} \left[ I(n, x, \delta_n) + J(n, x_0, \delta_n) \right] = o(1) \quad \text{as} \quad n \to \infty,$$

implying that $S[f]$ is $(C, \beta)$ summable to $f(x)$ for each $x$. If in addition $x_n \to x$, a point of continuity of $f$, then

$$n^{-\beta} \left[ I(n, x_n, \delta_n) + J(n, x_0, \delta_n) \right] = o(1) \quad \text{as} \quad n \to \infty.$$

Thus $S[f]$ is uniformly $(C, \beta)$ summable at $a_0$ and, therefore, on each closed interval of continuity.

5. Some remarks on Harmonic Bounded Variation. In our previous paper [3], we discussed the Fourier series of functions of class $\mathcal{H}B^V$, i.e., of $\mathcal{H}B$ with $\lambda = (w)$. We showed there that functions of that class satisfy the Lebesgue convergence test. Using the lemma of § 3, a more direct proof of this fact may be furnished.

We are required to show that

$$I = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \left| \varphi(t) - \varphi(t + \eta) \right| \, dt = o(1) \quad \text{as} \quad n \to \infty,$$

where $\eta = \pi/n$. Suppose $I_1$ and $I_2$ are defined in the same fashion as in § 4. Then

$$I_1 \leq \sum_{n=1}^{\infty} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} |f(x + t) - f(x + t + \eta)| \, dt \leq 2V(\{x + \eta, x + \delta + 2\pi\})$$

and, by the lemma, $I_1 = o(1)$ as $\delta \to 0, n \to \infty$. $I_2$ is treated similarly.

Now consider

$$J = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \left| \varphi(t) - \varphi(t + \eta) \right| \, dt \leq \frac{1}{\delta} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \left| \varphi(t) - \varphi(t + \eta) \right| \, dt.$$

The last integral is $o(1)$ as $n \to \infty$ uniformly in $x$. Now choose $\delta_n \to 0$ so slowly that $n\delta_n \to \infty$ and

$$\frac{1}{\delta_n} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \left| \varphi(t) - \varphi(t + \eta) \right| \, dt = o(1) \quad \text{as} \quad n \to \infty.$$

We see at once that $f$ satisfies the Lebesgue test at each point. That $f$ satisfies the test for uniform convergence on a closed interval of continuity may be shown by considering $I_1(n, x_0, \delta_n)$ and $I_2(n, x_0, \delta_n)$ as in § 4.