(L_p, L_q) mapping properties of convolution transforms

by

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Abstract. Let k and f be two Lebesgue measurable functions on R^n. Then the equation

\[ k * f(x) = \int k(x-t)f(t)dt \]

defines the convolution transform of k and f. Let T(f) = k * f. In this paper we give necessary as well as sufficient conditions for T to map L_p → L_q continuously. We show that our results are sharp in the sense that we exhibit a class of functions k such that the mapping interval we obtain is maximal except for endpoints. For example, for \( k(t) = \frac{\|f\|_q}{\|t\|^q} \) we give the exact mapping properties. We also give the exact mapping properties for a class of kernels in R^n.

Introduction. Let k and f be two Lebesgue measurable functions on R^n. Then the equation

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defines the convolution transform of k and f. Let T(f) = k * f. In this paper we give necessary as well as sufficient conditions for T to map L_p → L_q continuously. We show that our results are sharp in the sense that we exhibit a class of functions k such that the mapping interval we obtain is maximal except for endpoints. For example, for \( k(t) = \frac{\|f\|_q}{\|t\|^q} \) we give the exact mapping properties (see Cor. 3.22 and 4.29).

The most basic result in this direction is Young’s inequality [4]. It states

\[ \|T(f)\|_q \leq \|k\|_\lambda \|f\|_p, \]

where \( 1/p - 1/q = 1 - \lambda \), \( 0 < \lambda < 1 \).

Lardy and Littlewood [3] extended this theorem to include the functions \( k(x) = |x|^{-\lambda} \), \( 0 < \lambda < 1 \), as well as the Hilbert transform. Riesz, Thorin and then Marcinkiewicz [5] proved a general mapping theorem that not only included all the previous cases but also gave other proofs that the Hilbert transform maps \( L_p(R) \to L_q(R) \) for \( 1 < p < \infty \).

Hörmander [3] has weakened the condition \( k \in L^{\infty} \) (see (*)) by giv-
ing a strictly larger class $K'$ (see Def. 1.3) which includes the Hardy-Littlewood kernels and the Hilbert transform. His theorems apply also in $R^r$. For example, if $k \in K'$ and $T: L_p \to L_q$ for some $p_0$ and $q_0$, then $T$ maps $L_p$ into $L_q$ for all $p$ and $q$ such that \[
frac{1}{p} - 1/q = 1/p_0 - 1/q_0 = 1 - \lambda \quad \text{with} \quad 1 < p < q < \infty.
\]

For the case $L_p(R^n) \to L_q(R^n)$, Hirschman [2] has given conditions on $k$ and $\lambda$ which are sufficient for $T$ to map $L_p \to L_q$ over finite intervals of $p$, i.e., $1 < p < q < \infty$. He shows by an example that this $p$ interval cannot be extended.

We extend Hirschman's result; see Corollaries 1.3 and 1.4. Further, we obtain theorems for the $L_p \to L_q$ case when $1/p - 1/q = 1 - \lambda$; the methods of proof depend on $\lambda$. That is, the method used for $1/2 < \lambda \leq 1$ differs from the one for $0 < \lambda \leq 1/2$.

We also give, in a systematic way, necessary conditions on $k$ in order that $T$ maps $L_p(R^n) \to L_q(R^n)$ continuously. All the functions $f, g, \ldots, k$ that appear in this paper are locally integrable on $R^r$. Unless otherwise specified, we will use $f$ to denote a function with compact support, whose Fourier transform $\hat{f} \in L(R^n)$.

We begin with convolutions defined through limiting processes, such as the Hilbert transform. A suitable way to define the Hilbert transform $H(f)$ is to approximate $1/\bar{t}$ by locally integrable functions $g_\varepsilon(t)$ that are zero in a symmetric neighborhood of both the origin and infinity and coincide with $1/\bar{t}$ everywhere else. Then we define
\[
H(f) = \lim_{\varepsilon \to 0} g_\varepsilon * f.
\]

To show it maps $L_p \to L_p$ for $1 < p < \infty$ it is essential to get good estimates of $|a| H(f)(a) > \bar{y} \rvert \text{for each} \ y > 0$. We will start by looking at all transforms of $H(f)$ that are defined in this manner.

**Definition 1.1.** Given a sequence of bounded functions $\{g_n\}$ with compact support, then if $g_n * f$ converges in measure for each $f$, we define $H(f) = \lim g_n * f$, where the convergence is in measure.

**Definition 1.2.** For $0 < \lambda \leq 1$ we define the weak $1/\lambda$ norm of a function $g$ by
\[
\|g\|_{1/\lambda} = \sup_{y > 0} \left| \int_{|x| > y} g(x) \, dx \right|^\lambda.
\]
compositions and estimates could be used. For example, one may replace the Riesz-Thorin estimates by Marcinkiewicz or Hörmander or other suitable estimates.

We first decompose

$$k(t) = \sum_{n=1}^{\infty} U_{n}(t), \quad t \in \mathbb{R}^{n},$$

in which we assume that the series converges almost everywhere. In addition, for each $A > 0$ let

$$\sum_{n=1}^{\infty} \int_{|x|<A} |U_{n}(x)| dx < \infty$$

for almost all $x \in \mathbb{R}^{n}$. Also, we assume that $U_{m}$ exists for each $m$, and

$$\hat{U}_{m} = \mathcal{F}^{-1}(U_{m}),$$

where

$$\hat{U}(x) = \mathcal{F}(U)(x) = \frac{1}{(2\pi)^{n/2}} \lim_{R \to \infty} \int_{|t|<R} U(t) e^{-ixt} dt.$$

Remark. If $U_{m} \in L_1$ for $m = 1, 2, \ldots$, then the latter properties would easily be satisfied.

**Lemma 1.7.** If $k(t) = \sum_{n=1}^{\infty} U_{n}(t)$, then

(i) $\|U_{m} \ast f\|_{L_{p}} \leq \|\hat{U}_{m}\|_{L_{p}} \|f\|_{L_{1}}$, and

(ii) $\|k \ast f\|_{L_{p}} \leq \sum_{m=1}^{\infty} \|U_{m} \ast f\|_{L_{p}}$ for $1 \leq p \leq \infty$.

**Definition 1.8.** Let $\|U_{m}\|_{L_{p}} = \|\mathcal{F}(U_{m})\|_{L_{p}}$.

**Definition 1.9.** Let $\mathcal{E}_{p}$ be any one of the following functionals: $\|\cdot\|_{1}$, $\|\cdot\|_{1}$, or $\|\cdot\|_{1,1,1}$. Set

$$\mathcal{E}_{p}((U_{m}), \varphi) = \sum_{n=1}^{\infty} \|U_{m}(U_{m})^{(2-\alpha)}(U_{m})^{(2-\alpha)}\|_{p}.$$  

**Theorem 1.10.** Let $1 \leq p \leq 2$. If $\mathcal{E}_{p}((U_{m}), \varphi) < \infty$, then $k \in L_{p}^{c}(\mathbb{R}^{n})$.

In fact, $k \in L_{p}^{c}(\mathbb{R}^{n})$ for $1 \leq p \leq p'$.

Proof. We shall prove the theorem in the cases $\varphi = \|\cdot\|_{1}$, $\|\cdot\|_{1}$, of the proof of the other case is similar and will be omitted.

We have

$$\|U_{m} \ast f\|_{1} \leq \|U_{m}\|_{1} \|f\|_{1} \quad \text{and} \quad \|U_{m} \ast f\|_{1} \leq \|\hat{U}_{m}\|_{1} \|f\|_{1}.$$  

By a theorem of Marcinkiewicz, this implies

$$\|U_{m} \ast f\|_{p} \leq C_{p} \|\hat{U}_{m}\|_{L_{p}} \|f\|_{L_{p}},$$

where $C_{p}$ depends only on $p$. Therefore, by Lemma 1.7,

$$\|k \ast f\|_{L_{p}} \leq \sum_{m=1}^{\infty} \|U_{m} \ast f\|_{L_{p}} \leq C_{p} \|f\|_{L_{p}} \mathcal{E}_{p}((U_{m}), \varphi).$$

One can, in fact, prove that

$$\mathcal{E}_{p}((U_{m}), \varphi) < \infty \Rightarrow \mathcal{E}_{p}((U_{m}), \|\cdot\|_{1}) < \infty \Rightarrow \mathcal{E}_{p}((U_{m}), \|\cdot\|_{1}) < \infty.$$

**Lemma 1.11.** If $A_{i} > B_{i} > 0$, then

$$\sum_{i=1}^{n} \frac{(A_{i} - B_{i}) \prod_{j=1}^{n} A_{j}}{A_{i} - B_{i}} \leq \frac{n}{\sum_{i=1}^{n} A_{i} - B_{i}} \prod_{j=1}^{n} A_{j}.$$  

Proof. By induction we can show

$$A_{1}A_{2} \ldots A_{n} = \frac{(A_{1} - B_{1})(A_{1} + B_{1})A_{2} \ldots A_{n} + B_{2}B_{3} \ldots B_{n}}{A_{1} - B_{1}} + \frac{(A_{1} - B_{1})(A_{1} + B_{1})A_{2} \ldots A_{n} + B_{2}B_{3} \ldots B_{n}}{A_{1} - B_{1}} + \cdots + \frac{(A_{1} - B_{1})(A_{1} + B_{1})A_{2} \ldots A_{n} + B_{2}B_{3} \ldots B_{n}}{A_{1} - B_{1}} + \frac{(A_{1} - B_{1})(A_{1} + B_{1})A_{2} \ldots A_{n} + B_{2}B_{3} \ldots B_{n}}{A_{1} - B_{1}}.$$  

A useful decomposition of $k$ is given by

$$U_{m}(t) = k(t) (R(t)/S_{m} - R(t)/S_{m-1}) \quad \text{for} \quad m > 1,$$

$$U_{1}(t) = k(t) R(t)/S_{1}.$$  

with $R(0) = 1$, $R$ continuous at the origin and $S_{m} \to \infty$. For Corollaries 1.13–1.15 we will set

$$R(t) = R(t_{1}, t_{2}, \ldots, t_{m}) = 1/(1 + t_{1}^{n_{1}})(1 + t_{2}^{n_{2}})\ldots(1 + t_{m}^{n_{m}}).$$  

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**Lemma 1.12.** Let $t \in \mathbb{R}$. If $E_{k}(t) = 1/(1 + t^{n})$, then

(i) there exists an $A$ and $b > 0$ such that $|E_{k}(t)| \leq Ae^{-\alpha t}$, and

(ii) $\int_{-\infty}^{\infty} \hat{E}_{k}(t) dt = 0$ for $1 \leq j \leq 2m - 1$.

Proof. For (i) we use contour integration to show that there exists an $A$ and $b > 0$ such that $|E_{k}(t)| \leq Ae^{-\alpha t}$. To show (ii) we use the facts
\[ \psi R_i(t) \in L_1 \cap L_\infty \] for all \( j \) and \( R_i(t) \in L_1 \cap L_\infty \), which imply
\[
R_i(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itv} \psi R_i(v) dv
\]
and
\[
R_i^2(t) = \frac{i \cdots i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itv} \psi R_i(v) dv.
\]
But
\[
R_i^2(0) = \frac{i \cdots i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi R_i(v) dv = 0 \quad \text{for} \quad 1 \leq j \leq 2m - 1.
\]

**Corollary 1.13.** Let \( S_{m} \to \infty \),
\[
\phi_m(t) = \begin{cases} 1, & t \leq S_{m-1}, \\ \frac{|t|}{S_m}, & |t| \leq S_m, \\ 0, & |t| > S_m, \end{cases}
\]
and suppose for some positive integer \( a \)
\[
(1) \quad \int_{\mathbb{R}^n} |\phi_m(t)|^a |k(t)| dt \leq a_m,
\]
\[
(2) \quad \left\| \int_{\mathbb{R}^n} e^{iat} k(t) (R(t/S_m) - R(t/S_{m-1})) dt \right\|_m \leq b_m,
\]
and
\[
(3) \quad \sum_{|\alpha| \leq 2m} a_2^{2-2|\alpha|} b_2^{2-2|\alpha|} < \infty,
\]
then \( \mathcal{L}_2^p(\mathbb{R}^n) \).

**Proof.** Since \( k(t) = \sum_{m=1}^{\infty} U_m(t) \), then using (2) together with Theorem 1.10, it is sufficient to show \( \|U_m\| \leq C a_m \), where \( C \) is an absolute constant.
Let
\[
A_j = 1 + (t/S_{m-1})^{2m} \quad \text{and} \quad B_j = 1 + (t/S_m)^{2m}.
\]
Therefore, by Lemma 1.11,
\[
0 \leq R(t/S_m) - R(t/S_{m-1}) = \left( \prod_{j=1}^{n} A_j - \prod_{j=1}^{n} B_j \right) / \prod_{j=1}^{n} A_j B_j
\]
\[
\leq \sum_{|\alpha| \leq m} \left( \prod_{j=1}^{n} A_j - \prod_{j=1}^{n} B_j \right) / \prod_{j=1}^{n} A_j B_j \leq \sum_{|\alpha| \leq m} (t/S_m)^{2m} \leq (t/S_m)^{2m}.
\]

Therefore,
\[
\|U_m\| = \int |k(t)||R(t/S_m) - R(t/S_{m-1})| dt
\]

\[
\leq 2m \int |\phi_m(t)|^a |k(t)| dt \leq 2a_m m^a.
\]

In the following corollary, for the sake of simplicity, we assume
\[
\sup_{m \geq 2M} \int e^{at} |k(t)| dt \leq P(a),
\]
where \( P \) is a polynomial and \( M \) is sufficiently large.

**Corollary 1.14.** Let \( \hat{k} \in \mathcal{L}_p(\alpha), \alpha \geq [a] + 1 \), and \( S_m = 2^m \). If
\[
\int |\phi_m(t)|^a |k(t)| dt \leq A 2^m \left( \frac{n}{2} - \beta \right)^a,
\]
where \( A \) is independent of \( m \), then \( \mathcal{L}_2^p(\mathbb{R}^n) \) for
\[
2a + \left( \frac{n}{2} - \beta \right) < p < 2a + \left( \frac{n}{2} - \beta \right), \quad 0 < a, 0 < \beta \leq \frac{n}{2}.
\]

**Proof.** From the proof of Corollary 1.13 we see
\[
\|U_m\| \leq A 2^m \left( \frac{n}{2} - \beta \right)^a.
\]

Now we show \( \hat{k} \in \mathcal{L}_p(\alpha) \) implies \( \|U_m\| \leq A 2^{-m \alpha} \),
\[
\hat{U}_m(\alpha) = \int e^{iat} k(t) (R(t/S_m) - R(t/S_{m-1})) dt
\]
\[
= \int \hat{k}(\alpha + t) (S_m^\alpha \hat{R}(S_m t) - S_{m-1}^\alpha \hat{R}(S_{m-1} t)) dt
\]
\[
= \int \hat{k}(\alpha + t) (\hat{R}(S_m t) - \hat{R}(S_{m-1} t)) dt.
\]
Further,
\[
\hat{k}(\alpha + t) = \hat{k}(\alpha) + \sum_{\lambda \in \mathbb{N}^n} a_\lambda (\alpha)(t/S_m)^\lambda + \psi(\varphi a, t, S_m) \varphi^\mu / S_m^\mu
\]
where \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) is a multi-index, \( |\mu| = \mu_1 + \mu_2 + \cdots + \mu_n \), \( \varphi^\mu = \varphi_1^{\mu_1} \cdots \varphi_n^{\mu_n} \), and for a suitable constant \( A \), \( |\varphi a, t, S_m| \leq A \) for all \( \alpha, t \) and \( S_m \). Therefore,
\[
\|\hat{U}_m(\alpha)\| \leq \sum_{\lambda \in \mathbb{N}^n} a_\lambda (\alpha) \int |(t/S_m)^\lambda - (t/S_{m-1})^\lambda| \hat{R}(t) dt +
\]
\[
+ 2A \int |\hat{R}(t)| t^\mu / S_m^\mu dt.
\]
Since \( \hat{\tilde{R}}(t) = \hat{\tilde{R}}(t_1) \hat{\tilde{R}}(t_2) \ldots \hat{\tilde{R}}(t_n) \),
\[
\int_{\mathbb{R}^n} \nu^p \hat{\tilde{R}}(t) \, dt = 0, \quad 1 \leq |\nu| \leq |\alpha|.
\]
Therefore,
\[
\|\tilde{U}_{m} \|_{\mathfrak{F}} \leq (A / S_{m}^{n-1}) \int_{\mathbb{R}^n} \|\hat{\tilde{R}}(t)\| \, dt \leq A 2^{-m}.\]

Now to show (3) of Corollary 1.13
\[
\sum_{m=1}^{\infty} 2^{-m(n-1-2p)} S_{m}^{(n-1-2p)/p} = \sum_{m=1}^{\infty} 2^{-m(a+\beta-n/2)(2-p)/p}.
\]
This series converges if \( a(2-2/p) + (\beta-n/2)(2-p)/p > 0 \), or
\[
p > \frac{2(a+n/2-\beta)}{2a+n/2-\beta}.
\]

**Corollary 1.15 (Hirschman [2]).** If \( k \in \text{Lip}(\alpha) \) and \( \hat{k} \in \text{Lip}(2, \beta) \), then \( \hat{k} \in L^p_{\alpha}(\mathbb{R}^n) \) for
\[
2a+2n/2-\beta < p < \min \left\{ \frac{2a+n/2-\beta}{n/2-\beta}, 0 < a, \ 0 < \beta < n/2 \right\}.
\]

**Proof.** The proof follows from Corollary 1.14 by using the condition \( \hat{k} \in \text{Lip}(2, \beta) \).

**Corollary 1.16.** If there exists a sequence \( S_m \to 0 \) such that
\[
\int_{|\nu| \leq S_m} |k(t)| \, dt \leq a_m,
\]
(1)
\[
\left\| \int_{|\nu| \leq S_m} k(t) e^{\nu \cdot \theta} \, dt \right\|_{L^p} \leq b_m,
\]
(2)
\[
\sum_{n=1}^{\infty} 2^{-n(2p-\beta)} S_{n-1}^{(n-1-2p)/p} < \infty,
\]
(3)
then \( k \in L^p_{\alpha}(\mathbb{R}^n) \).

**Proof.** Take \( U_m(t) \) with \( E(t) \) the characteristic function on the closed unit ball. Then we find
\[
\| U_m \|_1 \leq a_m \quad \text{and} \quad \| U_m \|_{\mathfrak{F}} \leq b_m.
\]
The result follows by Theorem 1.10.

Remark. In Corollary 1.16 we could set
\[
E(t) = R_1(t_1) R_2(t_2) \ldots R_n(t_n),
\]
where
\[
R_1(t) = 1, \quad \|t\| \leq 1, \quad 0, \quad \text{elsewhere}.
\]

**Theorem 1.17.** If there exists a continuous \( f \geq 0 \) and \( f \not\equiv 0 \) such that
\[
\int_{|x| \leq \theta} k(t) \, dt = O((f(\theta))^\gamma)
\]
and
\[
\int_{|x| \leq \theta} e^{\alpha \cdot x} k(t) \, dt = O\left( \left( \frac{1}{f(\theta)} \right)^\gamma \right),
\]
where \( \alpha > 0, \beta > 0 \), then \( k \in L^p_{\alpha}(\mathbb{R}^n) \) for
\[
2(\alpha+\beta)/\alpha+2\beta < p < 2(\alpha+\beta)/\alpha.
\]

**Proof.** Choose \( S_m \) such that \( f(S_m) = 2^n \). Then there is a constant \( A \) such that
\[
\int_{S_m \cdot |\nu| \leq S_m} |k(t)| \, dt \leq A 2^{-m},
\]
and
\[
\int_{S_m \cdot |\nu| \leq S_m} e^{\alpha \cdot \theta} k(t) \, dt \leq A (2^{-m})^{\gamma}.
\]
But here Corollary 1.16 applies and gives the result.

Remark. It follows from Theorem 1.17 that the functions \( \frac{\sin \theta}{\theta} \) and \( 1 + \text{log}(1+|\theta|) \) are in \( L^p_{\alpha}(\mathbb{R}) \) for \( 1 < p < \infty \); and \( e^{\theta \cdot \theta} \) is in \( L^p_{\alpha}(\mathbb{R}) \) for \( 5/3 < p < 3 \).

2. Necessary conditions on \( k \) such that \( T(f) \) maps \( L_\alpha(\mathbb{R}^n) \to L_\beta(\mathbb{R}^n) \) continuously. It is well known that \( T(f) = k \circ f \) maps \( L_\alpha \to L_\beta \) continuously for \( 1 \leq p < \infty \) if and only if \( k \in L^p_{\alpha} \). Thus in this section we will only look at \( k \in L^p_{\alpha} \). As a matter of fact, the basic idea in this section is to study the interplay between the partial derivatives of \( k \) and the way the \( L^p \)-norm of \( k \) goes to infinity.

For a given function \( k(t) \), \( \hat{k} \in \mathbb{R}^n \), and with \( s_j = \pm 1, \ T_j > 0 \), we set
\[
s_j(t) = \begin{cases} 
0, & \text{if } 1 \leq |t| \leq T_j, \\
1, & \text{if } |t| > T_j,
\end{cases}
\]
then
\[
\| k \|_{L^p_{\alpha}} = \| \hat{k} \|_{L^p_{\alpha}} = \| k \|_{L^p_{\alpha}},
\]
and
\[
\| k \|_{L^p_{\alpha}} = \| k \|_{L^p_{\alpha}} = \| k \|_{L^p_{\alpha}}.
\]
We also set
\[ I(g) = \int_{x_1 < \alpha_1 < x_2} \ldots \int_{x_n < \alpha_n < x_{n+1}} dt_1 \ldots dt_n g(t). \]

**Lemma 2.18.** If
(1) there exist positive constants \( A \) and \( B = 1/2 \), and functions \( \alpha_j \) such that
\[ I[k(\cdot) - k(\cdot - v)] \leq A \sum_{j=1}^n \alpha_j I(\alpha_j) \quad \text{for} \quad 0 < \alpha_j < BT_j, \quad 1 \leq j \leq n, \]
and
(2) there exists a positive constant \( C < 2AB \) such that
\[ \lim_{x \to \infty} \frac{I(k)}{I(\alpha)} = C \quad \text{for} \quad 1 \leq j \leq n, \]
and
(3) \[ \lim_{x \to \infty} \left( \frac{I(k)}{I(\alpha)} \right)^{1/ \alpha_j} = \infty, \]
where \( Z(s) \) is a subset of \( \{1, 2, \ldots, n\} \) which includes \( s \), and \( T_j \to \infty \) as \( x \to \infty \), \( r \in Z(s) \), while \( T_j \) remains fixed for \( i \in \{1, 2, \ldots, n\} \setminus Z(s) \). Here \( A \) and \( B \) are absolute constants and \( s \) is exactly one of the integers \( 1, 2, \ldots, n \).

Then \( k \cdot I^2_R(R^d) \).

**Proof.** By (2), this implies for \( T_j \) large
\[ \frac{I(k)}{I(\alpha)} \leq CT_j \quad \text{for} \quad 1 \leq j \leq n. \]

Now consider only those \( v \)'s in \( R^d \) for which
\[ 0 < \alpha_j \leq \frac{I(k)}{2AB} I(\alpha_j). \]
Then for \( T_j \) large,
\[ \alpha_j \leq \frac{C}{2AB} T_j \quad \text{for} \quad 1 \leq j \leq n \]
\[ \leq BT_j. \]
Therefore for these \( v \)'s we have
\[ |I(k) - k + \alpha| \leq A \sum_{j=1}^n \alpha_j I(\alpha_j) \leq \frac{1}{2} I(k). \]
This implies
\[ \left| \alpha \in R^d: \left| k + \alpha \right| > \frac{I(k)}{2} \right| \geq D \frac{I(k)^n}{n! I(\alpha)}. \]

Therefore,
\[ \frac{I(k)}{2} \left| \left| v: \left| k + \alpha(v) \right| > \frac{I(k)}{2} \right| \right| \geq D \frac{I(k)^{1+\alpha}}{n! I(\alpha)}. \]

But if \( k \) is to map \( L_2 \to L_2 \) this would imply the existence of an absolute constant \( C \) such that
\[ C T_j^{1+\alpha} \ldots 1^{n+\alpha} > D \frac{I(k)^{1+\alpha}}{n! I(\alpha)}. \]

But on letting \( T_j \to \infty \) this contradicts the hypothesis (3) and thus \( k \cdot I^2_R(R^d). \)

**Theorem 2.19.** If
(1) \[ 0 \leq \alpha_j \leq 1 + \beta_j \quad \text{for} \quad 1 \leq i, j \leq n, \]
(2) \[ C_1 T_1 \ldots T_n \geq I(k) \geq C_2 T_1 \ldots T_n \]
and
(3) \[ \frac{\partial k}{\partial y} \leq C_1 T_1 \ldots T_n \quad \text{for} \quad 1 \leq j \leq n, \]
where these estimates hold for large \( t \) and \( T_j \), and \( C_1, C_2, C_3 \) are absolute constants.

Then \( k \cdot I^2_R(R^d) \) for
\[ n + \sum_{i=1}^n (1 + \beta_i - \alpha_i) \]
\[ p > \frac{\sum_{i=1}^n \alpha_i}{\sum_{i=1}^n \alpha_i} \]

**Proof.** We shall apply Lemma 2.18 with \( T_1 = T_2 = \ldots = T_n \). Set
\[ \alpha_j(t) = \alpha_j \ldots \alpha_j \]

Then by (3) there exists an \( A \) such that
\[ I[k(\cdot) - k(\cdot - v)] \leq A \sum_{j=1}^n \alpha_j I(\alpha_j) \quad \text{for} \quad 0 < \alpha_j \leq \frac{T_j}{2}, \quad 1 \leq j \leq n. \]

Since \( 0 \leq \alpha_j \leq 1 + \beta_j \) for \( 1 \leq i, j \leq n \), then
\[ \frac{I(k)}{I(\alpha)} \leq C_1, \]
and therefore for \( T_j \) large, \( 1 \leq j \leq n \), we find
\[ \lim_{T_j \to \infty} \frac{I(k)}{I(\alpha)} = 0. \]
Finally,
\[
\lim_{n \to \infty} \frac{\|I(k)\|^{1+\epsilon}}{T_{-1}^p \left( \frac{1}{I(n)} \right)^{1+\epsilon}} = G \lim_{n \to \infty} \frac{T_{-1}^p \left( \frac{1}{I(n)} \right)^{1+\epsilon}}{T_{-1}^p}.
\]

This implies \( k \ell L_p^R(\mathbb{R}) \) for \( p \) satisfying (1).

To see how Theorem 2.19 works, take \( k(t) = e^{i\theta(t)} \), \( \epsilon \geq 2 \) and \( n = 1 \). At this point we could also make some straightforward observations. It is well known that \( T(f) = k * f \) maps \( L_p \) continuously, \( L_p \) continuously, \( L_p \) continuously. Therefore, if the right-hand side of (1) is \( \epsilon < 2 \), then \( k \ell L_p^R(\mathbb{R}) \) for any \( p \).

A way of showing \( k \ell L_p^R(\mathbb{R}) \) is to first localize \( k \) and then study both the \( L_1 \)-norm as well as the Fourier transform of this localized version. We could give a theorem which is in a sense a partial converse, by investigating the \( L_1 \)-norm of the local version of \( k \) and the Fourier transform of \( k \) to determine when \( k \ell L_p^R(\mathbb{R}) \). Its proof would be similar to that of Theorem 2.19.

3. Examples to show our theorems are sharp. To show our theorems are best possible we give the exact mapping properties of a certain class of functions. For \( R^m (m \geq 1) \) we consider the class of functions \( e^{i\theta}(g(t)) \); while in \( R \) we consider other classes as well. We shall divide this section into parts A and B. In Part A we shall consider \( R \)-examples and in Part B we shall consider \( R^m \)-examples, \( m \geq 2 \).

A. EXAMPLES IN \( R \).

THEOREM 3.20. Let \( k(t) = e^{i\theta(t)} \) with \( \epsilon \geq R \), where \( f(t) \) is real-valued and \( g(t) \geq 0 \). Also, \( |f''(t)| \), \( g(t) \) for \( t > 0 \), \( |f''(t)| \), \( g(t) \) for \( t < 0 \), and \( |f''(t)| \) is larger than a fixed positive constant outside a compact set. Finally, we will assume \( f(0) \) is locally integrable and
\[
\lim_{t \to \infty} \frac{1}{g(t)} \int_{-t}^{t} 1 \, dt = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{g(t)} \int_{-t}^{t} 1 \, dt = \infty.
\]

If
\[
\lim_{n \to \infty} \frac{g(|S|^{1+\epsilon})}{g(S)|f''(S)|^{1+\epsilon}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} g(|S|^{1+\epsilon}) f''(S)|^{1+\epsilon} = \infty,
\]

\( j = 1, 2 \), where \( \epsilon_1 = 1, \epsilon_2 = -1 \), then
\[
k \ell L_p^R(\mathbb{R}) \quad \text{for} \quad p < q < p'.
\]

Proof. First we will show \( k \ell L_p^R(\mathbb{R}) \) for \( p < q < p' \). We define for \( m \geq 1 \),
\[
V_{2m}(t) = \begin{cases} \frac{e^{i\theta}}{g(t)} & 2^{m-1} \leq t < 2^m, \\ 0, & \text{elsewhere}, \end{cases}
\]

\[
V_{2m-1}(t) = \begin{cases} \frac{e^{i\theta}}{g(t)} & -2^m \leq t < -2^{m-1}, \\ 0, & \text{elsewhere}, \end{cases}
\]

and
\[
k(t) = \sum_{m=1}^{\infty} V_{2m}(t).
\]

Now
\[
\|V_{2m}\| \leq \frac{2^{-m}}{g(2^{m-1})}, \quad \|V_{2m-1}\| \leq \frac{2^{-m}}{g(-2^{m-1})},
\]

and
\[
\|V_{2m}\| \leq \frac{1}{g(2^{m-1})|f''(2^{m-1})|^{1+\epsilon}}, \quad \|V_{2m-1}\| \leq \frac{1}{g(-2^{m-1})|f''(-2^{m-1})|^{1+\epsilon}}.
\]

Thus, for \( 2 \leq q < p' \) consider
\[
\sum_{m=1}^{\infty} \|V_{2m}\|^{1+\epsilon} \leq \sum_{m=1}^{\infty} \frac{2^{m(1+\epsilon)}}{g(2^{m-1})|f''(2^{m-1})|^{1+\epsilon}}.
\]

But this series converges since
\[
\lim_{m \to \infty} \frac{g(2^{m-1})|f''(2^{m-1})|^{1+\epsilon}}{2^{m(1+\epsilon)}} < \infty.
\]

Similarly, we get the same estimates for the pair \( \|V_{2m-1}\|, \|V_{2m-1}\| \).

Hence, by Theorem 1.10, this implies \( k \ell L_p^R(\mathbb{R}) \).

To complete the theorem we shall show \( k \ell L_p^R(\mathbb{R}) \) for \( q > p' \). To do this we shall use Lemma 2.18. Without loss of generality we could assume \( k(t) = 0 \) for \( t < 0 \), and for \( t \) large \( f''(t) > 0 \).

Since \( f''(t) > 0 \) for \( t \) large we have
\[
|f''(t) - f''(A)| \geq C|t - A| \quad (A \text{ fixed}).
\]
Therefore,
\[
|f'(t)| \geq \frac{C}{2} T \quad \text{for} \quad t > T.
\]
Since \(f''(t) > 0\) and \(|f'(t)| \to \infty\) this implies for real large \(f'(t) > 0\) and in fact
\[
f'(t) \geq \frac{C}{2} T \quad \text{for} \quad t > T.
\]
Now we shall employ Lemma 2.18 and hence assume \(0 < v \leq T/2\), and \(T\) sufficiently large.

\[
I[k(\cdot) - k(\cdot - v)] = \int_{T = 0}^{T} k(t) - k(t - v) \, dt
\]
\[
\leq \int_{T = 0}^{T} \frac{dt}{g(t)} \left| \frac{e^{\theta t} - e^{\theta (t-v)}}{g(t)} + \frac{e^{\theta (t-v)} - e^{\theta t}}{g(t)} \right|
\]
\[
\leq \int_{T = 0}^{T} \frac{dt}{g(t)} \left( e^{\theta T} - e^{\theta (T-v)} \right) + \int_{T = 0}^{T} \frac{dt}{g(t)} \int_{T = 0}^{T} \frac{dt}{g(t)}
\]
\[
\leq \int_{T = 0}^{T} \frac{dt}{g(t)} \left( e^{\theta T} - e^{\theta (T-v)} \right) + \int_{T = 0}^{T} \frac{dt}{g(t)}
\]
\[
\leq \int_{T = 0}^{T} \frac{dt}{g(t)} \left( e^{\theta T} - e^{\theta (T-v)} \right) + \int_{T = 0}^{T} \frac{dt}{g(t)}
\]
\[
\leq 2v \int_{T = 0}^{T} \frac{d^2 f(t)}{g(t)} dt.
\]

Hence,
\[
I[k(\cdot) - k(\cdot - v)] \leq 2v \int_{T = 0}^{T} \omega(t) dt \quad \text{with} \quad \omega(t) = f'(t)/g(t)
\]
\[
\text{and} \quad 0 < v < T/2.
\]
Now we consider
\[
I(k) = \int_{T = 0}^{T} \frac{dt}{g(t)}.
\]

But since \(f'(t) \geq \frac{C}{2} T\) for \(t > T\), this implies
\[
\lim_{T \to \infty} \frac{I(k)}{TT(\omega)} \leq \lim_{T \to \infty} \frac{1}{CT} = 0.
\]
Finally, we must show for \(q > p'\),
\[
\lim_{T \to \infty} \frac{(f(k))^{1+\frac{1}{q}}}{T^{\frac{1}{q}(f(\omega))^{1/q}}} = \infty.
\]

Now,
\[
|f'(t) - f'(A)| = |f'(\zeta)||t - A| \quad (A \text{ is fixed}) \quad T \leq t \leq 2T
\]
\[
\leq 4Tf''(2T).
\]

Therefore,
\[
f(t) \leq S^2f''(2T) \quad \text{for} \quad T \leq t \leq 2T.
\]

Hence,
\[
\lim_{T \to \infty} \frac{T^{1/q}}{T} \left( \int_{T}^{T^2} \frac{d^2 f}{g(t)} \right)^{1/2}
\]
\[
\geq C \lim_{T \to \infty} \frac{T^{1/q}}{T} \left( \int_{T}^{T^2} \frac{d^2 f}{g(t)} \right)^{1/2}
\]
\[
\geq \lim_{T \to \infty} \frac{T^{1/q}}{T} \left( \int_{T}^{T^2} \frac{d^2 f}{g(t)} \right)^{1/2}
\]
\[
\geq \lim_{T \to \infty} \frac{T^{1/q}}{T} \left( \int_{T}^{T^2} \frac{d^2 f}{g(t)} \right)^{1/2}
\]
\[
= \infty.
\]

But by hypothesis
\[
\sum_{n=0}^{\infty} \frac{2^{\alpha(n-1/2)}}{g(2^m)} \to T^{-1/2}
\]
which implies for \(q > p'\),
\[
\lim_{T \to \infty} \frac{T^{1/q}}{T} \left( \int_{T}^{T^2} \frac{d^2 f}{g(t)} \right)^{1/2} = \infty
\]

and hence our result.

**Theorem 3.21.** Let \( k(t) = \frac{e^{\theta t}}{g(t)} \) with \( t \in \mathbb{R} \), where \( f \) is real-valued,
\( g(t) > 0 \), and \( 1/g(t) \) is locally integrable. Also,
\[
|f'(t)| \downarrow, \ g(t) \uparrow \quad \text{for} \quad t > 0, \quad |f'(t)|, \ g(t) \downarrow \quad \text{for} \quad t < 0.
\]

Finally, we assume
\[
\frac{1}{g(S/2)} \leq \left| \{ S \} \vert f'(S) \vert + \int_{S}^{T} \frac{dt}{g(t)} \right| \quad \text{for} \quad S \text{ large}.
\]
If
\[ \lim_{|\omega| \to \infty} \frac{|S|}{|\omega|^{\beta}} < \infty, \quad \lim_{|\omega| \to \infty} \frac{|S|}{|\omega|^{\beta} \langle f(2\beta) \rangle^{1/2}|} = \infty \quad \text{for some } \beta < 2, \]
and
\[ \sum_{n=1}^{\infty} \frac{2^{-(\beta-1)n}}{g(2^n|S|^{\beta})} \langle f(2^{(\beta-1)n}) \rangle^{1/2} = \infty, \]
\( j = 1, 2, \) where \( \epsilon_1 = 1, \epsilon_2 = -1, \) then
\[ k \in L^2_{\beta}(R) \quad \text{for } p < q < p', \]
and
\[ k \notin L^2_{\beta}(R) \quad \text{for } q < p \quad \text{and} \quad q > p'. \]

Proof. First we shall show \( k \in L^2_{\beta}(R) \) for \( p < q < p' \). With the same decomposition as in Theorem 3.20,
\[ \| \tilde{V}_{2m} \|_1 \leq \frac{2^{m-1}}{g(2^{m-1})}, \quad \| \tilde{V}_{2m} \|_{\infty} \leq \frac{2^{m-1}}{g(2^{m-1})}, \]
and
\[ \| \tilde{V}_{2m} \|_1 \leq \frac{1}{g(2^{(m-1)} f(2^{(m-1)} \omega))}, \quad \| \tilde{V}_{2m} \|_{\infty} \leq \frac{1}{g(2^{(m-1)} f(2^{(m-1)} \omega))}. \]

Thus, for \( 2 \leq q < p' \)
\[ \sum_{m=1}^{\infty} \frac{2^{m-1}}{g(2^{m-1})} \frac{1}{g(2^{m-1})} \langle f(2^{m-1}) \rangle^{1/2} \langle f(2^{m-1}) \rangle^{1/2} \]

Now for \( m \) large enough,
\[ \frac{2^{m-1}}{g(2^{m-1})} \frac{1}{g(2^{m-1})} \langle f(2^{m-1}) \rangle^{1/2} \langle f(2^{m-1}) \rangle^{1/2} \leq C \]
and
\[ \frac{1}{g(2^{m-1})} \frac{1}{g(2^{m-1})} \langle f(2^{m-1}) \rangle^{1/2} \langle f(2^{m-1}) \rangle^{1/2} \leq C. \]

Therefore,
\[ \sum_{m=1}^{\infty} \frac{2^{m-1}}{g(2^{m-1})} \langle f(2^{m-1}) \rangle^{1/2} \langle f(2^{m-1}) \rangle^{1/2} \leq C. \]

Hence,
\[ \langle f(2^{m-1}) \rangle^{1/2} \langle f(2^{m-1}) \rangle^{1/2} \leq C, \quad \text{for } 2 \leq q < p'. \]
and we get the same estimates for the pair \( \| \tilde{V}_{2m} \|_1 \| \tilde{V}_{2m} \|_{\infty} \). Hence, by

Theorem 1.10, this implies \( k \in L^2_{\beta}(R) \). To complete the theorem we shall show \( k \notin L^2_{\beta}(R) \) for \( q > p' \). To do this we shall use Lemma 2.18. Without

\((L_2, L_2)\) mapping properties of convolution transforms
But
\[ \sum_{n=1}^{\infty} \frac{2^{\alpha(n-1)b^2}}{2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}} = \infty \]
and
\[ \lim_{T \to \infty} 2^{\alpha} |f''(T)| = \infty \quad \text{for some } \beta < 2 \]
implies our result. For, if
\[ \lim_{T \to \infty} \frac{T^{1/2}}{g(2T)(T^{1/2} + |f''(T)|^{1/2})} = C < \infty, \]
then
\[ \frac{2^{\alpha(n-1)b^2} + 2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}}{2^{\alpha(n-1)b^2} + 2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}} \leq C + 1 \quad \text{for } m \text{ large.} \]
Hence,
\[ \frac{2^{\alpha(n-1)b^2}}{2^{\alpha(n-1)b^2} + 2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}} \leq \frac{(C+1)2^{\alpha(n-1)b^2}}{(C+1)2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}} \]
\[ \leq \frac{(C+1)2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}}{(C+1)2^{\alpha(n-1)b^2} + |f''(T)|^{1/2}} \]
which is a contradiction and hence our result follows.

**Corollary 3.22.** Let \( k(t) = e^{\phi(t)} \|t\| \) with \( t \in \mathbb{R}, a \neq 1, b < 1 \) and \((1/2)a + b > 1\), then
\[ k \in L^2_q(R) \quad \text{for} \quad \frac{a}{a+b-1} < q < \frac{a}{1-b} \]
and
\[ k \in L^2_q(R) \quad \text{for} \quad q > \frac{a}{1-b} \text{ and for } q < \frac{a}{a+b-1}. \]

**Proof.** Case 1. Here we apply Theorem 3.20. \( f(t) = \|t\|^a, a \geq 2, g(t) = \|t\|^b \) implies
\[ |f''(t)| = a(a-1) |t|^{a-2} \]
and since \((1-b)a < 2, 1/2, \)
\[ \sum_{n=1}^{\infty} \frac{a^{\alpha(n-1)b^2}}{2^{\alpha(n-1)b^2} + 2^{\alpha(n-1)b^2} + |t|^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{2^{\alpha(n-1)b^2} + 2^{\alpha(n-1)b^2} = \infty.} \]

Also,
\[ \lim_{T \to \infty} \frac{|T|^{1-(a-b)/a}}{|T|^{1-b/a}} = \lim_{T \to \infty} 1 = 1, \]
and hence we get our result.

Case 2. \( f(t) = \|t\|^a, 0 < a < 2, a \neq 1, \) and \( g(t) = \|t\|^b \)
\[ |f''(t)| = a(a-1)|t|^{a-2}. \]

Now we will apply Theorem 3.21:
\[ \frac{1}{g(T)^2} = \frac{1}{(a(a-1)|T|^{a-1} + a|T|^{a-1})} \frac{1}{(1-b)2^{b-1}|T|^{-b}}, \]
since
\[ 2^{b/|T|} \leq C |T|^{b-a} \quad \text{as } |T| \to \infty. \]

Also,
\[ \lim_{T \to \infty} \frac{1}{|T|^{1-b/a}} = \lim_{T \to \infty} 1 = 1 \]
and
\[ \lim_{T \to \infty} |T|^{(a-1)/a} = \infty \quad \text{if } \beta > b-a. \]

Finally,
\[ \sum_{n=1}^{\infty} 2^{n(a-1)/a} \frac{a^{n-1}}{2^{n-1}} = \left( \frac{1}{2} \right) \sum_{n=1}^{\infty} \frac{a^{n-1}}{2^{n-1}} = \infty. \]

When \( b < 0 \), a similar argument applies.

We would like to point out that these methods show
\[ k(t) = \begin{cases} \frac{1}{\log(|t|)}, & |t| > \epsilon, \\ 0, & |t| \leq \epsilon \end{cases} \]
maps \( L_p \to L_p \) for \( 1 < p < \infty \) (apply Theorem 1.17), but \( L^{\alpha/2} \) for any \( \alpha > 0 \). This extends a result of Hirschman ([2], Theorem 46).

If \( f \in L^2_q(R) \) for \( 1 < q < \infty \), then it follows that if \( \sum |a_n| < \infty \), then \( \sum a_n f_n(t) \in L^2_q(R) \) for \( 1 < p < \infty \). One can then ask whether \( \sum a_n f_n(t) \in L^2_q(R) \) for \( 1 < p < \infty \) when \( \sum |a_n| = \infty. \)
Consider the example
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{t} \cdot X[\cdot^{-n}a, b] = E(\cdot), \]
where
\[ \frac{1}{t} \cdot X[\cdot^{-n}a, b] \rightarrow f_n \]
is the Hilbert transform applied to the characteristic function of the interval \([-n^{1/2}, n^{1/2}]\). Setting
\[ U_n = \frac{(-1)^{n-1}}{(2n-1)} \int X[\cdot^{-n}a, b] \cdot (2n-1)^{1/2} \frac{1}{t} \cdot X[\cdot^{-n}a, b] dt, \]
we were able to show, by employing Theorem 1.10, that \( E(\cdot) \ast L_p^p(\mathbb{R}) \) for \( 1 < p < \infty \).

**B. Examples in \( \mathbb{R}^n \).**

**Corollary 3.33.** Let \( t \in \mathbb{R}^n \), \( n \geq 2 \), and \( 0 < a < 1 \), then
\[ k(t) = \frac{\varepsilon(t^n)}{|t_1|^{2a} + |t_2|^{2a} + \ldots + |t_n|^{2a}} \ast L_p^p(\mathbb{R}^n) \quad \text{for} \quad \frac{2a}{n+2a} < p < \frac{2a}{n-2a} \]
and
\[ k \ast L_p^p(\mathbb{R}^n) \quad \text{for} \quad p > \frac{2a}{n-2a} \quad \text{and} \quad p < \frac{2a}{n+2a}. \]

**Proof.** We apply the remark following Corollary 1.16; thus we need to estimate
\[ \left| \int \sum_{j=1}^{n} dt_1 \ldots dt_n e^{it^a k(t)} \right| \leq \sum_{j=1}^{n} \left| \sum_{j=1}^{n} dt_1 \ldots dt_n e^{it^a k(t)} \right| \leq \sum_{j=1}^{n} \left| \sum_{j=1}^{n} dt_1 \ldots dt_n e^{it^a k(t)} \right| \leq 0 \]
and
\[ \sum_{j=1}^{n} \left| \sum_{j=1}^{n} dt_1 \ldots dt_n k(t) \right| \leq C \varepsilon_m^{n-2a}. \]

Now set \( S_m = 2^n \). Then we just need to find those \( p \geq 2 \) such that
\[ \sum_{m=1}^{\infty} 2^{-n^{2a}} \frac{1}{2^{n-2a} + 2^{n-2a-2} + \ldots + 2^{n-2a}} < \infty. \]
Hence we need
\[ \frac{2n}{p} + 2a - n > 0 \quad \text{or} \quad p < \frac{2n}{n-2a}. \]
We apply Theorem 3.19 to show \( k \ast L_p^p(\mathbb{R}^n) \) for \( p > \frac{2n}{n-2a} \).

We set
\[ a_1(t) = \begin{cases} 0 & \text{if } t_1^a \leq a < 2a^2, \\ 1 & \text{elsewhere}, \end{cases} \]
\[ I(k) = \int \frac{dt_1}{t_1^a} \ldots \int \frac{dt_n}{t_n^a} = \frac{1}{t^n} \]
\[ \sum_{j=1}^{n} \left| \sum_{j=1}^{n} dt_1 \ldots dt_n e^{it^a k(t)} \right| \leq C_n T^{1-2a}. \]

Therefore
\[ D_n T^{1-2a} \leq I(k) \leq C_n T^{1-2a}. \]

Also,
\[ \frac{\partial k}{\partial t_j} = \frac{12t_j t^{a-1} e^{it_j}}{t_j^a + t_j^a + \ldots + t_j^a} - \frac{2a t_j^a t^{a-1} e^{it_j}}{t_j^a + t_j^a + \ldots + t_j^a} \]
and
\[ |\frac{\partial k}{\partial t_j}| \leq 3 |t_j|^{1-a}, \]
and thus with \( a_1 = \ldots = a_n = 1 - 2a/n \)
\[ \beta_{a} = 1 - 2a, \quad S = j, \quad S = f, \]
we get \( k \ast L_p^p(\mathbb{R}^n) \) for \( p > 2n/(n-2a) \). By the same methods we can also show \( e^{it^a k} \ast L_p^p(\mathbb{R}^n) \) for \( p \geq 2 \), while for \( p = 2 \), \( e^{it^a k} \ast L_p^p(\mathbb{R}^n) \).

**4. Conditions on \( k \) such that \( T(f) = f \ast k \) maps \( L_p(\mathbb{R}) \to L_q(\mathbb{R}) \).**

In this section we shall obtain some analogues of the results of the first chapter, without going into complete generality. However, we believe that these methods could be used to obtain essentially all of the results which are analogous to the \( L_p^p(\mathbb{R}) \) case. It turns out that the problem breaks into two parts. We will first study the cases where \( 1/p - 1/q = 1 - \lambda \) and \( \lambda > 1/2 \), and then the case where \( 0 \leq \lambda \leq 1/2 \). The reason \( \lambda = 1/2 \) is the dividing line depends heavily on the behavior of \( \|k\|_{L_1} \) as a function of \( \lambda \). That is, in the examples we study, where \( \lambda > 1/2 \), \( \|k\|_{L_1} \) is reasonably small; while for \( 0 \leq \lambda \leq 1/2 \) it is too large. Intuitively, one can see this by means of the Riesz–Thorin Theorem.
Part A. $1/2 < \lambda < 1$.

**Lemma 4.24.** Let $1/2 < \lambda < 1$. If

$$k(t) = \sum_{n=1}^{\infty} U_n(t) \text{ with } 1/p - 1/q = 1 - \lambda,$$

then

(i) $\|U_m*f\|_q \leq C_1 \|\hat{U}_m\|_{Lip, q} \|f\|_{Lip, q}$ and

(ii) $\|k*f\|_q \leq \sum_{n=1}^{\infty} \|U_m*f\|_q$ for $1 < q < \infty$.

**Proof.** Part (ii) of the theorem (which also appears as part (ii) in Lemma 1.7 of the first chapter) follows from Minkowski's inequality.

To show (i) we note,

$$\|U_m*f\|_q^q = \|\hat{U}_m*f\|_q^q \leq \left(\int \|\hat{U}_m\|_{Lip, q}^q \|f\|_{Lip, q}^q \right)^{\frac{1}{q}} \leq C_1 \left(\|\hat{U}_m\|_{Lip, q}^q \|f\|_{Lip, q}^q \right)^{\frac{1}{q}},$$

and hence our result.

**Corollary 4.25.** Let $1/2 < \lambda < 1$. If

$$\sum_{n=1}^{\infty} \|U_m*f\|_q^q \|f\|_{Lip, q}^q < \infty,$$

then

$$k*U_m^q(R^n) \text{ with } \frac{1}{q} = \frac{(1 - 2\lambda) + 2\lambda}{2} \text{ and } \frac{1}{p} - \frac{1}{q} = 1 - \lambda.$$

**Proof.** We know

$$\|U_m*f\|_q \leq C_1 \|\hat{U}_m\|_{Lip, q} \|f\|_{Lip, q},$$

and

$$\|U_m*f\|_q \leq \|\hat{U}_m\|_{Lip, q} \|f\|_q.$$

Therefore,

$$\|U_m*f\|_q \leq C_1 \|\hat{U}_m\|_{Lip, q} \|f\|_q,$$

with

$$\frac{1}{q} = \frac{(1 - 2\lambda) + 2\lambda}{2} \text{ and } \frac{1}{p} - \frac{1}{q} = 1 - \lambda.$$

But

$$\|k*f\|_q \leq \sum_{n=1}^{\infty} \|U_m*f\|_q \leq C_1 \|f\|_q \sum_{n=1}^{\infty} \|\hat{U}_m\|_{Lip, q} \|U_m\|_{Lip, q}^q,$$

and hence our result.

Part B. $0 \leq \lambda \leq 1/2$. It will help us to study the following picture:

$$\begin{array}{c}
\text{(Fig. 1)}
\end{array}$$

Here we will think of $y$ as $1/p$ and $x$ as $1/q$, and hence study the estimates along the line $y = x + (1 - \lambda)$.

**Lemma 4.26.** Let $0 \leq \lambda \leq 1/2$ with $1/p = 1/2 + v$ and $t = 2t/(3p - 2)$.

If

$$\sum_{n=1}^{\infty} \|U_m*f\|_p \|f\|_p < \infty,$$

then

$$k*U_m^q(R^n) \text{ with } s = \frac{3p - 2}{2p} \text{ and } \frac{1}{r} - \frac{1}{s} = 1 - \lambda.$$

**Proof.** Since $1/p = 1/2 + v$, we have

$$\|U_m*f\|_p \leq C_p \|U_m\|_p \|f\|_p,$$

and

$$\|U_m*f\|_p \leq \|U_m\|_p \|f\|_1.$$

Thus, we are interpolating along the line

$$y = \frac{2t - 2\lambda}{p} x + 1 \text{ (see Fig. 1).}$$

We are interested in that $x$ where

$$\frac{2t - 2\lambda}{p} x + 1 = x + (1 - \lambda).$$
This implies
\[ \omega = \frac{3p}{2} - \frac{t - \frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{t}{2}, \]
and therefore by Riesz-Thorin,
\[ \|U_m f k_t\|_p \leq C_{p,r} \|U_{m,r} f\|_p \|f\|_r, \]
where
\[ s = \frac{3p - 2}{3p} - \frac{1}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1}{s} + 1 - \lambda. \]
Putting this together with part (ii) of Lemma 4.24, we get our result.

**Examples in R.**

**Lemma 4.27.** Let 0 < \lambda < 1 and a > 0. If
\[ k(t) = \frac{e^{t^a}}{|t|^r}, \quad t \in \mathbb{R}, \]
with b < \lambda,
then
\[ k \ast I^{p,a}_q \quad \text{for} \quad q > \frac{a}{\lambda - b} \quad \text{with} \quad \frac{1}{p} - \frac{1}{q} = 1 - \lambda. \]
Proof. Let
\[ \sigma_{a}(-t) = e^{t^a} e^{-|t|^r}, \]
For
\[ k(t) = \frac{e^{t^a}}{|t|^r}, \]
\[ \frac{|dk|}{dt} \leq C|t|^{r-1}, \]
for |t| sufficiently large, where C depends only on a and b.
Then
\[ |k \ast \sigma_{a}(0) - k \ast \sigma_{b}(0)| = \int_{\mathbb{R}} |k(t) - k(t - v)| dt \leq C|v|^{r-1}, \]
which implies
\[ \left\{ v : |k \ast \sigma_{a}(0)| > \frac{T^{1-b}}{2} \right\} \leq C T^{1-\lambda}. \]
But, if \( k \ast I^{p,a}_q \), there exists \( C_{p,a} \), independent of \( T \), such that
\[ T^{1-\lambda} \left\{ v : |k \ast \sigma_{a}(0)| > \frac{T^{1-b}}{2} \right\} \leq C_{p,a} T^{1-\lambda}, \]
with \( 1/p - 1/q = 1 - \lambda \).
But that implies
\[ T^{1-\lambda} T^{1-\lambda/q} \leq C_{p,a} T^{1-\lambda}. \]
Thus, \( k \ast I^{p,a}_q \) if \((1-b)+(1-a)/q > 1/p \) (letting \( T \to \infty \)), or if \( q > a/(\lambda - b) \) and \( 1/p - 1/q = 1 - \lambda \).

**Theorem 4.28.** Let 0 < \lambda < 1 with t \in \mathbb{R} and \( k(t) = e^{t^a} |t|^r \). If a \neq 1, b < \lambda and \( \frac{1}{2} + \frac{1}{2} > 0 \), then
\[ k \ast I^{p,a}_q \quad \text{for} \quad \frac{a}{\lambda - b} < q < \frac{a}{\lambda - b} \]
and
\[ k \ast I^{p,a}_q \quad \text{for} \quad q > \frac{a}{\lambda - b} \quad \text{with} \quad \frac{1}{p} - \frac{1}{q} = 1 - \lambda. \]
Proof. By Lemma 4.27, we have \( k \ast I^{p,a}_q \) for \( q > a/\lambda - b \). Now we set
\[ \psi_{-n}(t) = \frac{e^{t^a}}{|t|^b} \left( \chi_{(a-1),m} + \chi_{(-a-1),-m-1} \right) \quad \text{for} \quad m \geq 1 \]
and
\[ \psi_{-n}(t) = \frac{e^{t^a}}{|t|^b} \chi_{(-a-1),m}. \]
Case 1. Assume 1/2 < \lambda < 1. To show \( k \ast I^{p,a}_q \) for \( q > a/\lambda - b \), we will first use Corollary 4.25, which applies to \( q < 2 \) or \( q > 2/(2\lambda - 1) \). We note
\[ \|V_{\psi_{-n}}\|_{L^q} \leq C_{n} \|V_{\psi_{-n}}\|_{L^q} < \infty. \]
Now,
\[ \|V_{\psi_{-n}}\|_{L^q} \leq 1 + II + III, \]
and for \( a > 1 \) we estimate as follows:
\[ \|V_{\psi_{-n}}\|_{L^q} \leq \left\{ \begin{array}{ll}
\int_{a - \lambda - (a - \lambda)}^{a - \lambda} \frac{dx}{x} & \text{if } a = 1 \\[a - \lambda - (a - \lambda)]^{1-1} \end{array} \right\} + 
\left\{ \begin{array}{ll}
\int_{a - \lambda - (a - \lambda)}^{a - \lambda} \frac{dx}{x} & \text{if } a = 1 \\[a - \lambda - (a - \lambda)]^{1-1} \end{array} \right\} + 
\left\{ \begin{array}{ll}
\int_{a - \lambda - (a - \lambda)}^{a - \lambda} \frac{dx}{x} & \text{if } a = 1 \\[a - \lambda - (a - \lambda)]^{1-1} \end{array} \right\}.
\]
While for 0 < a < 1 we decompose the x-axis into the intervals \((-\infty, 0), [0, 2a(\lambda - a - b)], \) and \([2a(\lambda - a - b), \infty)\), and estimate the corresponding integrals.
Note that the estimates for the intervals $[-2^n, -2^{n-1}]$ are similar. Thus, we obtain
\[
I \leq C \frac{1}{2} 2^{m(\lambda - a - b - \frac{a-2}{2})},
\]
and
\[
II + III \leq C 2^{m(-b-a-\delta)}.
\]

Now,
\[
b - (a-1) \lambda < (a-1) (1-\lambda) - b - \frac{a-2}{2} \text{ since } a > 0.
\]

Hence
\[
\|V_m\|_{L^q} \leq C 2^{m\left((a-1)\lambda - b - \frac{a-2}{2}\right)}, \quad \|V_m\|_{L^p} \leq C 2^{m(b-a)}
\]
where $C$ depends on $a, b$ and $\lambda$ but not on $m$. Thus,
\[
\sum_{m=1}^\infty \|V_m\|_{L^q} \|V_m\|_{L^p} \leq \sum_{m=1}^\infty 2^{m\left((a-1)\lambda - b - \frac{a-2}{2}\right)} 2^{m(b-a)}.
\]

We are interested in those $t$'s for which
\[
\left((a-1)(1-\lambda) - \frac{a-2}{2} - b\right) t + (\lambda - b) (1-t) < 0,
\]
or
\[
\left((a-1)(1-\lambda) - \frac{a-2}{2} - b\right) t + (\lambda - b) t < 0,
\]
or
\[
\left(\frac{a}{2} - a\lambda\right) t + (\lambda - b) < 0.
\]

Therefore,
\[
(\lambda - b) < t (a(\lambda - \frac{1}{2})), \quad \text{or} \quad t > \frac{\lambda - b}{a(\lambda - \frac{1}{2})}.
\]

From Corollary 4.23 we see the $q$'s we are looking for satisfy
\[
\frac{1}{q} = \frac{1}{(1-2\lambda) + 2\lambda} = \frac{2\lambda - (2\lambda - 1)}{2},
\]
and
\[
\frac{1}{2} < \frac{2\lambda - (\lambda - \frac{1}{2})(2\lambda - 1)}{2} = \frac{(\lambda - b)}{a} = \frac{\lambda(a-1) + b}{a},
\]
and
\[
\frac{1}{q} > \frac{\lambda - b}{\lambda(a-1) + b}.
\]

For the remaining values of $q$, the proof follows the same argument as worked out below in Case 2, but with $\frac{1}{2} < \lambda < 1$.

Case 2. Assume $0 < \lambda < \frac{1}{2}$. Since $h(t) = \sum_{n=0}^{\infty} V_n(t)$ we have by part (ii) of Lemma 4.24,
\[
\|k * f\|_{L^q} \leq \|V_h * f\|_{L^q} + \sum_{m=1}^\infty \|V_m * f\|_{L^q}
\]
and by Lemma 4.26,
\[
\|k * f\|_{L^q} \leq \|V_h * f\|_{L^q} + \sum_{m=1}^\infty \|V_m * f\|_{L^q}
\]
with $t = 2\lambda \log(3p - 2)$, $1/r - 1/t = 1 - \lambda$ and $1/p = 1/2 + v$.

As we have shown in Case 1,
\[
\|V_m\|_{L^q} \leq \frac{C}{2^{2m}} \cdot \frac{1}{m} \cdot \frac{2^{m(a-1)}}{m} \quad \text{and} \quad \|V_m\|_{L^p} \leq \frac{C}{2^{2m}}.
\]
where $C$ depends only on $a, b$ and $p$ but not on $m$. Now consider
\[
\sum_{m=1}^\infty \left(\frac{1}{2^{2m}} \cdot \frac{1}{m} \cdot \frac{2^{m(a-1)}}{m} \right)^{-t} \left(\frac{1}{2^{2m}}\right)^{-t} = \sum_{m=1}^\infty 2^{m(a-1)\frac{1}{2} - \frac{1}{2} - \frac{a-2}{2}}.
\]

Suppose
\[
\frac{t}{2} < \frac{\lambda(a-1) + b}{a} \quad \text{(see Fig. 1),}
\]

i.e. $2/t$ is a prospective $q$.

Set $t = \frac{\lambda(a-1) + b}{a} - \delta$ with $\delta > 0$ but $\delta$ small. Since
\[
\frac{t}{2} = \frac{\lambda p}{3p - 2} \quad \text{implies} \quad \frac{1}{2} - \frac{1}{2} = \frac{\lambda(a-1) + b - \delta a}{2\lambda(a-1) + b - \delta a}.
\]
then

\[
\left\{ (a-1) \frac{1}{p} - \frac{1}{2}, \frac{a-2}{2} \right\} - b
\]

\[
= \left\{ (a-1) \frac{b(a-2)+2b-\delta a}{a} - \frac{a-2}{2} \frac{2b(a-1)+2b-\delta a}{a} \right\} - b
\]

\[
= \frac{ba-\delta a^2}{2} \frac{2}{a} - b = \frac{-\delta a}{2} < 0.
\]

Therefore,

\[ k \in L_p^\infty \text{ for } \frac{a}{\lambda(a-1)+b} < q \leq \frac{a}{\lambda-b} \text{ and } \frac{1}{p} - \frac{1}{q} = 1-\lambda. \]

We note that by adding the Hardy–Littlewood kernels $1/|t|^4$, we get the following:

**Corollary 4.29.** Let $0 < \lambda < 1$ with $t \in R$ and $k(t) = \frac{\text{e}^{it^4}}{|t|^4}$. If $a \neq 1, b \leq \lambda$ and $\frac{a}{\lambda} + (b-\lambda) > 0$, then

\[ k \in L_p^\infty \quad \text{for} \quad \frac{a}{\lambda(a-1)+b} < q < \frac{a}{\lambda-b} \]

and

\[ k \not\in L_p^\infty \quad \text{for} \quad q > \frac{a}{\lambda-b}, \]

with $1/p - 1/q = 1 - \lambda$.

**Examples in $R^n$.**

**Example 4.30.** If

\[ k(t) = \frac{\text{e}^{it^4}}{|t_1|^{2b} + \cdots + |t_n|^{2b}}, \quad n \geq 2 \text{ and } 0 < b < \lambda, \]

then

\[ k \in L_p^\infty (R^n) \quad \text{for} \quad \frac{2n}{n\lambda+2b} < q < \frac{2n}{n\lambda-2b}, \]

and

\[ k \not\in L_p^\infty (R^n) \quad \text{for} \quad q > \frac{2n}{n\lambda-2b}, \]

with $1/p - 1/q = 1 - \lambda$.

**Proof.** Since this argument is very similar to the ones found in Corollary 3.23 and Theorem 4.28, we shall be brief.

We define

\[ V_n(t) = \frac{\text{e}^{it^4}}{|t_1|^{2b} + \cdots + |t_n|^{2b}} \prod_{j=1}^n \left| 1 - \text{e}^{it_j} \right|, \]

and for $m > 1$,

\[ V_n(t) = \frac{\frac{2}{n\lambda+2b} \cdots + \frac{2}{n\lambda-2b}}{2^{n-1}} \prod_{j=1}^n \left| 1 - \text{e}^{it_j} \right| \prod_{j=1}^n \left| 1 - \text{e}^{it_j} \right| \]

\[ \forall m \leq n \leq 1 - \lambda. \]

We first do the case $1/2 < \lambda < 1$ and $q < 2$ or $q > 2/(2\lambda-1)$. We can show

\[ \|V_n\|_{\ell^q} \leq C \frac{2^{n\lambda(q-1)}}{2^{n\lambda}} \] for $m > 1$,

\[ \|V_n\|_{\ell^q} \leq C \frac{2^{n\lambda(q-1)-2}}{2^{n\lambda}} \] for $m \geq 1$

then apply Corollary 4.25 to show that

\[ k \in L_p^\infty \quad \text{for} \quad \frac{2n}{n\lambda+2b} < q < \frac{2n}{n\lambda-2b} \]

and $1/p - 1/q = 1 - \lambda$. Here, $C$ depends only on $n, b, \lambda$ but not on $m$.

For the remaining cases, we show

\[ \|V_n\|_{\ell^q} \leq C \frac{2^{n\lambda(q-1)-2}}{2^{n\lambda}} \] for $m > 1$,

and

\[ \|V_n\|_{\ell^q} \leq C \frac{2^{n\lambda(q-1)-2}}{2^{n\lambda}} \] for $m > 1$,

where $1/p - 1/q = 1 - \lambda$, and here we apply the method of proof in Lemma 4.26 in conjunction with Lemma 4.24 to show

\[ k \in L_p^\infty \quad \text{for} \quad \frac{2n}{n\lambda+2b} < q < \frac{2n}{n\lambda-2b} \]

and $1/p - 1/q = 1 - \lambda$.

To show $k \not\in L_p^\infty$ for $q > 2n/(n\lambda-2b)$ and $1/p - 1/q = 1 - \lambda$, we apply Lemma 2.18 in conjunction with Theorem 2.19.

**References**

$L^p$-Struktur in Banachräumen

Von

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Abschnitt. Sei $V$ ein $K$-Banachraum, $1 \leq p < \infty$. Zwei abgeschlossene Unterräume $X, X'$ von $V$ heißen zueinander orthogonale $L^p$-Summanden (Schreibweise: $V = X \oplus X'$), wenn algebraisch $V = X \oplus X'$ gilt und für $v \in X, w \in X'$ stets

$$||v + w||^p = ||v||^p + ||w||^p$$

(bzw. für $p = \infty$: $||v + w|| = \max(||v||, ||w||)$) ist. Projektionen $e$ auf $L^p$-Summanden, die offensichtlich durch

$$||v||^p = ||ev||^p + ||e^*v||$$

(für $p = \infty$: $||v|| = \max(||ev||, ||e^*v||)$) für alle $v \in V$ charakterisiert sind, heißen $L^p$-Projektionen.

$L^p$-Summanden und $L^p$-Projektionen für $p = 1, \infty$ wurden — ausgehend von Arbeiten von Cunningham [4], [5] — in letzter Zeit besonders von [1], [2], [6], [8] untersucht. Die Arbeiten [9], [10] der den allgemeinen Fall $1 \leq p \leq \infty$ behandeln, setzen die Kommutativität von $L^p$-Pro-

Untersuchungen der Mengen sämtlicher $L^p$-Summanden im Falle klassischer Banachräume $V$ ergaben (7), (11a, c, f), daß dort für $p \neq 2$ $L^p$-Projektionen kommutieren und daß für $dim V > 2$ überhaupt nur ein Typ von nichttrivialen $L^p$-Projektionen existieren kann. In der vorlie-