THEOREM. *Sei f strikt konvex und* Epi *f flach konvex. Die Hyperebene* $H =$ Kern u_0 *sei f-Chebychev. Dann ist die metrische Projektion* $P_H(\cdot, f)$ *genau dann stetig, wenn die Abbildung*

$$\psi: \begin{array}{l} \lambda \mapsto x, \quad \textit{wobei} \quad f^*(\lambda u_0) + f(x) = \lambda u_0(x), \\ \mathring{A}(u_0) \mapsto (P_H(\cdot, f))^{-1}(0) \end{array}$$

stetig ist.

KOROLLAR. *f und H mögen die Voraussetzungen des Theorems erfüllen. Ist f außerdem differenzierbar, so ist* $P_H(\cdot, f)$ *genau dann stetig, wenn* $(f')^{-1}|\mathring{A}(u_0)\cdot u_0$ *stetig ist.*

Beweis. In [2], Seite 59, wird gezeigt, daß

$$\psi(\lambda) = (f')^{-1}(\lambda u_0).$$

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Some permanence properties of locally convex spaces defined by norm space ideals of operators

by

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Abstract. The concept of an $\mathscr{A}$-space, $\mathscr{A}$ an ideal of operators between normed spaces, as introduced by A. Pietsch allows one to treat simultaneously many important classes of locally convex spaces. In this paper necessary and sufficient conditions are given for the space $\mathscr{L}(E, F)$ of continuous linear maps between locally convex spaces E and F and the tensor product $E\otimes_\alpha F$ ($\alpha = \varepsilon$ or π) to be spaces of type $\mathscr{A}$ provided that $\mathscr{A}$ possesses certain stability properties. As an application of our results to "concrete" $\mathscr{A}$-spaces some known results of A. Grothendieck and D. Randtke on (strongly) nuclear and Schwartz spaces are derived.

1. INTRODUCTION

Let $\mathscr{A}$ be an ideal of operators between normed spaces. A. Pietsch [11] has introduced the concept of an $\mathscr{A}$-space, which unifies the treatment of various important classes of locally convex spaces. For example, if $\mathscr{A}$ is chosen to be the ideal of nuclear (resp. precompact) operators, the corresponding $\mathscr{A}$-spaces are the nuclear (resp. Schwartz) spaces. Nuclear and Schwartz spaces share many remarkable stability properties. Subspaces, quotients, arbitrary products and countable direct sums of these spaces are of the respective type. In fact, as pointed out by A. Pietsch [11], all the above mentioned properties are shared by all $\mathscr{A}$-spaces under certain general assumptions about the ideal $\mathscr{A}$.

In this paper spaces of linear operators and topological tensor products will be dealt with. We shall introduce the concepts of "Hom-stability" and "$\otimes_\alpha$-stability" of an ideal $\mathscr{A}$ and prove necessary and sufficient conditions for $\mathscr{L}(E, F)$ and $E\otimes_\alpha F$ ($\alpha = \varepsilon$ or π) to be $\mathscr{A}$-spaces provided that $\mathscr{A}$ possesses the respective stability.

We shall include the proof of Hom-stability of the ideals of nuclear and type s operators, which is essentially contained in an unpublished manuscript of K. Vala, who has kindly let me make use of his material. The $\otimes_\alpha$-stability ($\alpha = \varepsilon$ or π) of these ideals and the ideal of precompact operators and also the Hom-stability of the latter all follow from results of J. Holub [8], [9], K. Vala [14] and the author [1]. This shows in particular that our theorems (3.3 and 4.4) on $\mathscr{A}$-spaces generalize the corresponding results of Randtke [12] and Grothendieck [5] concerning (strongly) nuclear and Schwartz spaces.

2. NOTATIONS AND DEFINITIONS

Ideals of operators. The class of all bounded linear mappings (briefly called operators) between arbitrary normed linear spaces will be denoted by $\mathscr{L}$ and the set of all operators between specific normed spaces by $\mathscr{L}(E, F)$. Following Pietsch [11] we say that a class $\mathscr{A}$ of operators is an *ideal* if for each set

$$\mathscr{A}(E, F) = \mathscr{A} \cap \mathscr{L}(E, F)$$

one has

(1) $x' \otimes y \in \mathscr{A}(E, F)$ for each $x' \in E'$ and $y \in F$,

(2) $\mathscr{A}(E, F)$ is a linear subspace of $\mathscr{L}(E, F)$,

(3) if $T \in \mathscr{A}(E, F)$ and $S \in \mathscr{L}(F, G)$, then $ST \in \mathscr{A}(E, G)$,

(4) if $T \in \mathscr{L}(E, F)$ and $S \in \mathscr{A}(F, G)$, then $ST \in \mathscr{A}(E, G)$.

The operator $x' \otimes y$ in condition (1) means the rank one operator: $x \to \langle x, x' \rangle y$. The finite rank operators $\mathscr{F}$ obviously form the smallest ideal.

An ideal $\mathscr{A}$ is called *injective* if for any topological injection (i.e. an isomorphism into) $J \in \mathscr{L}(F, G)$ the following is true:

$$\text{if } S \in \mathscr{L}(E, F) \text{ and } JS \in \mathscr{A}(E, G), \text{ then } S \in \mathscr{A}(E, F).$$

The smallest injective ideal containing $\mathscr{A}$ is called the *injective hull* and denoted by $\mathscr{A}^J$.

The *composition* of two ideals $\mathscr{A}$ and $\mathscr{B}$ will be defined as follows: $T \in (\mathscr{A} \circ \mathscr{B})(E, G)$ if and only if there is a normed space F and operators $S \in \mathscr{B}(E, F)$ and $R \in \mathscr{A}(F, G)$ such that $T = RS$.

Two ideals $\mathscr{A}$ and $\mathscr{B}$ are said to be *equivalent* if there are integers n and m such that

$$\mathscr{A}^n \subset \mathscr{B} \quad \text{and} \quad \mathscr{B}^m \subset \mathscr{A},$$

where $\mathscr{A}^n = \mathscr{A} \circ \mathscr{A} \circ \ldots \circ \mathscr{A}$ (n times).

For example, the ideal of quasi-nuclear operators is an injective ideal equivalent to the ideal of nuclear operators.

An ideal $\mathscr{A}$ is called *symmetric* if

$$T \in \mathscr{A}(E, F) \quad \text{implies} \quad T' \in \mathscr{A}(F', E'),$$

where T' is the transpose of T.

Locally convex topologies and bornologies of type $\mathscr{A}$. Let E be a Hausdorff locally convex space and $\mathscr{N}_E$ a fundamental system of balanced convex neighbourhoods of the origin. The gauge of a set $U \in \mathscr{N}_E$ will be denoted by P_U and the factor space $E/\ker P_U$ by E_U. The norm of E_U is defined by the formula

$$\|\psi_U(x)\|_U = P_U(x), \quad x \in E,$$

where $\psi_U\colon E \to E_U$ is the canonical surjection.

Let $\mathscr{A}$ be an ideal of operators in normed spaces. We say (after Pietsch [11]) that a locally convex space E is of *type* $\mathscr{A}$ if there is a fundamental system $\mathscr{N}_E$ of balanced convex neighbourhoods of the origin such that:

for each $U \in \mathscr{N}_E$ there exists a $V \in \mathscr{N}_E$ such that $V \subset U$ and the canonical mapping $\psi_{V,U}\colon E_V \to E_U$ belongs to $\mathscr{A}(E_V, E_U)$.

Similarly, if E is a separated convex bornological vector space and $\mathfrak{S}$ a fundamental system of bounded convex balanced sets, we denote by E_A the vector subspace generated by $A \in \mathfrak{S}$. In this case the gauge P_A is a norm on E_A. The canonical injection $E_A \to E$ will be denoted by i_A.

A convex bornology $\mathfrak{S}$ is said to be of *type* $\mathscr{A}$ if there is a fundamental system $\mathfrak{S}_0$ of balanced convex bounded subsets such that:

for each $A \in \mathfrak{S}_0$ there exists a $B \in \mathfrak{S}_0$ such that $B \supset A$ and the canonical mapping $i_{A,B}\colon E_A \to E_B$ belongs to $\mathscr{A}(E_A, E_B)$.

For the basic concepts of bornology the reader is referred to [6]. We shall put our statements in such a form that in fact no aquaintance with the theory of bornology is necessary. By a *convex bornology* in a locally convex space we simply mean a family of bounded subsets which covers E and is saturated in the sense that it contains arbitrary subsets, scalar multiples, finite unions and balanced convex hulls of members of the family. In bornologic terms this means that we are dealing with convex bornologies compatible with a given locally convex topology. To avoid unnecessarily complicated use of language we take us the freedom of speaking about bornology even if we were dealing with a base of bornology.

3. SPACES OF LINEAR OPERATORS

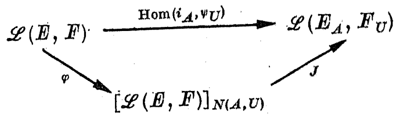
Let E and F be locally convex Hausdorff spaces and $\mathfrak{S}$ a bornology on E. We shall consider the space $\mathscr{L}_{\mathfrak{S}}(E, F)$ of continuous linear mappings from E into F equipped with the $\mathfrak{S}$-topology. The sets $N(A, V) = \{T \mid T(A) \subset V\}$, $A \in \mathfrak{S}$, $V \in \mathscr{N}_F$ form a fundamental system of neighbourhoods of the origin of $\mathscr{L}_{\mathfrak{S}}(E, F)$.

Let $E_1, \ldots, E_4$ be normed spaces and let $S \in \mathscr{L}(E_1, E_2)$ and $T \in \mathscr{L}(E_3, E_4)$. The mapping which to each $U \in \mathscr{L}(E_2, E_3)$ assigns the composition

TUS will be denoted by $\operatorname{Hom}(S, T)$. Thus $\operatorname{Hom}(S, T)$ is for fixed S and T a bounded linear mapping from $\mathscr{L}(E_2, E_3)$ into $\mathscr{L}(E_1, E_4)$.

LEMMA 3.1. *Let E and F be locally convex Hausdorff spaces, let A be a balanced convex bounded subset of E and let $U \in \mathscr{N}_F$. Then the normed space $[\mathscr{L}(E, F)]_{N(A,U)}$ can be isometrically imbedded into $\mathscr{L}(E_A, F_U)$.*

Proof. Let $i_A\colon E_A \to E$ be the canonical injection and $\psi_U\colon F \to F_U$ the canonical surjection. We claim that there is an isometry J which makes the following diagram commute:



(φ is the canonical surjection onto the factor space.)

It is enough to show that

$$P_{N(A,U)}(T) = \|\operatorname{Hom}(i_A, \psi_U)(T)\| \quad \text{for all } T \in \mathscr{L}(E, F),$$

where $P_{N(A,U)}$ is the gauge of the neighbourhood $N(A, U)$. For $T \in \mathscr{L}(E, F)$ we have:

$$\begin{aligned} P_{N(A,U)}(T) &= \inf\{\lambda > 0 \mid T(A) \subset \lambda U\} \\ &= \inf\{\lambda > 0 \mid P_U(Tx) \leqslant \lambda \text{ for all } x \in A\} \\ &= \sup_{x\in A} P_U(Tx) = \sup_{x \in A} \|(\psi_U T i_A)(x)\|_U \\ &= \|\operatorname{Hom}(\psi_U, i_A)(T)\| \quad \text{in } \mathscr{L}(E_A, F_U). \end{aligned}$$

So the required equality has been established. ■

In order to state our stability theorem concerning spaces of linear operators we shall need the following definition.

DEFINITION 3.2. An ideal $\mathscr{A}$ is said to be *Hom-stable* if the following holds: Given normed spaces $E_1, \ldots, E_4$ and operators $S \in \mathscr{A}(E_1, E_2)$, $T \in \mathscr{A}(E_3, E_4)$ it follows that

$$\operatorname{Hom}(S, T) \in \mathscr{A}\big(\mathscr{L}(E_2, E_3), \mathscr{L}(E_1, E_4)\big).$$

Let us point out that if $\mathscr{A}$ is Hom-stable, then it is in particular symmetric, because choosing $E_3 = E_4 = \boldsymbol{K}$ (= the scalar field) and $T = \mathrm{id}_K$ we have $T \in \mathscr{A}(E_3, E_4)$ and thus

$$S' = \operatorname{Hom}(S, T) \in \mathscr{A}(E_2', E_1')$$

as soon as $S \in \mathscr{A}(E_1, E_2)$.

THEOREM 3.3. *Let $\mathscr{A}$ be an ideal of operators which is Hom-stable and equivalent to its injective hull. Let E and F be locally convex Hausdorff spaces and let $\mathfrak{S}$ be a bornology on E. Then the space $\mathscr{L}_{\mathfrak{S}}(E, F)$ is a locally convex space of type $\mathscr{A}$ if and only if $\mathfrak{S}$ is a bornology of type $\mathscr{A}$ and F is a locally convex space of type $\mathscr{A}$.*

Proof. 1° Assume first that $\mathscr{L}_{\mathfrak{S}}(E, F)$ is of type $\mathscr{A}$. To prove that F is of type $\mathscr{A}$, imbed F into $\mathscr{L}(E, F)$ by assigning to each $y \in F$ the rank one operator: $x \to \langle x, x_0'\rangle y$ in $\mathscr{L}(E, F)$, where x_0' is any fixed non-zero element of E'. It is easy to see that the $\mathfrak{S}$-topology induces the original topology of F. As $\mathscr{A}$ is equivalent to its injective hull it follows exactly as in Pietsch [10], Proposition 5.1.1, that F as a subspace of an $\mathscr{A}$-space is itself of type $\mathscr{A}$.

To prove that $\mathfrak{S}$ is a bornology of type $\mathscr{A}$, imbed E' into $\mathscr{L}(E, F)$ by assigning to each $x' \in E'$ the operator: $x \to \langle x, x'\rangle y_0$, where y_0 is a fixed non-zero element of F. The $\mathfrak{S}$-topology of $\mathscr{L}(E, F)$ induces the $\mathfrak{S}$-topology of E', so we conclude as above that E' is a locally convex space of type $\mathscr{A}$. Thus, given $A \in \mathfrak{S}$, there is a $B \in \mathfrak{S}$ such that $B \supset A$ and the canonical mapping

$$\psi_{B^0, A^0}\colon E'_{B^0} \to E'_{A^0}$$

belongs to $\mathscr{A}(E'_{B^0}, E'_{A^0})$. As $\mathscr{A}$ is symmetric, it follows that

$$(\psi_{B^0, A^0})' \in \mathscr{A}\big((E'_{A^0})', (E'_{B^0})'\big).$$

If we identify E_A (resp. E_B) with a subspace of $(E'_{A^0})'$ (resp. $(E'_{B^0})'$), the restriction of $(\psi_{B^0, A^0})'$ to E_A coincides with $i_{A,B}$ (cf. Pietsch [10], 0.11). It follows that

$$i_{A,B} \in \mathscr{A}^J(E_A, E_B)$$

which implies that $\mathfrak{S}$ is a bornology of type $\mathscr{A}$, as $\mathscr{A}$ was assumed to be equivalent to its injective hull.

2° Assume that $\mathfrak{S}$ and F are of type $\mathscr{A}$. Consider the fundamental system

$$\mathscr{N} = \{N(A, U) \mid A \in \mathfrak{S},\ U \in \mathscr{N}_F\}$$

of neighbourhoods of the origin of $\mathscr{L}_{\mathfrak{S}}(E, F)$. We claim that for each $M(A, U) \in \mathscr{N}$ there is an $N(B, V) \in \mathscr{N}$ such that $N(B, V) \subset M(A, U)$ and the canonical mapping

$$\varphi_{N,M}\colon [\mathscr{L}(E, F)]_{N(B,V)} \to [\mathscr{L}(E, F)]_{M(A,U)}$$

belongs to $\mathscr{A}$.

Given $A \in \mathfrak{S}$ and $U \in \mathscr{N}_F$ there exist $B \in \mathfrak{S}$ and $V \in \mathscr{N}_F$ such that $B \supset A$, $V \subset U$ and the canonical mappings

$$i_{A,B}\colon E_A \to E_B \quad \text{and} \quad \psi_{V,U}\colon F_V \to F_U$$

belong to $\mathscr{A}$. The mapping

$$\mathrm{Hom}(i_{A,B}, \psi_{V,U}): \mathscr{L}(E_B, F_V) \to \mathscr{L}(E_A, F_U)$$

thus belongs to $\mathscr{A}(\mathscr{L}(E_B, F_V), \mathscr{L}(E_A, F_U))$, as $\mathscr{A}$ is Hom-stable. Let

$$\varphi_N: \mathscr{L}(E, F) \to [\mathscr{L}(E, F)]_{N(B,V)} \quad \text{and} \quad \varphi_M: \mathscr{L}(E, F) \to [\mathscr{L}(E, F)]_{M(A,U)}$$

be the canonical surjections and J_N and J_M the corresponding isometries according to Lemma 3.1. For $T \in \mathscr{L}(E, F)$ we have:

$$\begin{aligned}
[\mathrm{Hom}(i_{A,B}, \psi_{V,U}) \circ J_N](\varphi_N(T)) &= [\mathrm{Hom}(i_{A,B}, \psi_{V,U}) \circ \mathrm{Hom}(i_B, \psi_V)](T) \\
&= \psi_{V,U}\psi_V T i_B i_{A,B} = \psi_U T i_A = \mathrm{Hom}(i_A, \psi_U)(T) \\
&= J_M(\varphi_M(T)) = J_M(\varphi_{N,M}(\varphi_N(T))).
\end{aligned}$$

Hence:

$$\mathrm{Hom}(i_{A,B}, \psi_{V,U}) \circ J_N = J_M \circ \varphi_{N,M}.$$

It thus follows that

$$\varphi_{N,M} \in \mathscr{A}^J([\mathscr{L}(E, F)]_N, [\mathscr{L}(E, F)]_M)$$

and the conclusion follows from the equivalence of $\mathscr{A}$ and $\mathscr{A}^J$. ■

The question whether the structure of an $\mathscr{A}$-space is preserved under passing to the space of linear operators has thus been reduced to the question of Hom-stability of the given ideal $\mathscr{A}$.

In order to deal with the space $\mathscr{L}(E', F)$ it is more natural to assume a slight variant of Hom-stability of the ideal $\mathscr{A}$.

DEFINITION. An ideal $\mathscr{A}$ is said to be *Hom'-stable* if the following holds: Given normed spaces E_1, F_1, E_2, F_2 and operators $S \in \mathscr{A}(E_1, F_1)$, $T \in \mathscr{A}(E_2, F_2)$ it follows that

$$\mathrm{Hom}(S', T) \in \mathscr{A}(\mathscr{L}(E_1', E_2), \mathscr{L}(F_1', F_2)).$$

Observe that if $\mathscr{A}$ is Hom-stable, then it is Hom'-stable. Moreover' every Hom'-stable ideal $\mathscr{A}$ has the property that the bitranspose T'' belongs to $\mathscr{A}$ for every T in $\mathscr{A}$.

THEOREM 3.4. *Let $\mathscr{A}$ be an ideal of operators which is Hom'-stable and equivalent to its injective hull. Then $\mathscr{L}_e(E_b', F)$ is an $\mathscr{A}$-space if and only if E and F are $\mathscr{A}$-spaces. (e = the topology of equicontinuous convergence, b = the strong topology of the dual.)*

Proof. The necessity follows by imbedding E (resp. F) into $\mathscr{L}_e(E_b', F)$.

To prove sufficiency replace E by E_b' and take for $\mathfrak{S}$ the family of equicontinuous sets of E' in the proof of Theorem 3.3. Observe that $i_{A,B}$ is the transpose of ψ_{V_1,U_1}, where $V_1^0 = B$, $U_1^0 = A$ and V_1 and U_1 are chosen so that $\psi_{V_1,U_1} \in \mathscr{A}(E_{V_1}, E_{U_1})$. As $\mathscr{A}$ is Hom'-stable,

$$\mathrm{Hom}(i_{A,B}, \psi_{V,U}) \in \mathscr{A}(\mathscr{L}(E_B', F_V), \mathscr{L}(E_A', F_U)).$$

Then proceed exactly as in the proof of Theorem 3.3. ■

COROLLARY 3.5. *Let $\mathscr{A}$ be an ideal which is equivalent to its injective hull and such that T'' belongs to $\mathscr{A}$ for all T in $\mathscr{A}$. Then a locally convex Hausdorff space E is of type $\mathscr{A}$ if and only if E'' equipped with the topology of equicontinuous convergence is of type $\mathscr{A}$.*

Proof. Take $F = F_V = F_U = K$ and $\psi_{V,U} = \mathrm{id}_K$ in the proof of Theorem 3.4. ■

It is well known that the ideals of precompact, nuclear and type-s-operators satisfy the above condition and hence the conclusion of Corollary 3.5 holds for the corresponding $\mathscr{A}$-spaces (cf. also [13], (10), p. 77). We shall next show that the statement of Theorem 3.3 holds for strongly nuclear, nuclear and Schwartz spaces by proving the Hom-stability of the ideals in question. This will then show that Theorem 3.3 generalizes the corresponding results of Randtke ([12], Proposition 4.1).

PROPOSITION 3.6. *The ideals of*

(i) *finite rank,*

(ii) *type s,*

(iii) *nuclear,*

(iv) *precompact*

operators are Hom-stable.

Proof. The proofs of (i) and (iii) can be regarded as simplified versions of the proof of (ii), whereas (iv) is the content of [14], Theorem 3. So it sufficies to prove (ii).

Let $S: E_1 \to E_2$ and $T: E_3 \to E_4$ be mappings of type s between given normed spaces. (For the definition and properties of s-type mappings the reader is referred to [2], [3] and [10], 8.5.) Thus there exist sequences (x_i) in E_2, (x_i') in E_1', (y_k) in E_4 and (y_k') in E_3' such that

$$Sx = \sum_{i=1}^{\infty} \langle x, x_i' \rangle x_i \quad \text{for } x \in E_1,$$

$$Tz = \sum_{k=1}^{\infty} \langle z, y_k' \rangle y_k \quad \text{for } z \in E_3$$

and

$$\sup_{i \in N} i^p \|x_i'\| \|x_i\| < \infty, \qquad \sup_{k \in N} k^r \|y_k'\| \|y_k\| < \infty \tag{1}$$

for all p and r in N (= the set of positive integers).

A simple computation yields:

$$\mathrm{Hom}(S,T)(U)(x) = \sum_{k,i} \langle x, x_i' \rangle \langle U x_i, y_k' \rangle y_k$$

for $U \in \mathscr{L}(E_2, E_3)$ and $x \in E_1$. Let z_{ik}' denote the continuous linear form on $\mathscr{L}(E_2, E_3)$ defined by

$$\langle U, z_{ik}' \rangle = \langle U x_i, y_k' \rangle \quad \text{for } U \in \mathscr{L}(E_2, E_3).$$

Define further $V_{ik} \in \mathscr{L}(E_1, E_4)$ by

$$V_{ik} x = \langle x, x_i' \rangle y_k, \quad x \in E_1.$$

Thus we can write

(2) $$\mathrm{Hom}(S,T)(U)(x) = \sum_{k,i} \langle U, z_{ik}' \rangle V_{ik}(x).$$

It follows from (1) that the series representations of S and T converge in the norm of the respective spaces of linear operators. From this it follows easily that the double series representation (2) converges uniformly on the unit ball of E_1 for any fixed $U \in \mathscr{L}(E_2, E_3)$, i.e. we have the representation

$$\mathrm{Hom}(S,T)(U) = \sum_{i,k} \langle U, z_{ik}' \rangle V_{ik} \quad \text{for } U \in \mathscr{L}(E_2, E_3).$$

There remains to be shown that the sequence $(\|z_{ik}'\| \|V_{ik}\|)_{i,k}$ is rapidly decreasing. For this, observe that

$$\|z_{ik}'\| \leqslant \|x_i\| \|y_k'\| \quad \text{and} \quad \|V_{ik}\| \leqslant \|x_i'\| \|y_k\|.$$

Thus, for any positive integers p and r:

$$\sup_{i,k} i^p k^r \|z_{ik}'\| \|V_{ik}\| \leqslant (\sup_i i^p \|x_i\| \|x_i'\|)(\sup_k k^r \|y_k'\| \|y_k\|) < \infty$$

by (1), and the conclusion follows from Lemma 3.7 to be given below. ■

To see that the double-indexed sequence $(\|z_{ik}'\| \|V_{ik}\|)_{i,k}$ in the preceding proof can indeed be arranged to a rapidly decreasing sequence we prove the following simple lemma.

LEMMA 3.7. *Let $(b_{ik})_{i,k}$ be a double-indexed sequence of non-negative numbers such that*

$$\sup_{i,k} i^p k^r b_{ik} < \infty \quad \text{for all } p, r \in \mathbf{N}.$$

Then the sequence $a_n = b_{i_n, k_n}$, $n = 1, 2, \ldots$ is rapidly decreasing, where the mapping $n \to (i_n, k_n)$ is the "diagonal indexing" of $\mathbf{N} \times \mathbf{N}$.

Proof. From the estimate

$$n \leqslant 1+2+\ldots+i_n+k_n-1 = \tfrac{1}{2}(i_n+k_n)(i_n+k_n-1) < (i_n+k_n)^2$$

it follows that $n^q a_n \leqslant (i_n+k_n)^{2q} b_{i_n,k_n}$ for any $q \in \mathbf{N}$. So the conclusion follows by expanding according to the binomial formula. ■

The ideals (i), (ii) and (iv) of Proposition 3.6 are injective, whereas (iii) is equivalent to the injective ideal of quasi-nuclear operators. So these ideals indeed satisfy the requirements of Theorems 3.3, 3.4 (and Corollary 3.5).

If $\mathscr{A}$ is an ideal satisfying the assumptions of Theorem 3.3, then the space $\mathscr{L}(E, F)$ equipped with the topology of pointwise convergence is of type $\mathscr{A}$ if and only if F is of type $\mathscr{A}$.

Making use of the fact that the compact bornology of a Fréchet space is a Schwartz bornology ([7]) Proposition 3.1, p. 60) we get, as a consequence of Theorem 3.3 and Proposition 3.6, a result of the same kind concerning Schwartz spaces and compact convergence:

THEOREM 3.10. *Let E be a Fréchet space and F a locally convex Hausdorff space. Then $\mathscr{L}(E, F)$ equipped with the topology of compact convergence is a Schwartz space if and only if F is a Schwartz space.*

4. TENSOR PRODUCTS

In this section we shall consider the tensor product of two $\mathscr{A}$-spaces. Let us recall that the ε- and π-topologies of a tensor product of locally convex spaces can be defined by means of the following families of seminorms:

$$\varepsilon_{U,V}(z) = \sup\Big\{\Big|\sum_{k=1}^{n} \langle x_k, x' \rangle \langle y_k, y' \rangle\Big| \mid x' \in U^0, y' \in V^0\Big\},$$

where

$$z = \sum_{k=1}^{n} x_k \otimes y_k \in E \otimes F, \quad U \in \mathscr{N}_E, \; V \in \mathscr{N}_F;$$

$$\pi_{U,V}(z) = \inf\Big\{\sum_{k=1}^{n} P_U(x_k) P_V(y_k) \mid z = \sum_{k=1}^{n} x_k \otimes y_k, \; n \in \mathbf{N}\Big\},$$
$$U \in \mathscr{N}_E, \quad V \in \mathscr{N}_F.$$

Recall further that $E \otimes_\varepsilon F$ can be regarded as a subspace of $\mathscr{L}_e(E_b', F)$ (e = equicontinuous convergence) and $\pi_{U,V}$ equals the gauge of $\Gamma(U \otimes V)$ = balanced, convex hull of the set $\{x \otimes y \mid x \in U, y \in V\}$.

As an immediate consequence of Theorem 3.4 we get the following:

THEOREM 4.1. *Let $\mathscr{A}$ be a Hom'-stable ideal of operators which is equivalent to its injective hull. Then $E \otimes_\varepsilon F$ is an $\mathscr{A}$-space if and only if E and F are $\mathscr{A}$-spaces.*

Proof. Imbed $E \otimes_\varepsilon F$ into $\mathscr{L}_e(E'_b, F)$ and apply Theorem 3.4. For the converse imbed E (resp. F) into $E \otimes_\varepsilon F$. ■

This result generalizes Corollary 4.4 of Randtke [12] concerning strongly nuclear, nuclear and Schwartz spaces because of Proposition 3.6. In the first two cases the ε and π-topologies coincide and even in the case of Schwartz spaces it is known that their π-tensorproduct is again a Schwartz space ([5], Chapter I, p. 48). We shall show that for any ideal $\mathscr{A}$ which is "$\otimes_\alpha$-stable" (α is either ε or π) the tensorproduct $E \otimes_\alpha F$ of two $\mathscr{A}$-spaces E and F is of type $\mathscr{A}$. This generalizes the results of Grothendieck and Randtke mentioned above (and even Theorem 4.1). In particular, we get a unified treatment of the ε and π-tensorproducts of Schwartz spaces whose methods of proof look quite different in [5] and [12].

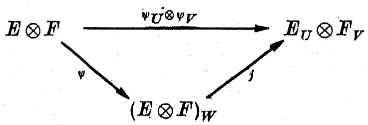
Definition 4.2. An ideal $\mathscr{A}$ is called *$\otimes_\alpha$-stable* if the following holds: Given normed spaces E_1, F_1, E_2, F_2 and operators $S \in \mathscr{A}(E_1, F_1)$, $T \in \mathscr{A}(E_2, F_2)$ it follows that

$$S \otimes T \in \mathscr{A}(E_1 \otimes_\alpha E_2, F_1 \otimes_\alpha F_2).$$

Lemma 4.3. *For any locally convex Hausdorff spaces E and F the space $(E \otimes_\alpha F)_W$ is isometric with $E_U \otimes_\alpha F_V$, where $U \in \mathscr{N}_E$, $V \in \mathscr{N}_F$ and*

$$W = \{z \mid \varepsilon_{U,V}(z) \leqslant 1\} \quad \text{if } \alpha = \varepsilon,$$
$$W = \{z \mid \pi_{U,V}(z) \leqslant 1\} \quad \text{if } \alpha = \pi.$$

Proof. 1° $\alpha = \varepsilon$. Let p_W denote the gauge of W, i.e. $p_W = \varepsilon_{U,V}$. We claim that there is an isometry j which makes the following diagram commute



where ψ_U, ψ_V and ψ are the canonical surjections. It is enough to show that

(1) $$p_W(z) = \|(\psi_U \otimes \psi_V)(z)\|_\varepsilon, \quad z \in E \otimes F.$$

The dual $(E_U)'$ can be identified with E'_{U^0} by means of the correspondence $u' \leftrightarrow x'$ defined by:

$$\langle \psi_U(x), u' \rangle = \langle x, x' \rangle, \quad x \in E,\ x' \in E'_{U^0}.$$

In the same way we identify $(E_V)'$ and E'_{V^0}. Let

$$z = \sum_{k=1}^{n} x_k \otimes y_k \in E \otimes F.$$

$$p_W(z) = \sup\Big\{\Big|\sum_{k=1}^{n} \langle x_k, x' \rangle \langle y_k, y' \rangle\Big| \mid x' \in U^0,\ y' \in V^0\Big\}$$
$$= \sup\Big\{\Big|\sum_{k=1}^{n} \langle \psi_U(x_k), u' \rangle \langle \psi_V(y_k), v' \rangle\Big| \mid \|u'\| \leqslant 1,\ \|v'\| \leqslant 1\Big\}$$
$$= \Big\|\sum_{k=1}^{n} \psi_U(x_k) \otimes \psi_V(y_k)\Big\|_\varepsilon = \|(\psi_U \otimes \psi_V)(z)\|_\varepsilon.$$

The proof of part 1° is thus complete.

2° $\alpha = \pi$. In this case $p_W = \pi_{U,V}$ and we have to prove equation (1) with ε replaced by π. The inequality

$$\|(\psi_U \otimes \psi_V)(z)\|_\pi \leqslant p_W(z), \quad z \in E \otimes F$$

follows directly from the fact that

$$\|(\psi_U \otimes \psi_V)(z)\|_\pi = \inf\Big\{\sum_k p_U(x_k) p_V(y_k) \mid (\psi_U \otimes \psi_V)(z) = (\psi_U \otimes \psi_V)\Big(\sum_k x_k \otimes y_k\Big)\Big\}.$$

To prove the reverse inequality we shall first show that

$$\ker(\psi_U \otimes \psi_V) \subset \ker p_W.$$

From the formula

$$\ker(\psi_U \otimes \psi_V) = \ker \psi_U \otimes F + E \otimes \ker \psi_V$$

(cf. [4], p. 25) it follows that every $\omega \in \ker(\psi_U \otimes \psi_V)$ has the representation

$$\omega = \sum_k x_k \otimes y_k + \sum_j u_j \otimes v_j, \quad \psi_U(x_k) = \psi_V(v_j) = 0.$$

Thus

$$p_W(\omega) \leqslant \sum_k p_U(x_k) p_V(y_k) + \sum_j p_U(u_j) p_V(v_j) = 0.$$

i.e. $\omega \in \ker p_W$.

Let now $(\psi_U \otimes \psi_V)\big(\sum_k x_k \otimes y_k\big)$ be an arbitrary representation of $(\psi_U \otimes \psi_V)(z)$ in $E_U \otimes E_V$. As

$$z - \sum_k x_k \otimes y_k \in \ker(\psi_U \otimes \psi_V) = \ker p_W,$$

it follows that

$$p_W(z) = p_W\Big(\sum x_k \otimes y_k\Big) \leqslant \sum p_U(x_k) p_V(y_k).$$

Thus

$$p_W(z) \leqslant \|(\psi_U \otimes \psi_V)(z)\|_\pi$$

and the proof is complete. ∎

THEOREM 4.4. *Let E and F be locally convex Hausdorff spaces and $\mathscr{A}$ an ideal of operators which is $\otimes_\varepsilon$-stable (resp. $\otimes_\pi$-stable). Then $E \otimes_\varepsilon F$ (resp. $E \otimes_\pi F$) is of type $\mathscr{A}$ if E and F are of type $\mathscr{A}$. The converse holds if $\mathscr{A}$ is equivalent to its injective hull.*

Proof. We shall treat both topologies simultaneously using the same notation as in Lemma 4.3. Consider the following fundamental system of balanced convex neighbourhoods of the origin of the tensor-product in question:

$$\{W(U, V) \mid U \in \mathscr{N}_E,\ V \in \mathscr{N}_F\}.$$

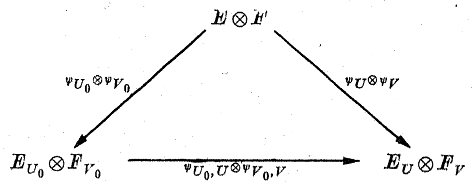
Let $U \in \mathscr{N}_E$ and $V \in \mathscr{N}_F$. As E and F are of type $\mathscr{A}$, there exist $U_0 \in \mathscr{N}_E$ and $V_0 \in \mathscr{N}_F$ such that $U_0 \subset U$, $V_0 \subset V$ and

$$\psi_{U_0,U} \in \mathscr{A}(E_{U_0}, E_U), \qquad \psi_{V_0,V} \in \mathscr{A}(F_{V_0}, F_V).$$

We claim that

$$\psi_{W_0,W}\colon (E \otimes F)_{W_0} \to (E \otimes F)_W$$

belongs to $\mathscr{A}$, where $W_0 = W(U_0, V_0)$, $W = W(U, V)$. Identify $(E \otimes F)_W$ with $E_U \otimes F_V$ and $(E \otimes F)_{W_0}$ with $E_{U_0} \otimes F_{V_0}$ according to Lemma 4.3. The corresponding canonical surjections from $E \otimes F$ into $(E \otimes F)_W$ (resp. into $(E \otimes F)_{W_0}$) will then be identified with $\psi_U \otimes \psi_V$ (resp. $\psi_{U_0} \otimes \psi_{V_0}$). Because of the commutativity of the diagram



it follows that $\psi_{W_0,W} = \psi_{U_0,U} \otimes \psi_{V_0,V}$, which by assumption belongs to $\mathscr{A}$.

The converse follows by identifying E (resp. F) with a subspace of the tensorproduct. ∎

All the ideals in Proposition 3.6 are both $\otimes_\varepsilon$ and $\otimes_\pi$-stable. The proof of (i) is immediate, (ii) follows from an obvious modification of the proof of (iii) by Holub [8], Theorem 3.7, (iv) follows from [9], Theorems 2 and 3 (or [1], Theorems 4.5 and 4.9). Thus we get the following:

COROLLARY 4.5. *$E \otimes_\alpha F$ is strongly nuclear, resp. nuclear, resp. Schwartz, if and only if E and F are both of the respective type, where α is either ε or π.*

Remark. The proof of Theorem 4.4 does not change if instead of $\otimes_\alpha$-stability it is assumed that $\mathscr{A}$ is equivalent to $\mathscr{A}^J$ and $S \otimes T \in \mathscr{A}^J$ as soon as S and T belong to $\mathscr{A}$. This implies that the "ε-part" of Theorem 4.4 generalizes Theorem 4.1, as is seen by the following observation: Let E_i, F_i be normed spaces and $T_i \in \mathscr{L}(E_i, F_i)$, $i = 1, 2$. If $E_1 \otimes_\varepsilon E_2$ (resp. $F_1 \otimes_\varepsilon F_2$) is regarded as a subspace of $\mathscr{L}(E_1', E_2)$ (resp. $\mathscr{L}(F_1', F_2)$), then $T_1 \otimes T_2$ equals the restriction of $\mathrm{Hom}(T_1', T_2)$ to $E_1 \otimes E_2$ (see [1], p. 24). Thus, if $\mathscr{A}$ satisfies the requirements of Theorem 4.1, then it also satisfies the (modified) requirements of Theorem 4.4.

Finally, we shall apply Theorem 3.3 to locally convex spaces whose strong duals are of type $\mathscr{A}$. Such spaces will be said to be of *type co-$\mathscr{A}$*. Our result will depend on the "problème des topologies" of A. Grothendieck ([5], Chapitre I, p. 33), i.e. whether every bounded subset of $E \otimes_\pi F$ is contained in some set of the form $\Gamma(A \otimes B) =$ balanced, convex hull of $\{x \otimes y \mid x \in A,\ y \in B\}$, where $A \subset E$ and $B \subset F$ are bounded. Grothendieck proves in [5], I, Proposition 5, that this condition is satisfied if E and F are (DF)-spaces (i.e. locally convex spaces with a countable fundamental system of bounded sets and with the property that every strongly bounded subset of the dual, which is a countable union of equicontinuous sets, is itself equicontinuous). We shall need the following lemma:

LEMMA 4.7. *The strong topology $\beta\big((E \otimes_\pi F)', E \otimes_\pi F\big)$ is finer than the topology induced by the inclusion*

$$(E \otimes_\pi F)' \to \mathscr{L}_b(E, F_b').$$

The question of equality of these topologies is equivalent to the "problème des topologies".

Proof. The dual of $E \otimes_\pi F$ can be indentified with the set of continuous bilinear forms on $E \times F$ by means of the formula

$$\langle x \otimes y,\ z' \rangle = f(x, y).$$

Denote by Tx the linear form: $y \to f(x, y)$ on F for $x \in E$. Thus we can assign to each $z' \in (E \otimes_\pi F)'$ an operator $T \in \mathscr{L}(E, F_b')$ such that

$$\langle x \otimes y,\ z' \rangle = \langle y, Tx \rangle.$$

Let $\mathscr{N}(A, B^0)$ be a neighbourhood of the origin of $\mathscr{L}_b(E, F_b')$, where $A \subset E$ and $B \subset F$ are bounded. Then its "intersection" with $(E \otimes_\pi F)'$ equals the set

$$\{z' \in (E \otimes_\pi F)' \mid T(A) \subset B^0\} = (A \otimes B)^0,$$

which is a neighbourhood of the origin in the strong topology of $(E \otimes_\pi F)'$, since $A \otimes B$ is bounded in $E \otimes_\pi F$ for bounded A and B. The rest of the statement is now obvious. ∎

Making use of the result of Grothendieck mentioned above that for (DF)-spaces the "problème des topologies" is settled in the affirmative, we get as an immediate consequence of Theorem 3.3 and Lemma 4.7 the following:

THEOREM 4.8. *Let E and F be locally convex spaces of type (DF) and let $\mathscr{A}$ be a Hom-stable ideal which is equivalent to its injective hull. Then $E\otimes_\pi F$ is of type co-$\mathscr{A}$ if and only if E and F are of type co-$\mathscr{A}$.*

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Two weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform

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Abstract. Necessary conditions are obtained on non-negative functions $U(x)$ and $V(x)$ so that $\int_{-\infty}^{\infty} |Tf(x)|^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx$, where $1 \leqslant p < \infty$, $Tf(x)$ denotes either the Hardy–Littlewood maximal function or the Hilbert transform of f and C is a constant independent of f. In the case $p = 1$, the necessary condition is also shown to be sufficient; in case $p > 1$ the necessary conditions are shown to be sufficient if various additional restrictions are placed on $U(x)$ and $V(x)$ or on $f(x)$.

1. Introduction. The first norm inequality of the form

$$\int_{-\infty}^{\infty} [f^*(x)]^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx, \tag{1.1}$$

where

$$f^*(x) = \sup_{y\neq x} \frac{1}{y-x} \int_x^y |f(t)|\,dt$$

is the Hardy–Littlewood maximal function of f and $1 < p < \infty$, was proved in [2] with $U(x) = V(x) = 1$. The first norm inequalities of the form

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx, \tag{1.2}$$

where

$$\tilde{f}(x) = \lim_{\varepsilon\to 0+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy$$

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