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On the integrability of a class of integral transforms

by

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Abstract. In [6] and [7], K. Soni and R. P. Soni proved L-integrability theorems and integrability theorems of Tauberian character for a class of integral transforms, where it includes the Hankel transform and so on. We prove some L^p-integrability $(1 < p < \infty)$ theorems for the class of integral transforms, and further give an answer of Boas's conjecture (see [2]).

1. Basic assumptions and definitions. Throughout this paper, the function $k(t)$ satisfies the following two assumptions.

(A1) $k(t)$ is real-valued, measurable and uniformly bounded in $0 \leqslant t < \infty$.

(A2)

$$k(t) = \begin{cases} k(0) + Bt^\beta + o(t^\beta) & \text{as } t \to +0 \text{ for } \beta > 0, \text{ where } B \neq 0, \\ k(0) + o(1) & \text{as } t \to +0 \text{ for } \beta = 0. \end{cases}$$

First suppose that the function $f(t)$ is real-valued in $0 < t < \infty$ and is of bounded variation in $T \leqslant t < \infty$ for every $T > 0$. In particular, if $f(+0)$ is not finite, then we define, for every measurable function $\eta(x)$,

$$\int_0^a \eta(x)\, df(x) = \lim_{\varepsilon \to +0} \int_\varepsilon^a \eta(x)\, df(x), \qquad a > 0,$$

whenever the limit exists and is finite. Now we define an integral transform as follows:

If $\int_0^1 |k(t) - k(0)|\, |df(t)| < \infty$, then the function $F(x)$ is defined by

$$F(x) = \int_0^\infty \{k(xt) - k(0)\}\, df(t), \qquad 0 < x < \infty,$$

and denotes the k-transform of $f(t)$.

Remark 1. In the definition of the k-transform, if $\beta > 0$, then the condition $\int_0^1 |k(t) - k(0)|\, |df(t)| < \infty$ can be replaced by

$$\int_0^1 t^\beta\, |df(t)| < \infty$$

from (A2).

K. Soni and R. P. Soni [6] defined the k-transform in a slightly different form than in our definition.

Throughout this paper, we put

$$\frac{1}{p}+\frac{1}{q}=1,\quad 1\leqslant p<\infty,\quad 1<q\leqslant\infty.$$

Also, the letter C, with or without a subscript, denotes a positive constant, not necessarily the same at each appearance.

2. Main results.

THEOREM 1. *Let $\beta>0$ and $1/p<\gamma<\beta+1/p$. Suppose that $f(t)$ is defined in $0<t<\infty$ and is of bounded variation in $T\leqslant t<\infty$ for every $T>0$, and that $\int_0^1 t^\beta |df(t)|<\infty$. If*

$$t^{\gamma-\beta-2/p}\int_0^t x^\beta |df(x)|\in L^p(0,\infty),$$

then $x^{-\gamma}F(x)\in L^p(0,\infty)$.

As a corollary of Theorem 1, we have the following theorem.

THEOREM 2. *Let $\beta\geqslant 0$ and $-1/q<\gamma<\beta+1/p$. Let $k(0)=0$ for $\beta>0$, and $k(0)\neq 0$ for $\beta=0$. Suppose that $k_1(t)$ is defined by*

$$k_1(t)=\int_0^t k(u)\,du,\quad 0\leqslant t<\infty, \tag{2.1}$$

and is uniformly bounded in $0\leqslant t<\infty$. Suppose that $f(t)$ is defined in $0<t<\infty$ and is of bounded variation in $T\leqslant t<\infty$ for every $T>0$, that $f(t)$ tends to zero as $t\to\infty$, and that $\int_0^1 t^{\beta+1}|df(t)|<\infty$. Then

$$\tilde{F}(x)=\int_0^{\to\infty} k(xt)f(t)\,dt \tag{2.2.}$$

converges for every $x>0$, and $x^{-\gamma}\tilde{F}(x)\in L^p(0,\infty)$ if

$$t^{\gamma-\beta-2/p}\int_0^t x^{\beta+1}|df(x)|\in L^p(0,\infty).$$

As a collorary of Theorem 2, we have a theorem as follows:

THEOREM 3. *Let $\beta\geqslant 0$ and $-1/q<\gamma<\beta+1/p$. Let $k(0)=0$ for $\beta>0$, and $k(0)\neq 0$ for $\beta=0$. Suppose that $k_1(t)$ is defined as in (2.1) and is uniformly bounded in $0\leqslant t<\infty$. Suppose that $g(t)$ decreases to zero in $0<t<\infty$, and that $t^\beta g(t)\in L(0,1)$. Then*

$$\tilde{G}(x)=\int_0^{\to\infty} k(xt)g(t)\,dt \tag{2.3}$$

converges for every $x>0$, and $x^{-\gamma}\tilde{G}(x)\in L^p(0,\infty)$ if $t^{\gamma+1-2/p}g(t)\in L^p(0,\infty)$.

As the inverse case to Theorem 1, we have the following theorem.

THEOREM 4. *Let $\beta>0$ and let $1/p<\gamma<\beta+1/p$. Suppose that $f(t)$ is monotone in $0<t<\infty$ and tends to a finite value as $t\to\infty$, and that $\int_0^1 t^\beta|df(t)|<\infty$. Suppose that there exists a function $\omega(x)$ such that*

(i) *$\omega(x)\in L(0,1)$, $x^\beta\omega(x)\in L(1,\infty)$ and $\int_0^\infty x^\beta\omega(x)\,dx\neq 0$,*

(ii) *$k^*(y)-k^*(0)$ has no change of sign in $0<y<\infty$, where*

$$k^*(y)=\int_0^\infty \omega(x)k(xy)\,dx. \tag{2.4}$$

If $x^{-\gamma}F(x)\in L^p(0,\infty)$, then $t^{\gamma-\beta-2/p}\int_0^t x^\beta df(x)\in L^p(0,\infty)$.

As a corollary of Theorem 4, we have the following theorem inverse to Theorem 3.

THEOREM 5. *Let $\beta\geqslant 0$ and $-1/q<\gamma<\beta+1/p$. Let $k(0)=0$ for $\beta>0$, and $k(0)\neq 0$ for $\beta=0$. Suppose that $k_1(t)$ is defined as in (2.1) and is uniformly bounded in $0\leqslant t<\infty$. Suppose that $g(t)$ decreases to zero in $0<t<\infty$, and that $t^\beta g(t)\in L(0,1)$. Then $\tilde{G}(x)$ is finite for every $x>0$, where it is of the form (2.3).*

Moreover, if there exists a function $\omega_1(x)$ such that

(i)′ *$\omega_1(x)\in L(0,1)$, $x^{\beta+1}\omega_1(x)\in L(1,\infty)$ and $\int_0^\infty x^{\beta+1}\omega_1(x)\,dx\neq 0$,*

(ii)′ *$k_1^*(y)$ has no change of sign in $0<y<\infty$, where*

$$k_1^*(y)=\int_0^\infty \omega_1(x)k_1(xy)\,dx,$$

and if $x^{-\gamma}\tilde{G}(x)\in L^p(0,\infty)$, then $t^{\gamma+1-2/p}g(t)\in L^p(0,\infty)$.

Remark 2. In Theorems 1 and 4, we attend to the case $p=1$. K. Soni and R. P. Soni [6], Lemma 1, gave a result as follows: *Let $\varphi(u)$ and $\psi(u)$ be two monotone functions ($\varphi\uparrow$, $\psi\downarrow$) defined in $0<u<\infty$, such that $\varphi(+0)$*

$= \psi(+\infty) = 0$. *If one of the integrals* $\int_0^\infty \varphi(u)\,d\psi(u)$ *or* $\int_0^\infty \psi(u)\,d\varphi(u)$ *is finite, then the other integral is finite and*

$$\lim_{u\to+0} \varphi(u)\psi(u) = 0, \quad \lim_{u\to\infty} \varphi(u)\psi(u) = 0,$$

$$\int_0^\infty \varphi(u)\,d\psi(u) = -\int_0^\infty \psi(u)\,d\varphi(u).$$

Let $\varphi(t) = \int_0^t x^\beta |df(x)|$ and $\psi(t) = t^{\gamma-\beta-1}$(notice $\gamma-\beta-1 < 0$). Then

$$(1+\beta-\gamma)\int_0^\infty t^{\gamma-\beta-2}\,dt\int_0^t x^\beta |df(x)| = \int_0^\infty t^{\gamma-\beta-1}t^\beta |df(t)| = \int_0^\infty t^{\gamma-1}|df(t)|.$$

Thus we see that Theorems 1 and 4 for the case $p = 1$ were proved by K. Soni and R. P. Soni [6], Theorems 1 and 2, respectively.

In Section 6, we shall show that Theorems 3 and 5 include an answer of Boas's conjecture.

3. Proofs of Theorems 1, 2 and 3. In order to prove Theorem 1, we need a lemma as follows:

LEMMA 1. *Let* $s > 0$ *and* $1 < m < sp+1$. *Suppose that* $\lambda(u)$ *increases in* $0 \leqslant u < \infty$, *and that* $\lambda(+0) = 0$. *Then* $u^{-m/p}\lambda(u) \in L^p(0, \infty)$ *if and only if*

$$u^{-m/p+s}\int_u^\infty x^{-s}\,d\lambda(x) \in L^p(0, \infty).$$

Proof. It is sufficient for us to prove that $u^{-s}\lambda(u)\to 0$ as $u\to\infty$ if $u^{-m/p}\lambda(u)\in L^p(0, \infty)$. Since $1 < m < sp+1$, we get $0 > (-m+1)p^{-1} > -s$. Since $\lambda(u)$ is non-negative and increasing in $0 < u < \infty$, and since $u^{-m/p}\lambda(u)\in L^p(0, \infty)$,

$$u^{-s}\lambda(u) \leqslant u^{(-m+1)/p}\lambda(u) \quad (u \geqslant 1)$$
$$\leqslant \left(\frac{1}{m-1}\int_u^\infty u^{-m}\lambda(u)^p\,du\right)^{1/p}$$
$$\to 0 \quad \text{as } u\to\infty.$$

The rest of the proof is indeed similar to Askey and Boas [1], Lemma 1, and so we omit it. Thus Lemma 1 is proved.

Proof of Theorem 1. When we put

$$\lambda(t) = \int_0^t x^\beta |df(x)|, \quad s = \beta, \quad m = -\gamma p+\beta p+2$$

in Lemma 1, we have

$$\int_0^\infty u^{\gamma p-2}\Big(\int_u^\infty |df(t)|\Big)^p du < \infty \quad \text{because} \quad u^{\gamma-\beta-2/p}\int_0^u t^\beta |df(t)| \in L^p(0, \infty).$$

Hence, by (A1) and (A2),

$$\int_0^\infty x^{-\gamma p}|F(x)|^p\,dx$$
$$\leqslant \int_0^\infty x^{-\gamma p}\Big(\int_0^\infty |k(xt)-k(0)|\,|df(t)|\Big)^p dx$$
$$\leqslant 2^{p-1}\Big\{\int_0^\infty x^{-\gamma p}\Big(\int_0^{1/x} |k(xt)-k(0)|\,|df(t)|\Big)^p dx +$$
$$+\int_0^\infty x^{-\gamma p}\Big(\int_{1/x}^\infty |k(xt)-k(0)|\,|df(t)|\Big)^p dx\Big\}$$
$$\leqslant C_1\int_0^\infty x^{-\gamma p+\beta p}\Big(\int_0^{1/x} t^\beta |df(t)|\Big)^p dx + C_2\int_0^\infty x^{-\gamma p}\Big(\int_{1/x}^\infty |df(t)|\Big)^p dx$$
$$= C_1\int_0^\infty u^{\gamma p-\beta p-2}\Big(\int_0^u t^\beta |df(t)|\Big)^p du + C_2\int_0^\infty u^{\gamma p-2}\Big(\int_u^\infty |df(t)|\Big)^p du < \infty.$$

Thus Theorem 1 is proved.

Proof of Theorem 2. Since $\int_0^1 t^{\beta+1}|df(t)| < \infty$, $\lim_{u\to+0}\int_0^u t^{\beta+1}|df(t)| = 0$ by the dominated convergence theorem. Since $\beta \geqslant 0$

$$\int_0^\delta t^{\beta+1}|df(t)| \geqslant \int_\varepsilon^\delta t^{\beta+1}|df(t)| \geqslant \varepsilon^{\beta+1}\int_\varepsilon^\delta |df(t)| \geqslant \varepsilon^{\beta+1}|f(\delta)-f(\varepsilon)|$$

for $0 < \varepsilon < \delta < 1$. When ε tends to zero and then δ tends to zero, we get

$$\varepsilon^{\beta+1}f(\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0.$$

Also, it is easily seen that $k_1(0) = 0$ and $k_1(t) = B_1 t^{\beta+1}+o(t^{\beta+1})$ as $t\to+0$, where $B_1 \neq 0$. Since $k_1(t)$ is uniformly bounded in $0 \leqslant t < \infty$, and since $f(t)$ tends to zero as $t\to\infty$ and $\int_0^1 t^{\beta+1}|df(t)| < \infty$, the integral transform

$$\int_0^\infty k_1(xt)\,df(t)$$

converges absolutely for every $x \geqslant 0$.

Thus, for $x > 0$,

$$\int_0^\infty k_1(xt)\,df(t) = \lim_{N\to\infty}\int_0^N k_1(xt)\,df(t) \tag{3.1}$$
$$= \lim_{N\to\infty}\Big\{[k_1(xt)f(t)]_0^N - x\int_0^N k(xt)f(t)\,dt\Big\}$$
$$= -x\int_0^{\to\infty} k(xt)f(t)\,dt = -x\tilde F(x).$$

Hence $\tilde F(x)$ is finite for $x > 0$. Since

$$t^{\gamma-\beta-2/p}\int_0^t x^{\beta+1}|df(t)| = t^{(\gamma+1)-(\beta+1)-2/p}\int_0^t t^{\beta+1}|df(t)|\in L^p(0,\infty),$$

we obtain, by Theorem 1,

$$x^{-\gamma}\tilde F(x) = x^{-(\gamma+1)}\int_0^\infty k_1(xt)\,df(t)\in L^p(0,\infty).$$

Thus Theorem 2 is proved.

Proof of Theorem 3. Since $g(t)$ decreases to zero in $0 < t < \infty$, it is of bounded variation in $T \leqslant t < \infty$ for each $T > 0$. We put

$$\varphi(u) = \begin{cases} g(u) & \text{for } 0 < u \leqslant t \\ 0 & \text{for } t < u \end{cases} \qquad (t > 0)$$

and $\psi(u) = u^{\beta+1}$ in Remark 2. Then, since $u^\beta g(u)\in L(0,t)$, the integral $\int_0^t u^{\beta+1}|dg(u)| = -\int_0^t u^{\beta+1}dg(u)$ is finite for every $t > 0$. Further we should notice that

$$\int_0^t u^\beta g(u)\,du = \frac{1}{\beta+1}\Big\{t^{\beta+1}g(t) - \int_0^t u^{\beta+1}dg(u)\Big\} \geqslant \frac{1}{\beta+1}\int_0^t u^{\beta+1}|dg(u)|. \tag{3.2}$$

Thus, using (3.1),

$$\int_0^\infty k_1(xt)\,dg(t) = -x\int_0^{\to\infty} k(xt)g(t)\,dt = -x\tilde G(x), \qquad x > 0, \tag{3.3}$$

and so $\tilde G(x)$ is finite for every $x > 0$. By Hardy, Littlewood and Pólya [5], Theorem 330, we get

$$\int_0^\infty t^{\gamma p-\beta p-2}\Big(\int_0^t x^\beta g(x)\,dx\Big)^p dt \leqslant C\int_0^\infty t^{\gamma p-2+p}g(t)^p\,dt < \infty.$$

Hence, from (3.2),

$$\int_0^\infty t^{\gamma p-\beta p-2}\Big(\int_0^t x^{\beta+1}|dg(x)|\Big)^p dt < \infty.$$

Now, by Theorem 2, $x^{-\gamma}\tilde G(x)\in L^p(0,\infty)$. Thus Theorem 3 is proved.

4. Proofs of Theorems 4 and 5. In order to prove Theorem 4, we need the following two lemmas.

LEMMA 2. *Suppose that $f(t)$ is non-negative and monotone in $0 < t \leqslant \delta$ for some $\delta > 0$, and that it is of bounded variation in $\delta \leqslant t < \infty$. If $\int_0^1 t^\beta|df(t)| < \infty$ for $\beta > 0$, then*

$$\int_0^1 |k(xt)-k(0)|\,|df(t)| = \begin{cases} O(x^\beta) & \text{as } x\to+0, \\ o(x^\beta) & \text{as } x\to\infty; \end{cases}$$

$$\int_1^\infty |k(xt)-k(0)|\,|df(t)| = \begin{cases} O(1) & \text{as } x\to+0, \\ O(1) & \text{as } x\to\infty. \end{cases}$$

When $f(t)$ increases in $0 < t \leqslant \delta$, Lemma 2 is trivial. Also, when $f(t)$ decreases in $0 < t \leqslant \delta$, it is due to K. Soni and R. P. Soni [6], Lemma 3.

LEMMA 3. *Let $\beta > 0$. Suppose that $\omega(x)$ satisfies condition* (i), *and that $k^*(y)$ is defined as in* (2.4). *Then*

(a) *$k^*(y)$ is uniformly bounded in $0 \leqslant y < \infty$,*

(b) *$k^*(y) - k^*(0) \sim Dy^\beta$ as $y\to+0$, where $D \neq 0$.*

Lemma 3 is due to K. Soni and R. P. Soni [6], Lemma 6 and p. 407.

Proof of Theorem 4. By assumption, $f(t)$ is of bounded variation in $T \leqslant t < \infty$ for every $T > 0$. Since $\omega(x)\in L(0,\infty)$, $k^*(y)$ is finite for every $0 \leqslant y < \infty$. Let, for $t > 0$,

$$\mu(t) = t^{-\gamma-1}\int_0^\infty \omega\Big(\frac{x}{t}\Big)F(x)\,dx = t^{-\gamma}\int_0^\infty \omega(u)\,du\int_0^\infty \{k(tuy)-k(0)\}\,df(y).$$

By Lemma 2, the repeated integral above converges absolutely. Interchanging the order of integration,

$$\mu(t) = t^{-\gamma}\int_0^\infty \{k^*(ty)-k^*(0)\}\,df(y). \tag{4.1}$$

By (i), we have $x^{\gamma-1/p}\omega(x)\in L(0,\infty)$, where $0 < \gamma-1/p < \beta$. Now, from assumption and a generalized form of Minkowski's inequality [8], p. 19, we get

$$\Big(\int_0^\infty |\mu(t)|^p\,dt\Big)^{1/p} \leqslant \Big\{\int_0^\infty t^{-(\gamma+1)p}\Big(\int_0^\infty \Big|\omega\Big(\frac{x}{t}\Big)F(x)\Big|\,dx\Big)^p dt\Big\}^{1/p}$$

$$= \Big\{\int_0^\infty \Big(\int_0^\infty |\omega(u)t^{-\gamma}F(tu)|\,du\Big)^p dt\Big\}^{1/p}$$

$$\leqslant \int_0^\infty \Big(\int_0^\infty |\omega(u)t^{-\gamma}F(tu)|^p\,dt\Big)^{1/p} du$$

$$= \int_0^\infty |\omega(u)|\,u^{\gamma}\Big(\int_0^\infty (tu)^{-\gamma p}|F(tu)|^p\,dt\Big)^{1/p} du$$

$$= \Big(\int_0^\infty |\omega(u)|\,u^{\gamma-1/p}\,du\Big)\Big(\int_0^\infty v^{-\gamma p}|F(v)|^p\,dv\Big)^{1/p} < \infty.$$

Hence $\mu(t)\in L^p(0,\infty)$. From (b) of Lemma 3, there exists a positive number $\theta<1$ such that

$$|k^*(y)-k^*(0)| \geqslant \tfrac{1}{2}|D|y^{\beta} \quad \text{for any } y,\ 0<y\leqslant\theta.$$

Now, since $f(t)$ is monotone, we have, by (4.1) and (ii),

$$\int_0^\infty |\mu(t)|^p\,dt = \int_0^\infty t^{-\gamma p}\Big(\int_0^\infty |k^*(ty)-k^*(0)|\,|df(y)|\Big)^p dt \tag{4.2}$$

$$\geqslant \int_0^\infty t^{-\gamma p}\Big(\int_0^{\theta/t} |k^*(ty)-k^*(0)|\,|df(y)|\Big)^p dt$$

$$\geqslant (\tfrac{1}{2}|D|)^p\int_0^\infty t^{-\gamma p+\beta p}\Big(\int_0^{\theta/t} y^{\beta}|df(y)|\Big)^p dt$$

$$= (\tfrac{1}{2}|D|)^p\,\theta^{-\gamma p+\beta p+1}\int_0^\infty u^{\gamma p-\beta p-2}\Big(\int_0^u y^{\beta}|df(y)|\Big)^p du$$

$$= C\int_0^\infty \Big|u^{\gamma-\beta-2/p}\int_0^u y^{\beta}\,df(y)\Big|^p du.$$

Thus Theorem 4 is proved.

Proof of Theorem 5. By assumption, $g(t)$ is of bounded variation in $T\leqslant t<\infty$ for every $T>0$, and $k_1(0)=k_1^*(0)=0$ and $k_1(t)=B_1t^{\beta+1}+o(t^{\beta+1})$ as $t\to+0$, where $B_1\neq 0$. Now, from (3.3),

$$\tilde{G}(x) = -x^{-1}\int_0^\infty k_1(xt)\,dg(t), \qquad x>0.$$

Since $x^{-\gamma}\tilde{G}(x) = -x^{-(\gamma+1)}\int_0^\infty k_1(xt)\,dg(t)\in L^p(0,\infty)$, we obtain, by Theorem 4,

$$t^{\gamma-\beta-2/p}\int_0^t x^{\beta+1}\,dg(x) = t^{(\gamma+1)-(\beta+1)-2/p}\int_0^t x^{\beta+1}\,dg(x)\in L^p(0,\infty).$$

Now, if we set

$$\lambda(t) = \int_0^t x^{\beta+1}|dg(x)|, \qquad m = -(\gamma+1)p+(\beta+1)p+2, \qquad s=\beta+1$$

in Lemma 1, then

$$\int_0^\infty t^{\gamma p+p-2}g(t)^p\,dt = \int_0^\infty t^{(\gamma+1)p-(\beta+1)p-2+(\beta+1)p}\Big(\int_t^\infty x^{-(\beta+1)}x^{\beta+1}|dg(x)|\Big)^p dt<\infty.$$

Thus Theorem 5 is proved.

5. Applications. In this section, we apply Theorems 1–5 to some well-known integral transforms.

1. *The Hankel transform.* Let

$$k(t) = t^{1/2}J_{\nu}(t), \qquad \nu\geqslant -\tfrac{1}{2}, \tag{5.1}$$

where $J_{\nu}(t)$ is the Bessel function of the first kind [3], p. 4(2). Then, in particular, we have

$$k(t) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\sin t & \text{for } \nu=\tfrac{1}{2},\\[2mm] \sqrt{\dfrac{2}{\pi}}\cos t & \text{for } \nu=-\tfrac{1}{2}.\end{cases} \tag{5.2}$$

K. Soni and R. P. Soni [7] pointed out the following three properties.

(H1) $k(t) = \dfrac{1}{2^{\nu}\Gamma(\nu+1)}t^{\nu+1/2}+o(t^{\nu+1/2})$ as $t\to+0$.

(H2) $k(t)$ and $k_1(t)$ are continuous and uniformly bounded in $0\leqslant t<\infty$.

(H3) If we put $\omega(x) = x^{\nu+1/2}e^{-x}$, then, for $0<y<\infty$,

$$k^*(y) = \int_0^\infty x^{\nu+1/2}e^{-x}(xy)^{1/2}J_{\nu}(xy)\,dx$$

$$= \pi^{-1/2}2^{\nu+1}\Gamma(\nu+\tfrac{3}{2})y^{\nu+1/2}(1+y^2)^{-\nu-3/2}>0, \qquad [4], \text{ p. } 29(4).$$

From (5.2), (H1), (H2) and (H3), we get three properties as follows:

(H4) $k(0)=0$ for $\nu>-\frac{1}{2}$, and $k(0)=\sqrt{2/\pi}$ for $\nu=-\frac{1}{2}$.

(H5) $k^*(y)-k^*(0) = k^*(y)>0$ for $\nu>-\frac{1}{2}$ and $0<y<\infty$.

(H6) Let $\omega_1(x)=\{-(\nu+1/2)x^{\nu-1/2}+x^{\nu+1/2}\}e^{-x}$. Since $\int_x^\infty \omega_1(u)\,du=\omega(x)$, we have, for $\beta=\nu+1/2$,

$$\int_0^\infty x^{\beta+1}\omega_1(x)\,dx=\left[x^{\nu+3/2}\left(-\omega(x)\right)\right]_0^\infty+(\nu+\tfrac{3}{2})\int_0^\infty x^{\nu+1/2}\omega(x)\,dx$$

$$=(\nu+\tfrac{3}{2})\int_0^\infty x^{2\nu+1}e^{-x}\,dx\neq 0$$

and

$$k_1^*(y)=\int_0^\infty \omega_1(x)\,dx\int_0^{yx}k(u)\,du=y\int_0^\infty \omega_1(x)\,dx\int_0^x k(yv)\,dv$$

$$=y\int_0^\infty \omega(v)k(yv)\,dv=yk^*(y)>0,\qquad 0<y<\infty.$$

We set $\beta=\nu+1/2$. Then, from (H1)–(H6), we may state Theorems 1–5 for the Hankel transform as follows:

(I) *Let $\nu>-1/2$ and $1/p<\gamma<\nu+1/2+1/p$. If $f(t)$ is defined in $0<t<\infty$ and is of bounded variation in $T\leqslant t<\infty$ for every $T>0$, if $\int_0^1 t^{\nu+1/2}|df(t)|<\infty$, and if*

$$t^{\gamma-\nu-1/2-2/p}\int_0^t x^{\nu+1/2}|df(x)|\in L^p(0,\infty),$$

then $x^{-\gamma}F(x)\in L^p(0,\infty)$ (Theorem 1). *Conversely, if $f(t)$ is monotone in $0<t<\infty$ and tends to a finite value as $t\to\infty$, if $\int_0^1 t^{\nu+1/2}|df(t)|<\infty$, and if $x^{-\gamma}F(x)\in L^p(0,\infty)$, then*

$$t^{\gamma-\nu-1/2-2/p}\int_0^t x^{\nu+1/2}df(x)\in L^p(0,\infty)$$

(Theorem 4).

(II) *Let $\nu\geqslant-1/2$ and $-1/q<\gamma<\nu+1/2+1/p$. Suppose that $f(t)$ is defined in $0<t<\infty$ and is of bounded variation in $T\leqslant t<\infty$ for every $T>0$, that $f(t)$ tends to zero as $t\to\infty$, and that $\int_0^1 t^{\nu+3/2}|df(t)|<\infty$. Then $\tilde F(x)$ is finite for every $x>0$, where it is of the form* (2.2), *and $x^{-\gamma}\tilde F(x)L^p(0,\infty)$ if*

$$t^{\gamma-\nu-1/2-2/p}\int_0^t x^{\nu+3/2}|df(x)|\in L^p(0,\infty)$$

(Theorem 2).

(III) *Let $\nu\geqslant-1/2$ and $-1/q<\gamma<\nu+1/2+1/p$. Suppose that $g(t)$ decreases to zero in $0<t<\infty$, and that $t^{\nu+1/2}g(t)\in L(0,1)$. Then $\tilde G(x)$ is finite for every $x>0$, where it is of the form* (2.3), *and $x^{-\gamma}\tilde G(x)\in L^p(0,\infty)$ if and only if $t^{\gamma+1-2/p}g(t)\in L^p(0,\infty)$* (Theorems 3 and 5).

2. *The Y-transform.* Let

$$k(t)=t^{1/2}Y_\nu(t),\qquad 0<|\nu|<\tfrac{1}{2},$$

where $Y_\nu(t)$ is the Bessel function of the second kind or Neuman's function and

$$Y_\nu(t)=(\sin\nu\pi)^{-1}\{J_\nu(t)\cos\nu\pi-J_{-\nu}(t)\},\qquad [3],\ \text{p. } 4(4).$$

Hence, from (H1),

(Y1) $k(t)=Bt^{1/2-|\nu|}+o(t^{1/2-|\nu|})$ as $t\to+0$, where

$$B=\begin{cases}-2^\nu\{(\sin\nu\pi)\Gamma(1-\nu)\}^{-1} & \text{for } 0<\nu<\tfrac{1}{2},\\ 2^{-\nu}(\cot\nu\pi)\{\Gamma(1+\nu)\}^{-1} & \text{for } -\tfrac{1}{2}<\nu<0.\end{cases}$$

K. Soni and R. P. Soni [7] pointed out a property as follows:

(Y2) $k(t)$ and $k_1(t)$ are continuous and uniformly bounded in $0\leqslant t<\infty$.

From Erdélyi [4], p. 105(1),

$$(5.3)\qquad \int_0^\infty x^{-1/2}e^{-x}(xy)^{1/2}Y_\nu(xy)\,dx=y^{1/2}(y^2+1)^{-1/2}(\sin\nu\pi)^{-1}\times$$

$$\times\left(y^\nu\{(y^2+1)^{1/2}+1\}^{-\nu}\cos\nu\pi-y^{-\nu}\{(y^2+1)^{1/2}+1\}^\nu\right),\ 0<y<\infty.$$

Hence, by (Y1) and (Y2), we have the following property.

(Y3) Let $\omega(x)=x^{-1/2}e^{-x}$. For $0<\nu<1/2$, $k^*(y)-k^*(0)=k^*(y)<0$ in $0<y<\infty$.

But, for $-1/2<\nu<0$, $k^*(y)-k^*(0)=k^*(y)$ has change of sign in $0<y<\infty$.

In the case of the Y-transform, it follows from (Y1)–(Y3) that Theorems 1–3 hold for $\beta=1/2-|\nu|$ and $0<|\nu|<1/2$, and that Theorem 4 holds for $\beta=1/2-\nu$ and $0<\nu<1/2$.

When we put $\omega_1(t)=-\dfrac{d}{dt}\omega(t)=(\tfrac{1}{2}x^{-3/2}+x^{-1/2})e^{-x}$, the function $\omega_1(t)$ does not exist in $L(0,1)$, and $k_1^*(y)=yk^*(y)$ in the same manner as (H6) but $k_1^*(y)$ is not bounded as $y\to\infty$ by (5.3). For the Y-transform, it is doubtful whether Theorem 5 holds.

3. *The K-transform.* Let

$$k(t)=t^{1/2}K_\nu(t),\qquad 0<\nu\leqslant\tfrac{1}{2},$$

where $K_\nu(t)$ is the modified Bessel function of the third kind and

$$K_\nu(t) = \frac{\pi}{2}(\sin\nu\pi)^{-1}\{I_{-\nu}(t)-I_\nu(t)\}, \quad I_\nu(t) = \sum_{n=0}^{\infty}\frac{(t/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

[3], p. 5 (13).

In particular, we have $k(t) = \sqrt{\frac{\pi}{2}}\, e^{-t}$ for $\nu = \frac{1}{2}$ (the integral transform is the Laplace transform). Now we get the following two properties.

(K1) $k(t) = \frac{2^{\nu-1}\pi}{(\sin\nu\pi)\Gamma(1-\nu)} t^{1/2-\nu} + o(t^{1/2-\nu})$ as $t\to+0$.

(K2) $k(0) = 0$ for $0<\nu<1/2$, and $k(0) = \sqrt{\frac{\pi}{2}} \neq 0$ for $\nu = 1/2$.

K. Soni and R. P. Soni [7] pointed out a property as follows:

(K3) $k(t)$ and $k_1(t)$ are non-negative, continuous and uniformly bounded in $0\leqslant t<\infty$.

Since $k(t)$ and $k_1(t)$ are non-negative in $0\leqslant t<\infty$ from (K3), the considerations of $\omega(x)$ and $\omega_1(x)$ are unnecessary. We see easily it from the calculations similar to (4.2). From (K1)–(K3), we may now sum up our results for the K-transform as follows.

Theorems 1 and 4 hold for $\beta = 1/2-\nu$ and $0<\nu<1/2$, but Theorem 4 need not consider $\omega(x)$. Theorems 2 and 3 hold for $\beta = 1/2-\nu$ and $0<\nu\leqslant 1/2$. Theorem 5 holds for $\beta = 1/2-\nu$ and $0<\nu\leqslant 1/2$ without the consideration of $\omega_1(x)$.

6. An answer of Boas's conjecture. For Fourier sine or cosine transforms, R. P. Boas [2] gave loosely a conjecture as follows:

(B1) *If h and H_0 are a pair of Fourier sine or cosine transforms, and if one of them is positive and decreasing in $0<t<\infty$, then $x^{-\gamma}H_0(x)\in L^p(0,\infty)$ if and only if $t^{\gamma+1-2/p}h(t)\in L^p(0,\infty)$ provided that $-1/q<\gamma<1/p$.*

Moreover, in [2], he proved a result for sine transform as follows:

(B2) *If $h(t)$ decreases to zero in $0<t<\infty$, if $t^{1/p}h(t)\in L^p(0,1)$, and if $H_s(x)$ is the sine transform of $h(t)$, then $x^{-\gamma}H_s(x)\in L^p(0,\infty)$ provided that $t^{\gamma+1-2/p}h(t)\in L^p(0,\infty)$, where $p>1$ and $-1/q<\gamma<2/p-1/q$.*

From (5.2) and 1(III) of Section 5, we obtain the following two results.

(III_s) *Let $-1/q<\gamma<1+1/p$. Suppose that $h(t)$ decreases to zero in $0<t<\infty$, and that $th(t)\in L(0,1)$. Then*

$$H_s(x) = \int_0^{\to\infty} h(t)\sin xt\,dt$$

converges for every $x>0$, and $x^{-\gamma}H_s(x)\in L^p(0,\infty)$ if and only if

$$t^{\gamma+1-2/p}h(t)\in L^p(0,\infty).$$

(III_c) *Let $-1/q<\gamma<1/p$. Suppose that $h(t)$ decreases to zero in $0<t<\infty$, and that $h(t)\in L(0,1)$. Then*

$$H_c(x) = \int_0^{\to\infty} h(t)\cos xt\,dt$$

converges for every $x>0$, and $x^{-\gamma}H_c(x)\in L^p(0,\infty)$ if and only if

$$t^{\gamma+1-2/p}h(t)\in L^p(0,\infty).$$

Now we see that (III_s) and (III_c) give an answer of (B1). Since

$$\int_0^1 th(t)\,dt \leqslant \Big(\int_0^1 th(t)^p\,dt\Big)^{1/p}\Big(\int_0^1 t\,dt\Big)^{1/q} = 2^{-1/q}\Big(\int_0^1 th(t)^p\,dt\Big)^{1/p} \quad (p>1)$$

by Hölder's inequality, it is clear that (III_s) includes (B2).

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