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Hull operators on a category of spaces of continuous functions on Hausdorff spaces*

by

HARRIET LORD (Philadelphia, U.S.A.)

Abstract. We define and study the notion of a hull operator on a category of spaces of continuous functions on Hausdorff spaces which we call **Funsp**. Both function space convex hull and function space affine hull are examples of hull operators. We show that the class of hull operators is in 1-1 correspondence with a subclass of all bicategory structures on **Funsp**. An application of the Freyd–Isbell theorem gives that a subcategory of **Funsp** is P_Q-reflective in **Funsp** if it is closed under the formation of products and I_Q-subobjects, where (I_Q, P_Q) is the bicategory structure corresponding to the hull operator Q. Furthermore, if f is $\mathscr{A}$-extendable, it is also $\mathscr{B}$-extendable, where $\mathscr{B}$ is the smallest P_Q-reflective subcategory of **Funsp** which contains all the objects in $\mathscr{A}$.

Introduction. In [4], Ky Fan generalized the notion of convexity to a function space setting. For X a Hausdorff space and H a linear space of continuous real-valued functions on X, Ky Fan defined an H-convex subset of X to be an intersection of sets of the form $\{x \in X : f(x) \geqslant a\}$, where $f \in H$. (The conditions on X and H in Ky Fan's definition were more general than this, but we do not need such generality for this paper.) The definition of H-convexity leads, in the usual way, to the idea of the H-convex hull of an arbitrary subset of X. (That is, the H-convex hull of $S \subseteq X$ is the intersection of all H-convex subsets of X that contain S.) The concept of the closed affine hull of a subset of a linear topological space can also be formulated in a more general function space setting.

In this paper, we introduce and study the notion of an abstract hull operator (Definition 2.1) on a certain category of spaces of continuous functions on Hausdorff spaces. In addition to numerous other examples, both function space convex hull and function space affine hull satisfy our axioms for a hull operator.

The axioms that we select for the definition of a hull operator are motivated, in part, by P. Hammer's paper [9] and H. Herrlich's paper

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[11], where each generalizes the concept of topological closure. In [11] Herrlich defines an abstract limit operator on the category of topological spaces. Our axiomatization of a hull operator is in the spirit of Herrlich's notion of a limit operator and in fact, certain limit operators induce hull operators.

Our approach to the study of hull operators is, for the most part, category theoretic. Categories of spaces of continuous functions on Hausdorff spaces have been studied by Grossman (see [6]).

For the basic results and terminology of category theory we refer the reader to [5], [10], [16], [18], and [19].

The category Funsp. We will denote by **Funsp** the category whose class of objects is all pairs (X, H), where X is a topological Hausdorff space, and H is a linear subspace of $BC(X)$ (the continuous, bounded, real-valued functions on X), such that H contains the constant functions and separates the points of X, and where a morphism $f\colon (X, H)\to(Y, K)$ in **Funsp** is a continuous function from X into Y with the property that $Kf \subseteq H$, where $Kf = \{k\circ f\colon k\in K\}$. This type of category was first considered by Grossman (see [6]).

DEFINITION 1.1. Suppose $(X, H)\in$ ob **Funsp** and S is a subset of X. The *H-affine hull* of S is the set $\{x\in X\colon$ for all $h\in H$, $h(S) = 0$ implies $h(x) = 0\}$. We denote the H-affine hull of S by H-aff S. S is said to be *H-affine* if it is equal to its H-affine hull. Since the functions in H are continuous, it is clear that H-aff S is closed in X.

If X is a subset of a locally convex linear topological Hausdorff space E, and H is the restriction of the continuous affine functions on E to X, then for $S \subseteq X$ the H-aff S coincides with the closed affine hull (in the geometric sense) of S in E intersected with X. The H-affine hull in the geometric setting has been studied by many authors. See Ellis [3] and the references there.

Proposition 1.3 below is due to Grossman ([6] and [8]). The characterization of an epimorphism in terms of affine hull is basic to our development of the notion of a hull operator on **Funsp**.

Since $(X, H)\in$ ob **Funsp** implies H separates points, the following lemma is immediate.

LEMMA 1.2. *Let $\alpha, \beta\colon (X_1, H_1)\to(X_2, H_2)$ be morphisms in* **Funsp**. *If α and β agree on $S \subseteq X_1$, then α and β agree on H_1-aff S.*

The following theorem characterizes monomorphisms, epimorphisms, and isomorphisms in **Funsp**.

PROPOSITION 1.3 (Grossman [6], p. 8, and [8]). *Let $\alpha\colon (X_1, H_1)\to(X_2, H_2)$ be a morphism in* **Funsp**. *Then:*

1) *α is a monomorphism in* **Funsp** *if and only if α is a 1-1 function from X_1 into X_2.*

2) *α is an epimorphism in* **Funsp** *if and only if H_2-aff $\alpha(X_1) = X_2$.*

3) *α is an isomorphism in* **Funsp** *if and only if α is a homeomorphism from X_1 onto X_2 and $H_2\alpha = H_1$.*

Proof. 1) and 3) are straightforward. By Lemma 1.2, it is enough to show that if $\alpha\colon (X_1, H_1)\to(X_2, H_2)$ is an epimorphism, then H_2-aff $\alpha(X_1) = X_2$. Suppose there is an $x\in X_2$ with $x\notin H_2$-aff $\alpha(X_1)$. Let $h\in H_2$ be such that $h(\alpha(X_1)) = 0$ and $h(x)\neq 0$. Consider the morphism $h\colon (X_2, H_2)\to([a, b], A[a, b])$, where $a = \inf h(X_2)$, $b = \sup h(X_2)$, and $A[a, b]$ denotes the continuous affine functions on $[a, b]$. Then $h\alpha = g\alpha$, where $g = 0$. Thus, α is not an epimorphism. ■

Remark. Grossman showed in [8] that Theorem 2.1 and consequently Theorem 2.4 in [6] are not valid. He pointed out that in the proof of Theorem 2.1 it is incorrectly stated that if α is an epimorphism in the category **Compconv**, then α is an onto map. He showed that the above characterization of an epimorphism in **Funsp** is the correct one for the subcategory of objects whose underlying space is compact. As was indicated in [8], the same arguments show that in the category **Compconv** a morphism $\alpha\colon K_1\to K_2$ is an epimorphism if and only if $A(K_2)$-aff $\alpha(K_1) = K_2$, where $A(K_2)$ denotes all continuous affine functions on K_2. For example, $i\colon [0, 1]\to[0, 2]$, where i is the inclusion map, is an epimorphism in **Compconv**.

PROPOSITION 1.4. **Funsp** *is well-powered and co-well-powered.*

Proof. Fix $(X, H)\in$ ob **Funsp**, and let $f\colon (X, H)\to(Y, B)$ be an epimorphism. Define $p\colon B\to Bf$ by $p(b) = bf$. Since B-aff $f(X) = Y$, p is 1-1. Hence $\operatorname{card} B^{\#} = \operatorname{card}(Bf)^{\#} \leqslant \operatorname{card} H^{\#}$ (where $B^{\#}$, $(Bf)^{\#}$, $H^{\#}$ are the vector spaces duals of B, (Bf), H). Define $G\colon Y\to B^{\#}$ by $G_y\cdot b = b(y)$. Since B separates the points of Y, G is 1-1. Consequently $\operatorname{card} Y \leqslant \operatorname{card} H^{\#}$. Thus, **Funsp** is co-well-powered.

Let $(X, H)\in$ ob **Funsp**. A representative set of monomorphisms is $\{i\colon (Y, K)\to(X, H)\}$, where $Y \subseteq X$, the topology on Y is stronger than the relative topology on Y, i is the inclusion, and $H|Y \subseteq K \subseteq BC(Y)$. So **Funsp** is also well-powered.

If $P \subseteq E$ (E is the class of all epimorphisms in **Funsp**), then **Funsp** is P-co-well-powered. Similarly, if $I \subseteq M$ (M is the class of all monomorphisms in **Funsp**), then **Funsp** is I-well-powered. ■

The category **Funsp** has products, coproducts, equalizers, and coequalizers. The next proposition gives the construction of products, equalizers, and intersections.

PROPOSITION 1.5 (Grossman [6], Section 3).

1. *Suppose $\{(X_i, H_i)\}$ is a family in* ob **Funsp**. *Let $X = \prod X_i$ be the cartesian product of the spaces in the family $\{X_i\}$. Let H be the subspace of $BC(X)$ spanned by $\bigcup\{h\varrho_i\colon h\in H_i\}$, where ϱ_i is the projection of X onto X_i.*

Then $\{\varrho_i: (X, H)\to(X_i, H_i)\}$ *is a product of the family* $\{(X_i, H_i)\}$ *in* **Funsp**.

2. *Suppose* $\alpha, \beta: (X, H)\to(Y, K)$ *are morphisms in* **Funsp**. *Let* $E = \{x\in X: \alpha(x) = \beta(x)\}$. *Then an equalizer for* α *and* β *is* $i: (E, H|E)\to(X, H)$, *where* i *is the inclusion map.*

3. *If* $\{i_j: (S_j, H|S_j)\to(X, H)\}$ *is a family of subobjects of* (X, H) *such that* i_j *is the inclusion map for each* j, *then the categorical intersection of the family is* $i: (Y, H|Y)\to(X, H)$, *where* $Y = \bigcap S_j$ *and* i *is the inclusion map.*

COROLLARY 1.6. **Funsp** *is complete and co-complete.*

Proof. This is immediate from Proposition 2.9, and its dual, in Mitchell [16].

DEFINITION 1.7. The morphism $\alpha: (X, H)\to(Y, K)$ is an *isomorphism into* if and only if $\alpha'(X, H)\to\big(\alpha(X), K|\alpha(X)\big)$ defined by $\alpha'(x) = \alpha(x)$ for all x is an isomorphism.

DEFINITION 1.8. (Isbell [14]). The morphism α is an *extremal monomorphism* if and only if α is a monomorphism, and whenever $\alpha = \gamma\beta$ and β is an epimorphism, then β is an isomorphism.

PROPOSITION 1.9. *If* $\alpha: (X, H)\to(Y, K)$ *is a morphism in* **Funsp**, *then the following three conditions are equivalent:*

1. α *is an isomorphism into and* K-aff $\alpha(X) = \alpha(X)$.
2. α *is an equalizer.*
3. α *is an extremal monomorphism.*

Proof. Assume α satisfies condition 1. Let (Z, J) be the coproduct of (Y, K) with (Y, K) and μ_i be the injections. Define an equivalence relation on Z by $x\sim y$ if $\mu_1^{-1}(x) = \mu_2^{-1}(y)\in\alpha(X)$ or $\mu_1^{-1}(y) = \mu_2^{-1}(x)\in\alpha(X)$. Let $J' = \{j\in J: x\sim y$ implies $j(x) = j(y)\}$, and $q: (Z, J)\to(Z/\sim, J')$ be the quotient map. Then α is the equalizer of $q\mu_1$ and $q\mu_2$.

That 2 implies 3 is proven in Satz 7.13 in Herrlich [10]. Assume α satisfies condition 3. Let $\beta: (X, H)\to\big(K\text{-aff }\alpha(X), K|K\text{-aff }\alpha(X)\big)$ be defined by $\beta(x) = \alpha(x)$, and $\gamma: \big(K\text{-aff }\alpha(X), K|K\text{-aff }\alpha(X)\big)\to(Y, K)$ be defined by $\gamma(X) = X$. Then $\alpha = \gamma\beta$ and β is an epimorphism. Since α is extremal, β is an isomorphism. ∎

Hull operators on Funsp. The definition of limit operators which appears in Herrlich [11] provides some of the motivation for our definition of a *hull operator*. Also see Hammer [9], pp. 305–316.

DEFINITION 2.1. A *hull operator* on the category **Funsp** is an operator Q that assigns to every pair $\big(S, (X, H)\big)$, where $(X, H)\in$ ob **Funsp** and $S\subseteq X$, a subset of X called the $H-Q$ *hull of* S, denoted by $H-Q$ hull S, which satisfies the following conditions:

1. If $(X, H)\in$ ob **Funsp** and $S\subseteq X$, then $S\subseteq H-Q$ hull $S\subseteq H$-aff S.
2. If $(X, H)\in$ ob **Funsp** and $A, B\subseteq X$, then $(H-Q\text{hull}A)\cup(H-Q$ hull $B)\subseteq H-Q$ hull $(A\cup B)$.
3. If $(X, H)\in$ ob **Funsp**, $S\subseteq Y\subseteq X$, and $Y = H-Q$ hull Y, then $(H-Q$ hull $S)\cap Y = H|Y-Q$ hull S.
4. If $\alpha: (X, H)\to(Y, K)$ is a morphism and $S\subseteq X$, then $\alpha(H-Q$ hull $S)\subseteq K-Q$ hull $\alpha(S)$.
5. If $(X, H)\in$ ob **Funsp** and $S\subseteq X$, then $H-Q$ hull $(H-Q$ hull $S) = H-Q$ hull S.

The following proposition, whose proof is straightforward, gives an equivalent characterization of hull operators.

PROPOSITION 2.2. Q *is a hull operator on* **Funsp** *if and only if* Q *satisfies the following conditions:*

1. *If* $(X, H)\in$ ob **Funsp** *and* $S\subseteq X$, *then* $S\subseteq H-Q$ hull $S\subseteq H$-aff S.
2. *If* $(X, H)\in$ ob **Funsp** *and* $A, B\subseteq X$, *then* $A\subseteq B$ *implies* $H-Q$ hull $A\subseteq H-Q$ hull B.
3. *If* $(X, H)\in$ ob **Funsp** *and* $S\subseteq Y\subseteq X$ *such that* $Y = H-Q$ hull Y, *then* $(H-Q$ hull $S)\cap Y = H|Y-Q$ hull S.
4. *If* $(X, H)\in$ ob **Funsp** *and* $S\subseteq Y\subseteq X$, *then* $H|Y-Q$ hull $S\subseteq(H-Q$ hull $S)\cap Y$.
5. *If* $\alpha: (X, H)\to(Y, K)$ *is a morphism in* **Funsp** *with* $\alpha(X) = Y$ *and* $S\subseteq X$ *is such that* $H-Q$ hull $S = X$, *then* $K-Q$ hull $\alpha(S) = Y$.
6. *If* $(X, H)\in$ ob **Funsp** *and* $S\subseteq X$, *then* $H-Q$ hull $(H-Q$ hull $S) = H-Q$ hull S.

The verifications that the following examples are in fact hull operators are either obvious or straightforward.

EXAMPLE 1. We call the hull operator defined for all $(X, H)\in$ ob **Funsp** and $S\subseteq X$ by $H-Q$ hull $S = S$, the *trivial hull operator*.

EXAMPLE 2. The *closure operator* on **Top** induces a hull operator on **Funsp**, where $H-Q$ hull $S = \mathrm{Cl}\,S$ (the closure of S in X), for all $(X, H)\in$ ob **Funsp** and $S\subseteq X$. In fact, any idempotent limit operator l on **Top** (in the sense of Herrlich [11]) that satisfies the additional property that if $X\in$**Top**, $S\subseteq Y\subseteq X$, and $Y = l_X Y$, then $(l_X S)\cap Y\subseteq l_Y S$, induces a hull operator on **Funsp** via the forgetful functor F: **Funsp**$\to$**Top**, where $F(X, H) = X$.

EXAMPLE 3. Let $(X, H)\in$ ob **Funsp** and $S\subseteq X$. The *H-convex hull of* S, denoted by H-conv$\,S$, is defined to be $\{x\in X$: for all $h\in H$, $h(x)\leqslant\sup h(S)\}$. This notion of function space convex hull has been used by many authors (see e.g., [4], [17]). The fact that this definition is consistent with the discussion in the introduction appears in Ky Fan [4].

If we define $H-Q$ hull $S = H$-conv S for all $(X, H) \in$ ob **Funsp** and $S \subseteq X$, then Q is a hull operator.

EXAMPLE 4. *Affine hull* is a hull operator. It has been studied in the geometric setting by many authors. See Ellis [3] and the references there.

EXAMPLE 5. The following proposition will provide a class of hull operators.

PROPOSITION 2.3. *Let* T: ob **Funsp** $\to$ ob **Funsp** *be such that*:

1. *For all* $(X, H) \in$ ob **Funsp**, $T(X, H) = (X, H')$, *where* $H \subseteq H'$.
2. *If* $\alpha: (X, H) \to (Y, K)$ *is a morphism in* **Funsp**, *then* $\alpha': T(X, H') \to (T(Y, K')$ *is a morphism in* **Funsp**, *where* $\alpha'(x) = \alpha(x)$ *for all* $x \in X$.

Then T induces a hull operator Q_T on **Funsp**, *where if* $(X, H) \in$ ob **Funsp** *and* $S \subseteq X$, *then* $H-Q_T$ hull $S = H'$-aff S, *where* $(X, H') = T(X, H)$.

The proof of this proposition follows immediately from Example 4. Examples of such a T can be obtained as follows:

Let $\mathscr{C}$ be a class of Hausdorff spaces that are preserved under continuous functions; i.e. $f: X \to Y$ continuous and $X \in \mathscr{C}$, implies $f(X) \in \mathscr{C}$. Let $\mathscr{C} \cap X$ denote the collection of subspaces of X that are in $\mathscr{C}$. Equip $BC(X)$ with the topology of uniform convergence on the sets in $\mathscr{C} \cap X$.

Each such $\mathscr{C}$ induces a T as above defined by $T_{\mathscr{C}}: (X, H) = (X, H_{\mathscr{C}})$, where $H_{\mathscr{C}}$ denote the closure of H in $BC(X)$ with respect to the topology of uniform convergence on the sets in $\mathscr{C} \cap X$.

We can choose for $\mathscr{C}$:

1. The class of all Hausdorff spaces (this gives $BC(X)$ the topology of *uniform convergence* on X).
2. The class of all *separable Hausdorff* spaces.
3. The class of all *Lindelöf Hausdorff* spaces.
4. The class of all *compact Hausdorff* spaces.
5. The class of all *connected Hausdorff* spaces.
6. The class of all *countably compact Hausdorff* spaces.
7. The class of all *sequentially compact Hausdorff* spaces.
8. The class of all *singleton* spaces (this gives $BC(X)$ the topology of *pointwise convergence* on X).

DEFINITION 2.4. Let Q be a hull operator on **Funsp**, $(X, H) \in$ ob **Funsp** and $S \subseteq X$. S is a $H-Q$ *subset* of X if and only if $S = H-Q$ hull S.

It is easy to prove that the intersection of a family of $H-Q$ subsets of X is also an $H-Q$ subset of X.

Consequently, if $S \subseteq X$, then $H-Q$ hull S is the intersection of all $H-Q$ subsets of X that contain S.

We show next how each hull operator Q induces a bicategory structure, (I_Q, P_Q), on **Funsp** (see, e.g. Kennison [15]). Define (I_Q, P_Q) as follows:

$I_Q = \{\alpha: (X, H) \to (Y, K) \mid \alpha$ is an isomorphism into and $K-Q$ hull $\alpha(X) = \alpha(X)\}$;

$P_Q = \{\alpha: (X, H) \to (Y, K) \mid \alpha$ is a morphism and $K-Q$ hull $\alpha(X) = Y\}$.

The I_Q-subobjects of (X, H), up to isomorphism, are the objects $(S, H|S)$ such that S is an $H-Q$ subset of X.

We note that if we choose Q to be the affine hull operator on **Funsp**, then P_Q coincides with all epimorphisms (Proposition 1.3) and I_Q coincides with all extremal monomorphisms (Proposition 1.9).

LEMMA 2.5. *If α, β are in P_Q, then $\beta\alpha$ is in P_Q whenever $\beta\alpha$ is defined.*

Proof. Suppose $\alpha: (X_1, H_1) \to (X_2, H_2)$ and $\beta: (X_2, H_2) \to (X_3, H_3)$ are in P_Q. Let $Y = H_3-Q$ hull $\beta\alpha(X_1)$. By condition 5 of Proposition 2.2, $H_3|\beta(X_2)-Q$ hull $\beta\alpha(X_1) = \beta(X_2)$. By condition 4, $\beta(X_2) \subseteq Y$. By 1 and 2, H_3-Q hull $\beta(X_2) \subseteq H_3-Q$ hull $Y = Y$. Therefore, $X_3 = Y$ and $\beta\alpha \in P_Q$. ∎

LEMMA 2.6. *Let Q be a hull operator on* **Funsp**. *If* $(X_2, H_2) \in$ ob **Funsp** *and $S \subseteq X_1 \subseteq X_2$, with H_2-Q hull $X_1 = X_1$, and $H_2|X_1-Q$ hull $S = S$, then S is an $H-Q$ subset of X.*

Proof. Let $B = H_2-Q$ hull S. $B = (H_2-Q$ hull $S) \cap B = H_2|B-Q$ hull $S = (H_2|X_1-Q$ hull $S) \cap B = S \cap B = S$. ∎

THEOREM 2.7. *If Q is a hull operator on* **Funsp**, *then (I_Q, P_Q) is a bicategory structure on* **Funsp**.

Proof. $(I_Q \cap P_Q)$ clearly contains all isomorphisms. By Lemma 2.5, P_Q is closed under composition.

Let $\alpha, \beta \in I_Q$, with $\alpha: (X_1, H_1) \to (X_2, H_2)$ and $\beta: (X_2, H_2) \to (X_3, H_3)$ $\beta\alpha(X_1) = H_3|(X_2-Q$ hull $\beta\alpha(X_1))$. By Lemma 2.6, H_3-Q hull $\alpha(X_1) = \beta\alpha(X_1)$. So I_Q is closed under composition.

Suppose $\alpha: (X_1, H_1) \to (X_2, H_2)$ is a morphism in **Funsp**. Let $X_3 = H_2-Q$ hull $\alpha(X_1)$, and $H_3 = H_2|X_3$. Define $\alpha_0: (X_1, H_1) \to (X_3, H_3)$ by $\alpha_0(x) = \alpha(x)$, and $\alpha_1: (X_3, H_3) \to (X_2, H_2)$ by $\alpha_1(x) = x$. Then $\alpha = \alpha_1\alpha_0$, where $\alpha_1 \in I_Q$ and $\alpha_0 \in P_Q$. If $\alpha = \beta\gamma$, $\beta \in I_Q$ and $\gamma \in P_Q$, then we let $e = \beta^{-1}|(X_3, H_3)$. So $e\alpha_0 = \gamma$ and $\beta e = \alpha_1$, so the factorization of α is unique up to isomorphism.

It is clear that P_Q is contained in the class of all epimorphisms and I_Q is contained in the class of all monomorphisms. ∎

The following corollary is an immediate consequence of Theorem 2.7 and the Freyd–Isbell theorem (see Isbell [14], p. 1276, Kennison [15], p. 356, and Herrlich [10], p. 96).

COROLLARY 2.8. *Let $\mathscr{A}$ be a full, replete subcategory of* **Funsp** *and Q a hull operator on* **Funsp**. *Then $\mathscr{A}$ is a P_Q-reflective in* **Funsp** *if and only* *if $\mathscr{A}$ is closed under the formation of products and I_Q-subobjects; that is, if and only if:*

1. *If $\{(X_i, H_i)\}$ is a family of objects in $\mathscr{A}$, then $\Pi(X_i, H_i)$ is in $\mathscr{A}$.*
2. *If $(X, H) \in \operatorname{ob} \mathscr{A}$ and E is an $H-Q$ subset of X, then $(E, H|E) \in \operatorname{ob} \mathscr{A}$.*

If $(X, H) \in \operatorname{ob}$ **Funsp**, then a P_Q-reflection can be realized as follows: Let $\{a_a | a_a: (X, H) \to (Y_a, H_a), a_a \in P, (Y_a, H_a) \in \operatorname{ob} \mathscr{A}\}$ be a representative set of all morphisms in P_Q from (X, H) to objects in $\mathscr{A}$ and define $a: (X, H) \to \Pi(Y_a, H_a)$ by $p_a = a_a$ for all a, $-p_a$ is the ath projection map. Let $(Y, K) = \Pi(Y_a, H_a)$. Then $a_0: (X, H) \to (Z, K|Z)$ is a P_Q-reflection of (X, H) in $\mathscr{A}$, where $Z = K-Q$ hull $a(X)$ and $a_0(x) = (x)$ for all $x \in X$.

For example, let Q be a hull operator on **Funsp** and let $\mathscr{A}_Q$ be the full subcategory consisting of all I_Q-subobjects of products of $([0,1], A[0,1])$ with itself, where $A[0,1]$ is the space of all real affine functions on $[0,1]$. $\mathscr{A}_Q$ is P_Q-reflective in **Funsp** and is called the *P_Q-reflective hull* of $([0,1], A[0,1])$. This example will be studied further in Section 4.

Characterizations of hull operators. In the previous section, we saw that each hull operator induces a bicategory structure on **Funsp**. In this section we prove that there is a 1-1 correspondence between certain bicategory structures and all hull operators on **Funsp**. Each hull operator Q assigns to each object (X, H) in **Funsp** a collection of subsets of X; namely, all $H-Q$ subsets of X. We also prove that there is a 1-1 correspondence between all hull operators on **Funsp** and a certain class of operators that assign to each object (X, H) a collection of subsets of X.

LEMMA 3.1. *Let Q be a hull operator on* **Funsp** *and $a: (X, H) \to (Y, K)$ a morphism. If $S \subseteq Y$ is such that $K-Q$ hull $S = S$, then $H-Q$ hull $a^{-1}(S) = a^{-1}(S)$. (That is, the inverse of a Q-subset is a Q-subset.)*

Proof. Let $T = H-Q$ hull $a^{-1}(S)$. Then $K|a(T)-Q$ hull $S = a(T)$. But $K|a(T)-Q$ hull $S \subseteq (K-Q \text{ hull } S) \cap a(T) = S \cap a(T) = S$. Therefore $a(T) \subseteq S$. Thus, $S = a(T)$, and so $T = a^{-1}(S)$. ∎

LEMMA 3.2. *Let (I, P) be a bicategory structure on* **Funsp**. *Let $a: (X, H) \to (Y, K)$ be a morphism in* **Funsp** *and $A \subseteq Y$ such that the inclusion $i_A: (A, K|A) \to (Y, K) \in I$. If $S = a^{-1}(A)$, then the diagram*

$$\begin{array}{ccc} (S, H|S) & \xrightarrow{a|S} & (A, K|A) \\ {\scriptstyle i_S}\downarrow & & \downarrow{\scriptstyle i_A} \\ (X, H) & \xrightarrow[a]{} & (Y, K) \end{array}$$

is a pullback of a and i_A. Consequently, $i_S \in I$.

Proof. It can be easily verified that this diagram is a pullback of a and i_A. That $i_S \in I$ follows from the dual of Proposition 1.1 in Kennison [15], p. 355.

LEMMA 3.3. *Let (I, P) be a bicategory structure on* **Funsp** *such that each morphism in I is an isomorphism into, and let $(X, H) \in \operatorname{ob}$* **Funsp**. *Let $(S_j, H|S_j)$, where $S_j \subseteq X$, be a family of I-subobjects of (X, H). Then $(Y, H|Y)$ is an I-subobject of (X, H), where $Y = \bigcap \{S_j\}$.*

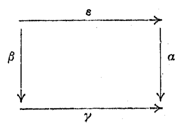
The lemma follows immediately from the fact that $(Y, H|Y)$ coincides with the categorical intersection of the family $\{i_{S_j}: (S_j, H|S_j) \to (X, H)\}$, where each i_{S_j} is the inclusion map, and the following proposition, which is the bicategory version of Baron [1], Corollary 2, p. 504; see also Herrlich [10], p. 72. The proof for the bicategory version is the same as that for the original corollary.

PROPOSITION 3.4. *If (I, P) is a bicategory structure on $\mathscr{C}$, a well-powered category with intersections and equalizers, then the intersection of a family of I-subobjects of a given object is again an I-subobject.*

THEOREM 3.5. *There exists a 1-1 correspondence between the following 4 families:*

1. $\{Q$: *Q is a hull operator on* **Funsp**$\}$.
2. $\{R$: *R is an operator on* **Funsp** *which assigns to each $(X, H) \in \operatorname{ob}$* **Funsp** *a collection of subsets of X, denoted by $R(X, H)$, and R satisfies the four conditions listed below*$\}$.
   (i) *If $(X, H) \in \operatorname{ob}$* **Funsp**, *then $R(X, H)$ is closed under arbitrary intersections.*
   (ii) *If $(X, H) \in \operatorname{ob}$* **Funsp**, *then $R(X, H)$ contains all H-affine subsets of X.*
   (iii) *If $(X, H) \in \operatorname{ob}$* **Funsp** *and $Y \in R(X, H)$, then $R(Y, H|Y) = \{T \cap Y: T \in R(X, H)\}$.*
   (iv) *If $f: (X, H) \to (Y, K)$ is a morphism and $T \in R(Y, K)$, then $f^{-1}(T) \in R(X, H)$.*
3. $\{(I, P)$: *(I, P) is a bicategory structure on* **Funsp**, *I contains all extremal monomorphisms and is contained in the class of all isomorphisms into*$\}$.
4. $\{I$: *I contains all extremal monomorphisms, is contained in the class of all isomorphisms into, and satisfies the three conditions listed below*$\}$.
   (i) *I is closed under composition.*
   (ii) *If $\{\varepsilon_i: (X_i, H_i) \to (Y_i, K_i)\}$ is a family of morphisms in I, then $\varepsilon = \prod \varepsilon_i \in I$, where $\varepsilon: \prod(X_i, H_i) \to \prod(Y_i, K_i)$ is defined by $\varepsilon(\{x_i\}) = \{\varepsilon_i(x_i)\}$.*

(iii) *Let the commutative diagram*



be a pullback of α and γ. Then $\alpha \in I$ implies $\beta \in I$.

Proof. Let Q be a hull operation on **Funsp**. If $(X, H) \in \text{ob}$ **Funsp**, then define $R_Q(X, H) = \{S: S \text{ is an } H-Q \text{ subset of } X\}$. Since $H-Q$ hull $S \subseteq H$-aff S, $R_Q(X, H)$ contains all H-affine subsets of X. From the comment following Definition 2.4, $R_Q(X, H)$ is closed under intersections. If $Y \in R_Q(X, H)$, and $S \subseteq Y$, then $H|Y-Q$ hull $S = (H-Q \text{ hull } S) \cap X$, by condition 3 of Proposition 2.2. Thus, $R_Q(Y, H|Y) = \{T \cap Y: T \in R_Q(X, H)\}$. Suppose $f: (X, H) \to (Y, K)$ is a morphism and $S \subseteq Y$ satisfies $S = K-Q$ hull S. By Lemma 3.1, $H-Q$ hull $f^{-1}(S) = f^{-1}(S)$. Thus, R_Q satisfies all four conditions of 2.

Suppose R satisfies 2. If $(X, H) \in \text{ob}$ **Funsp** and $S \subseteq X$, define $H-Q$ hull $S = \bigcap\{T: T \in R(X, H), S \subseteq T\}$. We will verify the conditions in Proposition 2.2. Clearly, $S \subseteq H-Q$ hull $S \subseteq H$-aff S, and $H-Q$ hull $(H-Q \text{ hull } S) = H-Q$ hull S. Also, $S \subseteq T$ implies $H-Q$ hull $S \subseteq H-Q$ hull T. If $Y \subseteq X$ and $Y \in R(X, H)$, then condition 3 implies $H|Y-Q$ hull $S = (H-Q \text{ hull } S) \cap Y$, for all $S \subseteq Y$. If $Y \subseteq X$ and $i: (Y, H|Y) \to (X, H)$ is the inclusion, then condition 4 yields $H|Y-Q$ hull $S \subseteq (H-Q \text{ hull } S) \cap Y$. It remains to prove that if $\alpha: (X, H) \to (Y, K)$ is a morphism, $\alpha(X) = Y$, and $H-Q$ hull $S = X$, then $K-Q$ hull $\alpha(S) = Y$. Let $T = K-Q$ hull $\alpha(S)$. $\alpha(S) \subseteq T$, and so $S \subseteq \alpha^{-1}(T)$. Since $\alpha^{-1}(T) = H-Q$ hull $\alpha^{-1}(T)$ by condition 4, we have $X = H-Q$ hull $S \subseteq \alpha^{-1}(T)$. Thus $T = Y$, and Q is a hull operation.

We have already shown that a hull operator induces a bicategory structure on **Funsp**, satisfying the conditions in 3. Suppose (I, P) is a bicategory structure on **Funsp** satisfying 3. If $(X, H) \in \text{ob}$ **Funsp**, let $R_I^P(X, H) = \{S: S \subseteq X, \text{ and the inclusion } i: (S, H|S) \to (X, H) \in I\}$. By Lemma 3.3, $R_I^P(X, H)$ is closed under intersections. Suppose $\alpha: (X, H) \to (Y, K)$ is a morphism and $V \in R_I^P(Y, K)$. Let $T = \alpha^{-1}(V)$. By Lemma 3.2, $T \in R_I^P(X, H)$. If E is an H-affine subset of X, then $i_E: (E, H|E) \to (X, H)$, the inclusion, is an extremal monomorphism. Therefore, $i_E \in I$ and $E \in R_I^P(X, H)$.

It remains to prove that if $Y \in R_I^P(X, H)$, then $R_I^P(Y, H|Y) = \{T \cap Y: T \in R_I^P(X, H)\}$. $i_Y: (Y, H|Y) \to (X, H)$, the inclusion, is in I. If $S \in R_I^P(Y, H|Y)$, then $i_S: (S, H|S) \to (Y, H|Y)$, the inclusion, is in I. $i_Y i_S: (S, H|S) \to (X, H) \in I$, since I is closed under composition. Therefore $S \in R_I^P(X, H)$,

and $R_I^P(Y, H|Y) \subseteq \{T \cap Y: T \in R_I^P(X, H)\}$. By condition (iv) of 2, $\{T \cap Y: T \in R_I^P(X, H)\} \subseteq R_I^P(Y, H|Y)$. So (I, P) induces R_I^P, which induces a hull operator.

Since each (I, P) corresponds to a unique I (see Kennison [15], Proposition 1.1, p. 355), it remains to prove that each I satisfying the three conditions in 4 corresponds to a unique bicategory structure (I, P) on **Funsp**. By the dual of Kennison's theorem ([15], p. 357), for each I, there exists P such that (I, P) is a left bicategory structure on **Funsp**. $P = \{\alpha: \alpha = \beta\gamma, \beta \in I \text{ implies } \beta \text{ is an isomorphism}\}$. We must show that $P \subseteq E$, the class of all epimorphisms. Each morphism α in **Funsp** can be written as $\alpha = \beta\gamma$, β an extremal monomorphism and γ an epimorphism. Thus, $\beta \in I$. Therefore, $\alpha \in P$ implies β is an isomorphism and so $\alpha \in E$. Consequently, (I, P) is a bicategory structure on **Funsp** satisfying condition 3.

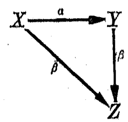
That there is a 1-1 correspondence between the families in 1, 2, 3 and 4 is straightforward from the constructions in the proof. ∎

A characterization of P_Q-refl hull $([0,1], A[0,1])$. In this section we apply some general category results in order to obtain a morphism characterization of the objects in the smallest P_Q-reflective subcategory of **Funsp** that contains $([0,1], A[0,1])$, where Q is a hull operator, and $A[0,1]$ denotes the continuous affine functions on $[0,1]$.

DEFINITION 4.1. Let $\mathscr{C}$ be a complete category, and (I, P) a bicategory structure on $\mathscr{C}$ such that $\mathscr{C}$ is P-co-well-powered. If $\mathscr{A}$ is a subclass of $\mathscr{C}$, then the *P-reflective hull of $\mathscr{A}$* is the full subcategory of $\mathscr{C}$ whose objects are all the I-subobjects of products of objects in $\mathscr{A}$. We will denote this subcategory by P-refl hull $\mathscr{A}$.

It is well known (Kennison [15], p. 306, Herrlich [10], p. 99, Baron [1], p. 500) that the P-reflective hull of $\mathscr{A}$ is the smallest P-reflective subcategory of $\mathscr{C}$ that contains all the objects in $\mathscr{A}$.

DEFINITION 4.2. (Herrlich [12], p. 100). Let $\mathscr{A}$ be a class of objects in the category $\mathscr{C}$. A morphism $\alpha: X \to Y$ is *$\mathscr{A}$-extendable* if and only if for each morphism $\beta: X \to Z$, where $Z \in \mathscr{A}$, there exists a morphism $\beta': Y \to Z$ such that $\beta = \beta'\alpha$.



We need the following bicategory version of a result that appears in Herrlich ([10], p. 101), for the case when P is the class of all epimorphisms and I is the class of all extremal monomorphisms. Since the proof is identical for the general bicategory structure, it will be omitted.

THEOREM 4.3 (cf. Herrlich [10], p. 101). *Let* $\mathscr{B} = P$-refl hull $\mathscr{A}$, *where* $\mathscr{A}$ *is a class of objects in a complete well-powered and co-well-powered category* $\mathscr{C}$, *with bicategory structure* (I, P). *If a morphism* $\alpha \in P$ *is* $\mathscr{A}$-*extendable, then it is* $\mathscr{B}$-*extendable.*

We note that if $(X, H) \in \mathrm{ob}$ **Funsp** and $h: X \to [0, 1]$ is a continuous map, then $h: (X, H) \to ([0,1], A[0,1])$ is a morphism if and only if $h \in H$. Consequently, we have the following Lemma.

LEMMA 4.4. *Let* $\alpha: (X, H) \to (Y, K)$ *be a morphism in* **Funsp**, *and for each* $(Z, J) \in \mathrm{ob}$ **Funsp**, *let* $J_1 = \{j \in J, j(Z) \subseteq [0, 1]\}$. *Then* α *is* $([0, 1], A[0, 1])$-*extendable if and only if* $K_1\alpha = H_1$.

The following is an immediate consequence of Lemma 4.5 and Theorem 4.3.

THEOREM 4.5. *If* Q *is a hull operator on* **Funsp**, (I_Q, P_Q) *is the associated bicategory structure on* **Funsp**, *and* $\alpha: (X, H) \to (Y, K) \in P_Q$, *then* α *is* P_Q-refl hull $([0, 1], A[0, 1])$-*extendable if and only if* $K_1\alpha = H_1$.

COROLLARY 4.6. *Let* Q *be a hull operator on* **Funsp**. *If* $(X, H) \in \mathrm{ob}$ **Funsp**, *then* $(X, H) \in \mathrm{ob} P_Q$-refl hull $([0, 1], A[0, 1])$ *if and only if whenever* $\alpha: (X, H) \to (Y, K)$ *is a morphism with* $K-Q$ hull $\alpha(X) = Y$ *and* $K_1\alpha = H_1$, *then* α *is an isomorphism.*

Proof. Since $\alpha \in P_Q$, the corollary follows from Theorem 4.5 and the fact that $(X, H) \in P_Q$-refl hull of $([0, 1], A[0, 1])$ if and only if every $([0, 1], A[0, 1])$-extendable morphism α with domain (X, H) and $\alpha \in P_Q$ is an isomorphism (see H. Herrlich [10], p. 102). We are using the bicategory version of Satz 11.2.4. The proof for the bicategory version is the same as that for the original theorem. ∎

COROLLARY 4.7. *Let* Q *be a hull operator on* **Funsp** *such that for all* $(X, H) \in \mathrm{ob}$ **Funsp** *and for all* $S \subseteq X$, $H-Q$ hull $S \subseteq H$-conv S. *Then Theorem* 4.5 *and Corollary* 4.6 *remain valid if we replace the condition* $K_1\alpha = H_1$ *by* $K\alpha = H$.

Proof. If suffices to prove that with the above condition on Q, $K\alpha = H$ if and only if $K_1\alpha = H_1$. Then we can use Theorem 4.5.

If $K\alpha = H$, then $K_1\alpha \subseteq H_1$. Let $h_1 \in H$. There exists $k \in K$ so that $k\alpha = h$. Since $\alpha \in P_Q$, K-conv $\alpha(X) = Y$. So $k\alpha(X) \subseteq [0, 1]$ and therefore $k(Y) \subseteq [0, 1]$. Thus, $k \in K_1$, and $K_1\alpha = H_1$.

If $K_1\alpha = H_1$, we know $K\alpha \subseteq H$ since α is a morphism. Let $h \in H$. There must be real numbers a and b so that $h' = ah + b$ and $h'(X) \subseteq [0, 1]$. Thus there exists $k' \in K_1$ with $k'\alpha = h'$. Let $k = ((k'/a) - b)$. Then $k\alpha = h$. ∎

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