It can also be shown, by calculation, that

$$G(C') = \{x' \in C' : x' = (e^{i\alpha}),\ \alpha \in \mathbf{R}\}$$

and

$$G(C'') = \{x'' \in C'' : x'' = (e^{i\beta n})\}.$$

It is easy to show that

$$\big(G(C'),\ G(C'')\big) = G(A).$$

Hence

$$T\big((e^{i\alpha}),\ (e^{i\beta n})\big) = \big((e^{i\alpha}),\ (k_n)\big),$$

where

$$k_1 = e^{i\alpha} \quad \text{and} \quad k_n = e^{i\beta n} \quad \text{for} \quad n > 1.$$

Since not every element of $G(A)$ is of the form $\big((e^{i\alpha}),\ (k_n)\big)$, $TG \neq G$.

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A converse to some inequalities and approximations in the theory of Stieltjes and stochastic integrals, and for nth derivatives

by

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Abstract. The object of this report is to establish by counter-examples the best possible character of theorems recently obtained about stochastic integrals and Stieltjes integrals, and about nth derivatives and finite differences. The hypotheses involve a pair of estimate functions subject to the convergence of a corresponding integral or Y-series, and it is shown that the divergence of this integral or series render in each case the conclusion false.

1. Our notation will be largely that of [6], [7]. Let n be a positive integer, and let $\varphi(u)$, $\psi(u)$ be functions defined for $0 \leqslant u \leqslant 1$, such that φ is non-negative and Borel measurable, while ψ is continuous and monotone increasing, and takes the value $\psi(u) = 0$ only at $u = 0$; further suppose that, for $0 < \lambda < 1$,

$$\varphi(\lambda u) \geqslant (\tfrac{1}{2}\lambda)^n \varphi(u), \qquad \psi(\lambda u) \geqslant \tfrac{1}{2}\lambda\psi(u). \tag{1.1}$$

We denote by $\{h\}$ a decreasing sequence h_ν ($\nu = 0, 1, \ldots$) with limit 0 and with initial term $h_0 \leqslant 1$. We write

$$Y = \sum_{\nu=0}^{\infty} (h_\nu)^{-n} \varphi(h_\nu)\psi(h_\nu), \tag{1.2}$$

and we denote by Y_N its partial sum for $0 \leqslant \nu < N$. For $n = 1$, $h_\nu = 2^{-\nu}$, the series (1.2) occurs in [2] and it then converges or diverges with the sum $\sum\varphi(1/\nu)\psi(1/\nu)$ previously introduced in [5]. This last sum has been termed Y-series by Leśniewicz and Orlicz [1]. We prefer here to term *Y-series* the series (1.2): it was itself introduced, for $n = 1$, in [3].

We say that the sequence $\{h\}$, or the Y-series Y, satisfies the condition $C(1)$, if for each ν the ratio $h_{\nu-1}/h_\nu$ is an integer expressible as an integer power of 2, and satisfies the condition $C(2)$ if

$$2\psi(h_\nu) \leqslant \psi(h_{\nu-1}) \leqslant 8\psi(h_\nu). \tag{1.3}$$

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We say it satisfies partially $C(2)$, if it satisfies the first of the two inequalities (1.3).

In the case of a Y-series subject to $C(1)$, $C(2)$, the convergence or divergence of Y is clearly equivalent to that of the series

$$S_{\{h\}} = \sum_{\nu=1}^{\infty} (h_\nu)^{-n} \varphi(h_\nu)\psi(h_{\nu-1}), \tag{1.4}$$

and, by [7], Appendix A, § A3, equivalent to that of the integral

$$S(h) = \int_0^h u^{-n}\varphi(u)\,d\psi(u). \tag{1.5}$$

We recall that, by the same reference and [6], § 3, there always exists a Y-series subject to $C(1)$ and $C(2)$.

(1.6) THEOREM. *Let F be a continuous real, comlpex, or Banach-valued, function of the real variable t, and suppose*

$$|\Delta_h^{*n}\Delta_k^{*2}F(t)| < \varphi(h)\psi(k) \qquad (0 < \tfrac{1}{2}h \leqslant k \leqslant 1), \tag{1.7}$$

where Δ_h^* is the difference operator given by

$$\Delta_h^* F(t) = F(t+\tfrac{1}{2}h) - F(t-\tfrac{1}{2}h).$$

Further suppose the integral (1.5) *convergent for some $h > 0$. Then F has a continuous n-th derivative $F^{(n)}$, satisfying uniformly in t, for $0 < h \leqslant 1$, the inequality*

$$|F^{(n)}(t) - h^{-n}\Delta_h^{*n}F(t)| \leqslant K_n S(h), \tag{1.8}$$

where K_n denotes a constant depending only on n.

The above statement is established by the slighest of changes in the proof of [7], Appendix A, Theorem A2.1. In that reference it was assumed that (1.7) held for $0 < h \leqslant 1$, $0 < k \leqslant 1$, instead of for $0 < \frac{1}{2}h \leqslant k \leqslant 1$, but the proof nowhere used this. In addition we have here subjected ψ to the second inequality in (1.1), and therefore defined the condition $C(2)$ by (1.3), whereas in the reference ψ fulfilled the weaker inequality $\psi(\lambda u) \geqslant (\frac{1}{2}\lambda)^2\psi(u)$, and the constant 8 in (1.3) was consequently changed to 32. Our present stronger condition merely weakens the statement of our theorem, but we prefer this weaker from in order to obtain a more precise converse, in which, incidentally, the dissymmetry we have introduced between h and k, and therefore between φ and ψ, is essential.

The main object of this note is to establish such a converse to (1.6), and at the same time to related theorems about Stieltjes and stochastic integrals. *In these converses the integral* (1.5) *is supposed divergent.* The converses were suggested by a counterexample due to Leśniewicz and

Orlicz [1], and in the case of Stieltjes integrals our results cover cases these authors found it necessary to exclude.

Henceforth we assume φ monotone increasing. Later, in the converse results relating to convolutions we shall strengthen this assumption, because the dissymmetry between h and k will then disappear. However, our assumptions will always remain necessary for the validity of our results in the most natural context, which is that in which φ, ψ are orders of magnitude, such that every product or ratio of powers of φ, ψ, multiplied or divided by a power of the identity function, is automatically monotone.

2. Our basic converse theorem is as follows:

(2.1) THEOREM. *Let φ, ψ be subject to the conditions stated at the beginning of Section* 1, *and suppose now φ monotone increasing. Then if the integral* (1.5) *is divergent, there exists a continuous F subject to* (1.7), *such that, at $t = 0$, $F(t)$ does not possess a finite n-th derivative.*

Explicit specification of F. Since (1.5) diverges by hypothesis, there exists, by what has been said, a divergent Y-series subject to $C(1)$ and $C(2)$. We only use the fact that there is a divergent Y-series subject to $C(1)$ and partially to $C(2)$. We fix the corresponding sequence $\{h\}$ and we write

$$F(t) = (-i)^{n+2}c\sum_{\nu=0}^{\infty}\varphi(h_\nu)\psi(h_\nu)e^{2\pi i t/h_\nu}.$$

Here c is a positive constant to be specified later. The function F is continuous and periodic, since the series on the rightconverges absolutely and uniformly: this last is because $\sum\varphi(h_\nu)\psi(h_\nu)$ is dominated, on account of the first inequality (1.3) and of the monotony of φ, by a multiple of the geometric series $\sum 2^{-\nu}$. Thus we must verify that

(2.2) *F satisfies* (1.7),

(2.3) *F has no finite n-th derivative at $t = 0$.*

Proof of (2.2). We have at $t = 0$,

$$2^{-n-2}c^{-1}\Delta_h^{*n}\Delta_k^{*2}F(t) = \sum_\nu \varphi(h_\nu)\psi(h_\nu)\sin^n(\pi h/h_\nu)\sin^2(\pi k/h_\nu)$$
$$= \sum a_\nu \quad \text{say}.$$

Here, as t varies, the quantity on the left attains a greatest absolute value $\leqslant \sum|a_\nu|$. Therefore (2.2) reduces to estimating the sum

$$\sum|a_\nu| = \sum\nolimits_1 + \sum\nolimits_2,$$

where $\sum_1$ is the sum for $h_\nu \leqslant \mathrm{Max}(h,k)$, and $\sum_2$ for $h_\nu > \mathrm{Max}(h,k)$. In $\sum_1$ we have

$$|a_\nu| \leqslant \varphi(h_\nu)\psi(h_\nu) \leqslant \varphi(h)\psi(h_\nu) \quad \text{if } h_\nu \leqslant h,$$
$$|a_\nu| \leqslant \varphi(h_\nu)\psi(h_\nu)(\pi h/h_\nu)^n \leqslant (2\pi)^n \varphi(h)\psi(h_\nu) \quad \text{otherwise.}$$

This second estimate is therefore valid in all cases, and if N is the smallest value of ν in $\sum_1$ we find that, if $h \leqslant k$,

$$\sum_1 \leqslant (2\pi)^n \varphi(h) \sum_{r=0}^{\infty} \psi(h_{N+r}) \leqslant (2\pi)^n \varphi(h)\psi(k) \sum 2^{-r} \leqslant 2(2\pi)^n \varphi(h)\psi(k),$$

or similarly, if $\frac{1}{2}h \leqslant k < h$,

$$\sum_1 \leqslant 2\varphi(h)\psi(h) \leqslant 2\varphi(h)\psi(2k) \leqslant 4\varphi(h)\psi(k) \quad \text{by (1.1)},$$

so that the previous estimate is still valid. On the other hand, in $\sum_2$ we have, since $|\sin\theta| \leqslant |\theta|$,

$$|a_\nu| \leqslant \varphi(h_\nu)\psi(h_\nu)(\pi h/h_\nu)^n (\pi k/h_\nu)^2 \leqslant 2^{n+1}\pi^{n+1}\varphi(h)\psi(k)(\pi k/h_\nu),$$

on account of (1.1). Thus by $C(1)$,

$$\sum_2 \leqslant \pi(2\pi)^{n+1}\varphi(h)\psi(k) \sum_{h_\nu > k} (k/h_\nu) \leqslant \pi(2\pi)^{n+1}\varphi(h)\psi(k) \sum 2^{-r}$$
$$= (2\pi)^{n+2}\varphi(h)\psi(k).$$

Evidently (2.2) follows by choosing c so that

$$2^{-n-2}c^{-1} = (2+4\pi^2)(2\pi)^n.$$

Proof of (2.3). We shall make h describe the sequence $\{h\}$. We therefore set now $h = h_N$. By $C(1)$ the quantity $\sin(\pi h/h_\nu)$ then vanishes for $\nu \geqslant N$, and lies, for $\nu < N$, between 1 and $2h/h_\nu$, since the minimum of $(\sin\theta)/\theta$ in $0 < \theta \leqslant \pi/2$ is $2/\pi$, attained at $\theta = \pi/2$.

Thus at $t = 0$ we have

$$|h^{-n}\Delta_h^{*n}F(t)| = 2^n c h^{-n} \sum_{\nu=0}^{N-1} \varphi(h_\nu)\psi(h_\nu)|\sin^n(\pi h/h_\nu)|$$
$$\geqslant 2^n c h^{-n} \sum_{\nu=0}^{N-1} \varphi(h_\nu)\psi(h_\nu)(2h/h_\nu)^n \geqslant c \sum_{\nu=0}^{N-1} h_\nu^{-n}\varphi(h_\nu)\psi(h_\nu).$$

This means that, in norm, the symmetric difference ratio of order n exceeds a constant multiple of the partial sum Y_N of our divergent Y-series, and so tends to infinity as $h \to 0$ along the sequence $\{h\}$. The possiblity that F has a finite nth derivative at $t = 0$ is clearly excluded, and this completes the proof of (2.3), and so of Theorem (2.1).

3. Let $\varrho = \varphi/\psi$, $\chi(u) = u^{-n}\varphi(u)\sqrt{\psi(u)}$. As stated earlier, we now strengthen our assumptions about φ: we shall suppose ϱ monotone increasing, or more generally subject to an inequality of the type

$$\varrho(\lambda u) < K\varrho(u) \quad \text{for } 0 < \lambda < 1,$$

where K is independent of u, λ. It ϱ is supposed monotone increasing it is clear that the divergence of $S(h)$ implies that of the twin integral

$$S^*(h) = \int_0^h u^{-n}\psi(u)\,d\varphi(u),$$

since we have then

$$S^*(h) = S(h) + \int u^{-n}\varphi\psi\,d\log\varrho \geqslant S(h).$$

This is the "reason behind the requirement on ϱ", but we do not use it in any explicit way.

In addition we shall make an assumption on χ in relation to the sequence $\{h\}$, but this is not really an assumption, because we can arrange, as will be seen below, for it to be satisfied, simply by choosing $\{h\}$ conveniently. We shall assume that the sequence

$$\chi(h_\nu) = \chi_\nu \quad (\nu = 1, 2, \ldots)$$

increases monotonely. Of course, if the sequence is monotone, it has to be monotone increasing, since

$$\sum (\psi(h_\nu))^{1/2} < \infty, \quad \text{while} \quad \sum \chi_\nu (\psi(h_\nu))^{1/2} = Y = \infty.$$

The function F will now be chosen as before except for the choice of the constant factor c, which we write in the form $c = c_1 c_2$, where $c_1 > 0$, $c_2 > 0$ are still to be determined. In that case F is now the convolution $f * g$ of the pair of functions

$$\text{(3.1)} \qquad f(t) = (-i)^n c_1 \sum_{\nu=0}^{\infty} \varphi(h_\nu) e^{2\pi i t/h_\nu},$$
$$g(t) = (-i)^2 c_2 \sum_{\nu=0}^{\infty} \psi(h_\nu) e^{2\pi i t/h_\nu}.$$

To see this it is sufficient to verify the convergence of $\sum\varphi(h_\nu)$, $\sum\psi(h_\nu)$. The former is majorized by the multiple $K\varrho(h_0)$ of the latter, which is in turn majorized, on account of the first relation (1.3), by a multiple of the geometric series $\sum 2^{-\nu}$.

The convolution $F = f * g$, on the period h_0, which we take to be 1, satisfies the identity

$$\Delta_h^{*n}\Delta_k^{*2}F = (\Delta_h^{*n}f) * (\Delta_k^{*2}g).$$

It will follow that relation (1.7) must remain valid in the wider range $0 < h \leqslant 1$, $0 < k \leqslant 1$, if we verify, as we now shall, that by choice of c_1, c_2,

$$(3.2)\qquad |\Delta_h^{*n} f(t)| \leqslant \varphi(h), \qquad |\Delta_h^{*2} g(t)| \leqslant \psi(h).$$

Verification of (3.2). We have, at $t = 0$,

$$2^{-n} c_1^{-1} \Delta_h^{*n} f(t) = \sum \varphi(h_\nu) \sin^n(\pi h/h_\nu) = \sum b'_\nu \quad \text{say},$$
$$2^{-2} c_2^{-1} \Delta_h^{*2} g(t) = \sum \psi(h_\nu) \sin^2(\pi h/h_\nu) = \sum b''_\nu \quad \text{say},$$

where as before the left-hand sides attain, as t varies, a greatest absolute value $\leqslant \sum |b'_\nu|$, or $\leqslant \sum |b''_\nu|$. We write

$$\sum |b'_\nu| = \sum\nolimits_1' + \sum\nolimits_2', \qquad \sum |b''_\nu| = \sum\nolimits_1'' + \sum\nolimits_2'',$$

where $\sum_1'$, $\sum_1''$ are sums for $h_\nu \leqslant h$, and $\sum_2'$, $\sum_2''$ for $h_\nu > h$. We write λ_ν for $(\psi(h_\nu))^{-1/2}$, and N for the smallest ν such that $h_\nu \leqslant h$.

In $\sum_1'$, $\sum_1''$ we have

$$|b'_\nu| \leqslant \varphi(h_\nu) \leqslant K\varrho(h)\psi(h_\nu), \qquad |b''_\nu| \leqslant \psi(h_\nu).$$

Consequently

$$\sum\nolimits_1'' \leqslant \sum \psi(h_{N+r}) \leqslant \psi(h_N) \sum 2^{-r} \leqslant 2\psi(h),$$
$$\sum\nolimits_1' \leqslant K\varrho(h) \sum\nolimits_1'' \leqslant 2K\varrho(h)\psi(h) = 2K\varphi(h).$$

On the other hand, in $\sum_2'$, $\sum_2''$ we have $\nu = N-1-r$ $(r \geqslant 0)$, so that

$$|b''_\nu| \leqslant \psi(h_\nu)(\pi h/h_\nu)^2 \leqslant 2\pi^2 \psi(h) h/h_\nu \leqslant 2\pi^2 \psi(h) 2^{-r},$$
$$|b'_\nu| \leqslant \varphi(h_\nu)(\pi h/h_\nu)^n = (\pi h)^n \chi_\nu \lambda_\nu \leqslant (\pi h)^n \chi_{N-1} \lambda_\nu$$
$$\leqslant 2^{-r/2} (\pi h)^n \chi_{N-1} \lambda_{N-1} = 2^{-r/2} (\pi h/h_{N-1}) \varphi(h_{N-1})$$
$$\leqslant 2^{-r/2} (2\pi)^n \varphi(h).$$

Therefore

$$\sum\nolimits_2'' \leqslant 2\pi^2 \psi(h) \sum 2^{-r} = 4\pi^2 \psi(h),$$
$$\sum\nolimits_2' \leqslant (2\pi)^n \varphi(h) \sum 2^{-r/2} = (2+\sqrt{2})(2\pi)^n \varphi(h),$$

and to derive (3.2) it is enough to choose $c_2^{-1} 2^{-2} = 2 + 4\pi^2$, $c_1^{-1} 2^{-n} = 2K + (2+\sqrt{2})(2\pi)^n$.

4. Thinning the Y-series. We denote by y_ν the general term $\chi_\nu \sqrt{\psi(h_\nu)}$ of the Y-series (1.2), where by hypothesis $C(1)$ is fulfilled, and partially $C(2)$, and the series $\sum y_\nu$ diverges. For this series we shall now suppose the first inequality (1.3) strengthened to

$$(4.1)\qquad 16\psi(h_\nu) \leqslant \psi(h_{\nu-1}), \quad \text{i.e.} \quad 4\sqrt{\psi(h_\nu)} \leqslant \sqrt{\psi(h_{\nu-1})}.$$

This means that, for any ν,

$$(4.2)\qquad \sum_{r=1}^{\infty} \sqrt{\psi(h_{\nu+r})} \leqslant \sqrt{\psi(h_\nu)} \cdot \sum_1^{\infty} 4^{-r} = \tfrac{1}{3}\sqrt{\psi(h_\nu)}.$$

We arrange for (4.1) to be satisfied, simply by choosing s from the values 0, 1, 2, 3 in such a manner that the series

$$\sum_\nu y_{4\nu+s}$$

be divergent, and by then rewriting h_ν for $h_{4\nu+s}$, and the other symbols correspondingly.

However, the new divergent Y-series thus obtained will now be thinned out further, and for this purpose we denote by E the set of suffixes ν for which

$$\nu' < \nu \quad \text{implies} \quad \chi_{\nu'} \leqslant \chi_\nu.$$

We write further E_ν for the set of positive integers r such that

$$\chi_\nu > \chi_{\nu+r}.$$

Evidently by (4.2)

$$\sum_{r \in E_\nu} y_{\nu+r} < \chi_\nu \sum_{r=1}^{\infty} \sqrt{\psi(h_{\nu+r})} \leqslant \tfrac{1}{3}\chi_\nu \sqrt{\psi(h_\nu)} = \tfrac{1}{3} y_\nu.$$

It follows easily that $\sum_{\nu \in E} y_\nu$ diverges, since its partial sum for $\nu < N$ differs by at most a third of itself from the partial sum Y_N. If we now rename h_ν the νth term of the restriction to $\mu \in E$ of the sequence h_μ $(\mu = 1, 2, \ldots)$, we see that χ_ν now increases monotonely with ν. The corresponding new Y-series is still divergent, and is subject to $C(1)$ and partially to $C(2)$.

5. We consider specially the case $n = 1$. We shall then show that by modifying the constant factor in $g(t)$ the second inequality (3.2) can be replaced by the first difference condition

$$|\Delta_h^* g(t)| \leqslant \psi(h).$$

Evidently, by writing $\frac{1}{2}g$ for g, we could then satisfy also the second inequality (3.2) as it stands, so that we are now, in effect, obtaining a stronger result than in the preceding section in our special case.

We choose this time

$$g(t) = -ic_2 \sum_\nu \psi(h_\nu) e^{2\pi i t/h_\nu}.$$

At $t = 0$ we have

$$(2)^{-1}c_2^{-1}\Delta_h^* g(t) = \sum \psi(h_\nu)\sin(\pi h/h_\nu) = \sum b_\nu \quad \text{say},$$

and as before

$$\sum |b_\nu| = \sum\nolimits_1 + \sum\nolimits_2,$$
$$\sum\nolimits_1 = \sum_{\nu\geqslant N} |b_\nu| \leqslant \sum_{\nu\geqslant N} \psi(h_\nu) \leqslant 2\psi(h),$$
$$\sum\nolimits_2 = \sum_{\nu<N} |b_\nu| \leqslant \sum_{\nu<N} \psi(h_\nu)(\pi h/h_\nu).$$

However, the last sum is now

$$\leqslant K\frac{\psi(h)}{\varphi(h)} \sum_{\nu<N} \varphi(h_\nu)(\pi h/h_\nu),$$

and this we have already estimated. It is therefore

$$\leqslant K\frac{\psi(h)}{\varphi(h)}(2+\sqrt{2})2\pi\varphi(h) = K(2+\sqrt{2})2\pi\psi(h),$$

and our assertion follows by choosing $c_2^{-1}2^{-1} = 2+(2+\sqrt{2})2\pi K$.

We remark that the symmetric differences in (3.2), or in its modification above, can of course be replaced by ordinary differences Δ_h, defined by setting, for an arbitrary function G, $\Delta_h G(t) = G(t+h)-G(t)$. For this, it is sufficient to change the variable t to $t-\frac{1}{2}h$. Thus our functions f, g now satisfy

$$(5.1) \qquad |\Delta_h f(t)| \leqslant \varphi(h), \quad |\Delta_h g(t)| \leqslant \psi(h),$$

and the convolution $F = f*g$ has no finite derivative at $t = 0$. We note that the second inequality (4.1) can of course be written $|\Delta_h \bar{g}(t)| \leqslant \psi(h)$, where $\bar{g}$ is the conjugate of g.

6. The functions f, g of the preceding section provide converses to existence theorems for Stieltjes integrals of functions subject to conditions of the type (5.1), and also for corresponding stochastic integrals. It is important that that f, g are periodic and that $\bar{g}(t) = g(-t)$. According to [6] ((5.4), p. 188), this implies, in view of the continuity of f, g that the existence of

$$I = \int_0^1 f d\bar{g}$$

as a Riemann–Stieltjes integral ensures that I be the derivative at $t = 0$ of the convolution $F = f*g$, and in particular that this derivative be finite. Indeed the existence of such a finite derivative at $t = 0$ is shown to be equivalent in such a case to each of the generalized definitions $\mathscr{I}, \mathscr{J}, \mathscr{L}$ of the integral I. Thus for the pair $f, \bar{g}$ of the preceding section, not only does I not exist as a Riemann–Stieltjes integral, but it also does not exist as the limit of

$$\hat{I} = \int_0^1 \hat{f} d\bar{g}, \quad \text{where} \quad \hat{f}(t) = \int_{t-h}^{t} f(u)\,du/h,$$

as $h \to 0$. Here $\hat{I}$ is an elementary integral easily evaluated by parts on account of periodicity: We have in fact

$$\hat{I} = -\int \hat{f}'\bar{g}\,dt = \int \big(f(t-h)-f(t)\big)\bar{g}(t)\,dt/h.$$

From the same pair of functions f, g we can construct a pair $f(t)$, $\bar{g}(t, \omega)$ and a pair $f(t, \omega)$, $\bar{g}(t, \omega)$, where functions of t only are deterministic, and functions of (t, ω) are stochastic. We do so by the simple expedient of making $f(t, \omega)$ or $g(t, \omega)$ independent of ω. Evidently the covariance of g satisfies

$$|x(\Delta, \Delta^*)| = \left|\int d\omega \Delta\bar{g}\Delta^* g\right| \leqslant \psi(|\Delta|)\psi(|\Delta^*|).$$

Similarly g satisfies a nigh-martingale condition. (We use the notation of [6], [7].)

Thus, with our hypotheses on φ, ψ, in all these cases, the divergence of the integral (1.5) implies the existence of a pair $f, \bar{g}$ for which the Stieltjes integral, or the stochastic integral with deterministic or with stochastic integrand, does not exist. This means that the results of [6], [7] can now be regarded as being, in the appropriate sense, the best of their kind.

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* For additional references, see [6], [7] and [1].

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