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### STUDIA MATHEMATICA

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### STUDIA MATHEMATICA, T. LV. (1976)

Norm-decreasing isomorphism on hermitian elements and the group of isometric and invertible multipliers of a Banach algebra

by

E. O. Osho, Bi (Ibadan, Nigeria)

Abstract. Let $A_i (i = 1, 2)$ be a complex Banach algebra, $T$, a norm decreasing algebra isomorphism of $A_1$ onto $A_2$, $H(A)$, the Banach space of hermitian elements in $A_i$ and $G(A)$, the group of isometric and invertible multipliers in $A_i$. We show that

(i) If $A_1$ is unital, $TH(A_1) = H(A_2)$, and $TG(A_1) = G(A_2)$. But if $A_1 = A_2 = A$ and $TG(A) = G(A)$, then $TH(A) = H(A)$.

(ii) If $A_1$ is a *-algebra, then $T$ is a *-isomorphism.

(iii) If $A_1$ has a minimal approximate identity $I$, the induced map of the multiplier algebra $A_2^*$ onto $A_1^*$ is a norm decreasing extension of $T$ and $T^* G(A)^* = G(A_1)^*$.

We finally construct an example to show that $T$ does not in general preserve $G(A)$ and $H(A)$.

1. Introduction. We shall investigate, in this paper, the effect of norm-decreasing algebra isomorphism on hermitian elements $H(A)$ and the group of isometric and invertible multipliers $G(A)$ of a Banach algebra $A$. The motivation for this work is Wendel’s paper in [7] on the preservation of $G(A)$ by a norm-decreasing $T$ when $A$ is a group algebra, Rigelhof’s in [5] when $A$ is a measure algebra on a locally compact group, and Wood’s in [8] where $A$ is $L^p(G)$ for compact group. We shall indicate that $G(A)$ and $H(A)$ are not, in general, preserved by a norm-decreasing $T$. But when $G(A)$ is preserved, $H(A)$ is also preserved.

This work is a part of the author’s Ph.D. thesis and I wish to express my thanks to Dr. G. V. Wood of the University College of Swansea for his help and advice as my Supervisor throughout my three years stay in Swansea.

2. Notations and definitions. We shall always consider Banach algebras $A$ over the complex field $C$ (A, assumed to be without order (i.e. $\forall a \in A$, $a^2 = 0$ or $A^2 = 0$ or $a = 0$)). We shall denote by $R$ the real scalars and by $I$, the identity in $A$ if it has one. $g(a)$ denotes the spectral radius of $a A$.

2.1. Definition: Hermitian Elements (see [1]). Let $A$ be a complex unital Banach algebra (i.e. $1 \in A$ and $\|1\| = 1$). We denote by $A^*$ the dual
space of $A$ and by $S(A)$ the unit sphere of $A$. Given $x \in S(A)$, we define

$$D(A, x) = \{ f \circ A^* : f(x) = 1 = \| f \| \}.$$  

Given $a \in A$ and $x \in S(A)$, let

$$V(A, a, x) = \{ f(ax) : f \in D(A, x) \}$$

and

$$V(A, a) = \bigcup \{ V(A, a, x) : x \in S(A) \}.$$  

$V(A, a)$ is called the numerical range of $a$. $a \in A$ is hermitian if $V(A, a) \subset \mathbb{R}$ and we shall denote by $H(A)$ the set of all hermitian elements of $A$.

$B(A)$ denotes the Banach algebra of all bounded linear operators in $A$. $L_a$ (defined by $L_a a = ax$ $\forall a \in A$ $x \in A$) denotes the left multiplication operator. The right multiplication operator $R_a$ is similarly defined.

If $A$ has no identity, then $H(A)$ is hermitian if $L_a \in B(A)$ is hermitian. We shall need the following results on hermitian elements.

2.3. Proposition. Let $A$ be a complex unital Banach algebra. Given $a \in A$, the following statements are equivalent:

(i) $ha \in H(A)$;

(ii) $\lim_{n \to \infty} \frac{1}{n+1} \| [a, h] \| = 0$;

(iii) $\| [a, h] \| = 1$ $\forall a \in R$ (see Lemma 2.2 of [1]).

2.4. Proposition. $H(A)$ is a Banach space (see Lemma 4.5 of [1]).

2.5. Definition. Multiplier algebras (see [3]). The bounded linear operator $\varphi$ on $A$ is a multiplier of $A$ if $\varphi(ax) = (\varphi a)x = \varphi(xa) \forall a, x \in A$. If $\varphi(ax) = (\varphi a)x = \varphi(xa)$, then $\varphi$ is a left multiplier of $a$. If $\varphi(ax) = \varphi(\varphi a)x = \varphi(xa)$, then $\varphi$ is a right multiplier of $a$. To see repetition, we shall deal with Banach algebras of left multipliers only in this paper; we denote this by $A^\star$. Clearly, $L_a \in A^\star$.

If $A$ has an identity, then $A = A^\pi$, $\varphi \in \mathfrak{G}(A)$ if $\varphi \in A^\pi$ and $[\varphi, x] = [\varphi x, x] = [x, x] \forall x \in A$. If $A$ has an identity, then $\mathfrak{G}(A) = \{ x : [x, x] = [x, x] = 1 \}$.

In fact, $\mathfrak{G}(A)$ is a topological group in the strong operator topology (SOT). (see Lemma 1.6.1 of [2]). A net $(a_\alpha)$ converges to $\varphi$ in the SOT iff $\lim_{\alpha} [a_\alpha, x] = 0 \forall a \in A$. It converges in the weak operator topology (WOT) iff for each $\varphi \in A$ and $\varphi^* A^\pi$ we have

$$\lim_{\alpha} \| \varphi^* (\varphi_\alpha, x) - \varphi^* (\varphi_{\alpha}) \| = 0.$$  

2.6. Definition. A net $(a_\alpha)$ in a Banach algebra $A$ is a left approximate identity if $\lim_{\alpha} [a_\alpha, x] = 0$ for each $a \in A$. It is a right approximate identity if $\lim_{\alpha} [x, a_\alpha - x] = 0$ for each $a \in A$. It is an approximate identity if $\lim_{\alpha} [a_\alpha, x] = 0$ for each $a \in A$. It is minimal if $\lim_{\alpha} [a_\alpha, x] = 1$ in addition.

$[L_a, \varphi_a]$ is dense in $A^\pi$ in the SOT iff $A$ has an approximate identity (see Theorem 1.1.6 of [3]).

3. Isomorphism of $H(A)$ and $G(A)$.

3.1. Theorem. Let $T$ be a norm decreasing algebra isomorphism of a complex unital Banach algebra $A$ onto another $A$.

(i) $TH(A) = H(A)$ and $TH(A) \cap H(A)$ and $TGH(A) \supseteq G(A)$.

(ii) $TGH(A) \supseteq G(A)$.

Proof. (i) Let $h \in H(A)$; then

$$\| \exp iah \| = 1, \quad a \in R.$$  

By 2.2.

Since $T$ is algebraic, $T \exp iah = \exp iTh$. Hence

$$\| \exp iTh \| = \| T \exp iah \| \leq \| \exp iah \| = 1 \quad \forall a \in R,$$

i.e.

$$\| \exp iTh \| = 1$$  

and $TH(A) = H(A)$.

This proves (i).

(ii) Let $a \in G(A)$. Then $[a, [a]] = 1$ since $A = A^\pi$. Hence $\| Ta \| \leq \| [a, a] \| = 1$ and $\| Ta^{-1} \| \leq \| [a, a] \| = 1$. But $Ta^{-1} = (Ta)^{-1}$. Therefore $[Ta^{-1}] = [Ta^{-1}] = 1$ and (ii) is proved.

Before proceeding with our investigation, we shall use 3.1 to show that norm decreasing is sufficient for Corollary 4.5.6 of [3] to hold.

3.2. Theorem. Any norm-decreasing isomorphism $T$ between two $B^\pi$ algebras $A$ and $A$ is a *.

Proof. It is known that if $A$ is a Banach algebra with an approximate identity and $\tilde{A}$ is the unitization of $A$, then

$$H(A) = A \cap H(\tilde{A})$$  

(1)  

Let $T$ be an algebra isomorphism of a Banach algebra $A$ onto another $A$ and $T$ be defined thus:

$$\tilde{T}(a, a) = (Ta, a), \quad \varphi \circ A^\pi \circ \varphi \circ A^\pi \circ \varphi.$$  

Clearly, $\tilde{T}$ is an algebra isomorphism of $\tilde{A}$ and $\tilde{A}$ and it is norm-decreasing. The norm in $\tilde{A}$ is defined by

$$\| (a, a) \| = \| a \| + |a|.$$  

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Norm-decreasing isomorphism
Hence, if $A_i$ ($i = 1, 2$) has an approximate identity, we have
\[
TH(A_i) = \tilde{H}(A_i) = \tilde{H}(A_i) \ast H(A_i)
\]
(1)
\[
\leq T(A_i) \ast TH(A_i)
\]
(2)
\[
\leq A_i \ast H(A_i)
\]
by 3.1
(3)
\[
= H(A_i)
\]
(4)

Since a $B^\ast$ algebra has an approximate identity, $T$ then maps a hermitian element to a hermitian element. But an element of a $B^\ast$ algebra is hermitian iff it is self-adjoint (see 2.1.3 of [3]). Hence $T$ is a $\ast$.

3.5. Lemma. Let $A_i$ ($i = 1, 2$) be a Banach algebra with a minimal approximate identity. Suppose that $T$ is a norm decreasing algebra isomorphism of $A_i$ onto $A_i$. Then $T^{\ast m}$, the induced map of $A_i$ onto $A_i$, is a norm-decreasing extension of $T$.

Proof. Let $\varphi_i A_i$, $T^{\ast m}$ be given by
\[
T^{\ast m} \varphi_i = T \varphi_i T^{-1}
\]
(2)

We shall only show that $T^{\ast m}$ is a norm-decreasing extension of $T$ as other properties of an algebra isomorphism can easily be verified. Let $\{a_i\}$ be a minimal approximate identity in $A_i$. Then $\{T(a_i)\}$ is an approximate identity in $A_i$ and $\|T(a_i)\| \leq 1$. Let $a' \in A_i$; having
\[
(T^{\ast m} \varphi_i)(T(a_i) \ast a') = (T \varphi_i T^{-1})(T(a_i) \ast a') = (T \varphi_i T^{-1} T(a_i)) \ast a' = (T \varphi_i a_i) \ast a'.
\]

Hence
\[
\|T^{\ast m} \varphi_i(T(a_i) \ast a')\| = \|T \varphi_i a_i \ast a'\|
\]
\[
\leq \|T \| \|T \varphi_i\| \|a_i\| \|a'\|
\]
(5)
\[
\leq \|T \| \|a'\|.
\]

Also
\[
\liminf \|T^{\ast m} \varphi_i(T(a_i) \ast a')\| = \|T \varphi_i a_i \ast a'\|
\]
\[
\leq \|T \varphi_i\| \|a'\|
\]

Therefore $\|T^{\ast m} \varphi_i\| \leq \|T \varphi_i\|$.

Since $\{L_a\}$ is a strong operator dense in $A_i$ (see 2.5) and $\|a\| = \|L_a\|$, $\{a\}$ is strong operator dense in $A_i$.

3.4. Theorem. Suppose $T$ is a norm-decreasing isomorphism of $A_i$ onto $A_i$, as in Lemma 3.3 above. Then

(i) $TH(A_i) \subseteq H(A_i)$ and

(ii) $T^{\ast m} G(A_i) \subseteq G(A_i)$.
Preservation of $G(A)$ by a norm-decreasing $T$ implies the preservation of $H(A)$ as we now show.

3.7. **Theorem.** Let $T$ be a norm-decreasing algebra automorphism of a complex unital Banach algebra. Then $TG(A) = G(A)$ implies $TH(A) = H(A)$.

**Proof.** Let $G_t(A)$ be the group generated by $E = \{\exp\{it\} : t \in H(A)\}$. Then $G_t(A) \subseteq G(A)$ since $\|\exp\{it\}\| = 1$. In fact, $T(G_t(A)) \subseteq G_t(A)$, for let $e = \exp\{it\}$. Then $Te = T(\exp\{it\}) = \exp\{iTh\}$. But $Th \in H(A)$ by 3.1. Hence $Te \in G_t(A)$. Since $G_t(A) \subseteq TG_t(A)$ by hypothesis, $T^{-1}a \in G_t(A)$ and $T^{-1}a = \exp\{i^{-1}h\}$. Therefore, $\|\exp\{i^{-1}h\}\| = \|T^{-1}a\|^{-1} = 1$, i.e. $T^{-1}a \in H(A)$ by 3.2. Combining this with 3.1, we have $TH(A) = H(A)$ and $TG_t(A) = G_t(A)$.

3.8. **Remark.** The converse of 3.7 is not known. It is, however, clear that $G_t(A)$ is not necessarily dense in $G(A)$ for, let $A = C[0, 1]$ (the unit circle in the complex plane). $G(A) = \{f \in C[0, 1] : \|f\| = 1\}$ and $E = \{\exp\{ig\} : g \in C[0, 1] \}$ is a real-valued function in $G(A)$. Let $x \in X$.

We shall conclude this paper with an example to show that norm decreasing isomorphism does not in general, preserve $G(A)$ and $H(A)$.

3.9. **Example.** Let $C'$ be the Banach space of all sequences of bounded variation with norm defined by

$$\|x'\| = \|x\| + \sum_{n=1}^{\infty} \|x_{n+1} - x_n\|$$

and $C''$, the Banach space of all convergent sequences with supremum norm

$$\|x''\| = \sup_{n} |x'_n|$$

for $x' \in C'$.

$C'$ is clearly a Banach algebra under pointwise multiplication. $C'$ is also a Banach algebra under pointwise multiplication, for it can be shown by induction that

$$\left(\|x\| + \sum_{n=1}^{k} \|x_{n+1} - x_n\|\right) \left(\|y\| + \sum_{n=1}^{k} \|y_{n+1} - y_n\|\right) \geq \|x\|\|y\| + \sum_{n=1}^{k} \|x_{n+1} - x_n\| \|y_{n+1} - y_n\|$$

for all positive integers $k$.

If we allow $k$ to tend to infinity, then

$$\|x'y'\| \leq \|x\|\|y\|,$$

$C', y' \in C'$.

All other properties are easily verified. $C'$ and $C''$ are then complex unital Banach algebras with sequence (1) as the unit element and $e_n$ as the basic elements in each of them ($e_n$ is the sequence with 0 in every entry but the nth which is 1 and (1) is the sequence with 1 in every entry).

We now define the Banach algebra $A$ as the direct sum of $C'$ and $C''$ ($A = C' \oplus C''$) with norm defined by

$$\|(x', y')\| = \max \{\|x'\|, \|y'\|\}, \quad (x', y') \in A, \quad x' \in C', \quad y' \in C''.$$

We define a map $T$ on $A$ thus:

$$T(x', y') = (U', U''),$$

where

$$U'_n = a'_{n+1}, \quad n \geq 1,$$

$$U''_n = a''_{n-1}, \quad n > 1.$$

Clearly, $T$ is linear, multiplicative and one-to-one. It is also easy to show that $T$ is onto $U$ is norm-decreasing since

$$\|(x', y')\| \geq \max \left\{ |x'_n| + \sum_{n=1}^{m} |x'_{n+1} - x'_n|, \sup |x'_n| \right\}$$

$$\geq \max \left\{ |x'_n| + \sum_{n=1}^{m} |x'_{n+1} - x'_n|, \max |x'_n|, \sup |x'_n| \right\}$$

$$= \max \{\|U'\|, \|U''\|\}$$

$$= \|T(x', y')\|.$$

Using the fact that $x \in H(C')$ implies $\lim_{n \to \infty} |\langle x, e_{n+1} \rangle - 1| = 0$, it is easy to show that $H(C')$ is the set of all real scalar multiples of the identity in $C'$ and $H(C'')$ is the set of all real convergent sequences in $C''$.

Using the fact that $x \in H(C) \Rightarrow \langle x, e_{n+1} \rangle = 1$, it is also easy to show that $H(C') \cong H(C'')$.

Suppose

$$\langle \lambda, (x) \in H(C) \rangle$$

Then $(\langle \lambda, (x) \in H(C) \rangle$ and, by definition,

$$T((\langle \lambda, (x) \in H(C) \rangle,$$

where

$$\lambda \in \mathbb{C}$$

and

$$\left\{ v_n = \mu_n, \quad n > 1. \right.$$
It can also be shown, by calculation, that
\[ G(C') = \{ x' \in C': x' = (e^{\alpha}) \}, \alpha \in \mathbb{R} \]
and
\[ G(C'') = \{ x'' \in C'' : x'' = (e^{2\alpha}) \}. \]
It is easy to show that
\[ \{ G(C') \cap G(C'') \} = G(A). \]
Hence
\[ T(e^{\alpha}), (e^{2\alpha}) = \{(e^{\alpha}), (k_{n})\}, \]
where
\[ k_{1} = e^{\alpha} \quad \text{and} \quad k_{n} = e^{2^n} \quad \text{for} \quad n > 1. \]
Since not every element of \( G(A) \) is of the form \( \{(e^{\alpha}), (k_{n})\}, TG \neq G. \)

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A converse to some inequalities and approximations in the theory of Stieltjes and stochastic integrals, and for \( n \)th derivatives

by

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Abstract. The object of this report is to establish by counter-examples the best possible character of theorems recently obtained about stochastic integrals and Stieltjes integrals, and about \( n \)th derivatives and finite differences. The hypotheses involve a pair of estimate functions subject to the convergence of a corresponding integral or \( Y \)-series, and it is shown that the divergence of this integral or series render in each case the conclusion false.

I. Our notation will be largely that of [6], [7]. Let \( m \) be a positive integer, and let \( \varphi(u), \psi(u) \) be functions defined for \( 0 \leq u \leq 1 \), such that
\[ \varphi(u) = \text{non-negative and Borel measurable}, \psi \text{ is continuous and monotone increasing, and takes the value } \psi(u) = 0 \text{ only at } u = 0; \]
and suppose that, for \( 0 < \lambda < 1 \),
\[ \psi(\lambda u) \geq (\lambda^p) \psi(u), \quad \varphi(\lambda u) \leq \frac{1}{\lambda} \psi(u). \]

We denote by \( \{a\} \) a decreasing sequence \( a_{n} \) \( (n = 0, 1, \ldots) \) with limit 0 and with initial term \( a_{0} \leq 1 \). We write
\[ Y = \sum_{n=0}^{\infty} \varphi(a_{n}) \psi(a_{n}), \]
and we denote by \( Y_{v} \) its partial sum for \( 0 < v < N \). For \( n = 1, a_{n} = 2^{-n} \), the series (1.2) occurs in [3] and it then converges or diverges with the sum \( \sum_{1}^{\infty} \varphi(1/n) \psi(1/n) \) previously introduced in [3].

We say that the sequence \( \{a\} \), or the \( Y \)-series \( Y \), satisfies the condition \( C(1) \), if for each \( v \) the ratio \( a_{n+1}/a_{n} \) is an integer expressible as an integer power of 2, and satisfies the condition \( C(2) \) if
\[ 2 \psi(a_{n}) < \psi(a_{n+1}) \leq 8 \psi(a_{n}). \]

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