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On isometries in linear metric spaces

by

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Abstract. It is proved that if (X, d) is a Montel locally convex linear metric space and T is an isometry from X onto another linear metric space with $T(0) = 0$, then T is linear. Also, some other problems related to the question whether every isometry is linear are discussed.

Introduction. We shall be concerned with

(H) CONJECTURE. *Let (X, d) be a linear metric space. Then every isometry from (X, d) onto an arbitrary linear metric space (Y, h) with $T(0) = 0$ is linear.*

This conjecture has been substantiated under some additional assumptions. For example: Mazur and Ulam [5] have proved that (H) holds provided that X is a linear normed space and the metric d is that induced by the norm on X (cf. Lemma 1). Charzyński [1] has proved that (H) holds under the assumption that (X, d) is a finite-dimensional linear metric space. The original (complicated) proof of Charzyński has essentially been simplified by Wobst in [6]. In this paper, using an argument similar to that of Wobst, we establish (H) under the assumption that (X, d) is an arbitrary Montel locally convex space.

The complete list of references on the subject can be found in [6].

Preliminaries. By a *linear metric space* (X, d) we shall mean a linear space X endowed with a translation invariant metric d, i.e., with a metric d satisfying the condition

$$d(x, 0) = d(x+y, y) \quad \text{for every } x, y \in X$$

and such that X endowed with the topology induced by the metric d is a linear topological space.

Let (X, d) and (Y, h) be linear metric spaces. By a *surjective isometry* between (X, d) and (Y, h) we shall mean a mapping T from X onto Y such that the equality $d(u, v) = h\big(T(u), T(v)\big)$ holds for every $u, v \in X$. The set of all surjective isometries between (X, d) and (Y, h) which map the origin onto the origin will be denoted by $I\big((X, d), (Y, h)\big)$ and the set of all isometries from (X, d) onto itself will be denoted by $I(X, d)$. If the metrics d and h are fixed, we shall often write simply $I(X, Y)$ and $I(X)$ instead of $I\big((X, d), (Y, h)\big)$ and $I(X, d)$, respectively.

For every linear metric space (X, d), by $\mathrm{Inv}(X)$ we shall denote the family of all isometrically invariant subsets in (X, d), i.e., $A \in \mathrm{Inv}\,X$ if and only if $T(A) = A$ for every $T \in I(X)$.

By $\boldsymbol{R}$, $\boldsymbol{Z}$, $\boldsymbol{N}$ we shall denote the sets of reals, integers and natural numbers, respectively.

We shall consider only the linear spaces over reals.

We begin with the following lemma (Charzyński [1]), which reduces the conjecture (H) to the problem concerning the linearity of isometries in $I(X)$.

LEMMA 1. *Let (X, d) be a linear metric space. Then the following two statements are equivalent:*

(i) *for every linear metric space (Y, h), the condition $T \in I(X, Y)$ implies that T is linear,*

(ii) *$T \in I(X)$ implies that T is linear.*

Proof. Obviously, (i)⇒(ii). To prove that (ii)⇒(i), let (X, d) and (Y, h) be arbitrary linear metric spaces and fix an arbitrary isometry $T \in I(X, Y)$. For every $y \in X$ define

$$U_y(x) = T^{-1}\big(T(x)+T(y)\big)-y \quad \text{for} \quad x \in X.$$

Set

$$S(x, y) = U_y(x)-x \quad \text{for} \quad x, y \in X.$$

We have

$$(1) \quad S(x, y) = T^{-1}\big(T(x)+T(y)\big)-(x+y) = S(y, x) \quad \text{for every } x, y \in X.$$

Obviously, $U_y \in I(X)$ for every $y \in X$. Hence, by (ii), $U_y\colon X \to X$ is linear for each fixed $y \in X$. Therefore $S(x, y)$ is linear in x. By (1), $S(x, y)$ is also linear in y. We shall prove that $S(x, y) = 0$ for every $x, y \in X$. To this end take an arbitrary $t \in \boldsymbol{R}$. Then we have

$$\begin{aligned} d(tx, 0) &= d\big(U_{t^{-1}y}(tx), 0\big) \geqslant d\big(U_{t^{-1}y}(tx)-tx, 0\big)-d(tx, 0) \\ &= d\big(S(tx, t^{-1}y), 0\big)-d(tx, 0) = d\big(S(x, y), 0\big)-d(tx, 0). \end{aligned}$$

Letting t tend to zero, we obtain

$$S(x, y) = 0 \quad \text{for every } x, y \in X.$$

By (1), this means that

$$T^{-1}\big(T(x)+T(y)\big) = (x+y) \quad \text{for every } x, y \in X.$$

Hence

$$T(x+y) = T(x)+T(y) \quad \text{for every } x, y \in X.$$

Since T is continuous and additive, we infer that T is linear, which comdletes the proof of the lemma.

Isometrically invariant sets in (X, d).

LEMMA 2. *Let (X, d) be a linear metric space and let $A \in \mathrm{Inv}(X)$. Then*

(i) $T(x+A) = T(x)+T(A) = T(x)+A$ *for* $x \in X$ *and* $T \in I(X)$,

(ii) *A is symmetric* (i.e., $A = -A$).

Proof. (i). Fix $x \in X$ and $T \in I(X)$. The statement follows immediately from the definition of the family $\mathrm{Inv}(X)$ applied to the isometry $\tilde{T} \in I(X)$ defined by

$$\tilde{T}(u) = T(x+u)-T(x) \quad \text{for} \quad u \in X.$$

(ii) follows from the fact that $-I \in I(X)$, where $-I(x) = -x$ for $x \in X$.

LEMMA 3. *Let (X, d) be a linear metric space. Then*

(i) *every ball B open or closed with the centre at the origin is isometrically invariant* (i.e., $B \in \mathrm{Inv}(X)$),

(ii) *if $A_t \in \mathrm{Inv}(X)$ for $t \in \mathscr{T}$, where $\mathscr{T}$ is an arbitrary set of indices, then $\bigcap_{t \in \mathscr{T}} A_t \in \mathrm{Inv}(X)$,*

(iii) *if $A \in \mathrm{Inv}(X)$, then $\overline{A} \in \mathrm{Inv}(X)$ ($\overline{A}$ denotes the closure of A),*

(iv) *if $A_1, A_2, \ldots, A_n \in \mathrm{Inv}(X)$, then $A_1+A_2+\ldots+A_n \in \mathrm{Inv}(X)$.*

Proof. (i), (ii) and (iii) are trivial. To prove (iv) it suffices to show that $A+B \in \mathrm{Inv}(X)$, provided that $A, B \in \mathrm{Inv}(X)$. To this end take $A, B \in \mathrm{Inv}(X)$ and observe that, by Lemma 2 (i), for every $T \in I(X)$, we have

$$T(A+B) = \bigcup_{x \in A} T(x+B) = \bigcup_{x \in A} T(x)+B = T(A)+B = A+B,$$

which means that $A+B \in \mathrm{Inv}(X)$, which completes the proof of the lemma.

We recall that a subset A of a linear topological space $(X, \mathscr{T})$ is said to be *bounded* if and only if for every neighbourhood U of the origin we have $A \subset tU$ for sufficiently large $t \in \boldsymbol{R}$.

The following lemma shows that in every locally convex linear metric space the family of isometrically invariant subsets is relatively rich.

LEMMA 4. *Let (X, d) be a locally convex linear metric space. Then for every bounded subset B of X there exists a closed bounded isometrically invariant subset A of X which contains B.*

Proof. Take a bounded subset B in (X, d). For every $t > 0$, define

$$K_t = \{x \in X\colon d(x, 0) \leqslant t\},$$

and

$$n_t = \min\{n \in \boldsymbol{N}\colon B \subset \underbrace{K_t+K_t+\ldots+K_t}_{n \text{ times}}\}.$$

Since B is bounded, we infer that for every $t>0$ the number n_t is finite. Next, we define

$$W_t = \underbrace{K_t+K_t+\ldots+K_t}_{n_t \text{ times}}$$

for every $t>0$ and finally we set

$$B' = \bigcap_{t>0} W_t.$$

It follows from the definition of B' and from the previous lemma that B' is isometrically invariant and that $B\subset B'$. We shall show that B' is bounded. To this end take any convex symmetric neighbourhood V of the origin in (X,d) and let $t_0>0$ be such that $K_{t_0}\subset V$. Then we have

$$B' = \bigcap_{t>0} W_t \subset W_{t_0} = \underbrace{K_{t_0}+K_{t_0}+\ldots+K_{t_0}}_{n_{t_0} \text{ times}} \subset \underbrace{V+V+\ldots+V}_{n_{t_0} \text{ times}} = n_{t_0}V,$$

which completes the proof of the boundedness of B'. To complete the proof it suffices, by Lemma 3 (iii), to define A as the closure of B'.

2-extremal points. Let A be a subset of a linear space X. Then a point $x\in A$ is said to be a *2-extremal point of* A if and only if whenever $x=2^{-1}(x_1+x_2)$ for some $x_1,x_2\in A$, then $x_1=x_2=x$. The set of all 2-extremal points of A will be denoted by $\mathrm{Ex}_2(A)$.

LEMMA 5. *Let (X,d) be a locally convex linear metric space and let A be an isometrically invariant subset of X. Then for every 2-extremal point x of A and for every isometry $T\in I(X)$ we have*

$$T(x+u) = T(x)+T(u) \quad \textit{for all } u\in X.$$

Proof. Fix an arbitrary $x\in \mathrm{Ex}_2(A)$. We have

$$(x+A)\cap(-x+A) = \{0\}.$$

Indeed, put $S=(x+A)\cap(-x+A)$. By Lemma 2 we infer that $-x\in A$. Hence $0\in S$. On the other hand,

$$-S = -\big((x+A)\cap(-x+A)\big) = (-x-A)\cap(x-A) = (-x+A)\cap(x+A) = S.$$

Thus S is symmetric. Let $u\in S$. Then also $-u\in S$. Hence $x_1=x+u\in A$ and $x_2=x-u\in A$. But $2^{-1}(x_1+x_2)=x$, and therefore, by the definition of 2-extremal points, $x_1=x_2=x$. This implies that $u=0$. Hence

$$S=(x+A)\cap(-x+A)=\{0\}.$$

Now, fix an arbitrary isometry $T\in I(X)$. We have

$$T(S) = \{0\} = T(x+A)\cap T(-x+A).$$

Since $A\in \mathrm{Inv}(X)$ and by Lemma 2 (i), we obtain

$$\big(T(x)+T(A)\big)\cap\big(T(-x)+T(A)\big) = \{0\}. \tag{2}$$

On the other hand, by the symmetry of $A=T(A)$, we have $T(-x)\in A$ and

$$T(x)+T(-x)\in \big(T(x)+T(A)\big)\cap\big(T(-x)+T(A)\big). \tag{3}$$

(2) and (3) imply

$$T(-x) = -T(x).$$

Now, define

$$\tilde{T}(u) = T(x+u)-T(x) \quad \text{for} \quad u\in X.$$

Then $\tilde{T}\in I(X)$. Applying the same argument as before to the isometry $\tilde{T}$, we obtain

$$\tilde{T}(-x) = -\tilde{T}(x). \tag{4}$$

But $\tilde{T}(-x)=-T(x)$ and $\tilde{T}(x)=T(2x)-T(x)$. This and (4) give

$$T(2x) = 2T(x).$$

In the same manner, putting $T_n(u)=T(nx+u)-T(nx)$, one can prove by induction that

$$T(nx) = nT(x) \tag{5}$$

for every $n\in \mathbf{Z}$.

Applying the same argument to the isometry U_y defined by

$$U_y(u) = T(u+y)-T(y) \quad \text{for} \quad u\in X,$$

where y is an arbitrary fixed point in X, we get by (5)

$$T(nx+y) = n\big(T(x+y)-T(y)\big)+T(y) \tag{6}$$

for every $n\in\mathbf{Z}$ and every $y\in X$.

Take an arbitrary $u\in X$. By Lemma 4, there exists a bounded subset $A_1\in \mathrm{Inv}(X)$ which contains u. Hence $mx+u\in mx+A_1$ for every $m\in\mathbf{Z}$. By Lemma 2 (i)

$$T(mx+u)\in T(mx)+A_1 \quad \text{for every } m\in\mathbf{Z}. \tag{7}$$

By (5), (6) and (7) we infer that, for every $m\in\mathbf{Z}$,

$$\begin{aligned} m\big(T(x+u)-T(x)-T(u)\big) &= m\big(T(x+u)-T(u)\big)+T(u)-T(u)-mT(x) \\ &= T(mx+u)-T(u)-T(mx)\in T(mx)+A_1+A_1-T(mx) \\ &= A_1+A_1. \end{aligned} \tag{8}$$

Since A_1 (and therefore A_1+A_1) is bounded, (8) implies that the set $\{m(T(x+u)-T(x)-T(u)): m\in \boldsymbol{Z}\}$ is bounded. Hence $T(x+u)-T(x)-T(u)=0$, i.e.,

$$T(x+u) = T(x)+T(u)$$

for every $u\in X$, which completes the proof of the lemma.

Main results. We begin with

LEMMA 6. *Let (X, d) be a linear metric space and let*

$$B = \{x\in X: T(x+u) = T(x)+T(u) \text{ for every } T\in I(X) \text{ and } u\in X\}.$$

Then B is closed and isometrically invariant in (X, d).

Proof. It follows from the continuity of isometries that B is closed. We shall show that $T(B) = B$ for every $T\in I(x)$. Indeed, fix $T\in I(X)$ and $x\in B$ and consider an arbitrary isometry $I\in I(X)$. Since $I\circ T\in I(X)$, we have

$$(9)\qquad (I\circ T)(x+z) = (I\circ T)(x)+(I\circ T)(z)$$

for every $z\in X$. On the other hand,

$$(10)\qquad (I\circ T)(x+z) = I\big(T(x+z)\big) = I\big(T(x)+T(z)\big).$$

Putting $z = T^{-1}(u)$, we obtain, by (9) and (10), that

$$I\big(T(x)+u\big) = I\big(T(x)\big)+I(u)$$

for every $u\in X$ and every $I\in I(X)$. This means that $T(x)\in B$ for every $T\in I(X)$. Hence $T(B)\subset B$ for every $T\in I(X)$. Putting $T = U^{-1}$, we infer that $B\subset U(B)$ for every $U\in I(X)$. Hence $B\in \operatorname{Inv}(X)$, which completes the proof of the lemma.

The main result of the paper is the following

THEOREM 1. *Let (X, d) be a locally convex linear metric space and let (Y, h) be a linear metric space. Assume that, for every $x\in X$, at least one of the following conditions holds:*

($*$) *the set the 2-extremal points of*

$$A(x) = \{Tx\in X: T\in I(X)\}$$

is non-empty,

($**$) *the set of the 2-extremal points of the closure of $A(x)$ is non-empty,*

($_*{}^*{}_*$) *the set of the 2-extremal points of the closure of the set $\{\tilde{T}(x)\in \tilde{X}: \tilde{T}\in I(\tilde{X}, \tilde{d})\}$ is non-empty, where $(\tilde{X}, \tilde{d})$ denotes the completion of (X, d).*

Then every surjective isometry between (X, d) and (Y, h) which maps the origin onto the origin is linear.

Proof. By Lemma 1, it suffices to prove that every $T\in I(X)$ is linear. To this end, it is enough to prove that, for every $T\in I(X)$,

$$T(x+u) = T(x)+T(u) \quad \text{for every } x, u\in X.$$

Fix arbitrary $x, u\in X$ and $T\in I(X)$.

Assume that ($*$) holds for the x fixed above. Let $z\in \operatorname{Ex}_2(A(x))$. Obviously, $A(x)\in \operatorname{Inv}(X)$. Hence, by Lemma 5,

$$T(z+y) = T(z)+T(y) \quad \text{for all} \quad y\in X.$$

Since $z\in A(x)$, we have $x = U(z)$ for some $U\in I(X)$. Thus, by Lemma 6

$$T(x+u) = T(x)+T(u),$$

which completes the proof of the first part of the theorem.

Now, assume that ($**$) holds for the x fixed above. Let $z\in \operatorname{Ex}_2(\overline{A(x)})$. Then there exist $\{z_n: n\in \boldsymbol{N}\}\subset X$ and $\{T_n: n\in \boldsymbol{N}\}\subset I(X)$ such that

$$(11)\qquad T_n(x) = z_n \quad \text{for every } n\in \boldsymbol{N},$$

$$(12)\qquad z_n\to z \quad \text{as} \quad n\to\infty.$$

Define $x_n = T_n^{-1}(z)$ for $n\in \boldsymbol{N}$. By (11), we have

$$d(x, x_n) = d\big(T_n(x), T_n(x_n)\big) = d(z_n, z)$$

and, by (12), we conclude that $x_n\to x$ as $n\to\infty$. Since $z\in \operatorname{Ex}_2(\overline{A(x)})$, by Lemma 5, we deduce that

$$T(z+y) = T(z)+T(y)$$

for every $y\in X$ (by Lemma 3 (iii), we infer that $\overline{A(x)}\in \operatorname{Inv}(X)$). Hence, by Lemma 6,

$$(13)\qquad T(x_n+y) = T(x_n)+T(y)$$

for every $y\in X$ and every $n\in \boldsymbol{N}$. Finally, by the continuity of T and because of $x_n\to x$ and (13), we obtain

$$(14)\qquad T(x+u) = T(x)+T(u),$$

which completes the second part of the proof of the theorem.

Finally, assume that ($_*{}^*{}_*$) holds for the x fixed above. To establish (14), extend T tothe isometry $\tilde{T}\in I(\tilde{X}, \tilde{d})$ and apply the same argument as in the case ($**$). This completes the proof of the theorem.

We recall that a linear topological space X is said to be a *Montel space* iff every bounded subset of X is precompact.

PROPOSITION 1. *Let (X, d) be a Montel locally convex linear metric space. Then for every $x\in X$ the condition ($_*{}^*{}_*$) of Theorem 1 is satisfied.*

Proof. It suffices to show that every closed bounded subset A of a Montel, complete, locally convex linear metric space admits a 2-extremal point. But then A is compact and the desired result follows from the Krein–Milman Theorem (cf. [2]).

By Theorem 1 and Proposition 1 we obtain

THEOREM 2. *Let (X, d) be a Montel locally convex linear metric space and let (Y, h) be a linear metric space. Then every isometry between (X, d) and (Y, h) which maps the origin onto the origin is linear.*

Theorem 1 shows that the technique of studying 2-extremal points is very useful in dealing with the problem of isometries. It can be shown that under some assumptions on the space (X, d), the condition $(*)$ of Theorem 1 is equivalent to the fact that every isometry from (X, d) onto another linear metric space which maps the origin to the origin is linear. This is the case where the topology of (X, d) can be generated by a uniformly convex norm on X. To prove this we shall need some additional results.

We shall say that an ordered pair (x_1, x_2) in a normed linear space is a *$(1, t)$-part of a tree* iff $\|x_1 - x_2\| \geqslant t$. Now if (n, t)-part of a tree is defined, we shall say that a 2^{n+1}-tuple $(x_1, x_2, \ldots, x_{2^{n+1}})$ is an *$(n+1, t)$-part of a tree* iff $\|x_{2j-1} - x_{2j}\| \geqslant t$, $1 \leqslant j \leqslant 2^n$ and the 2^n-tuple

$$\left(\frac{x_1+x_2}{2}, \frac{x_3+x_4}{2}, \ldots, \frac{x_{2^{n+1}-1}+x_{2^{n+1}}}{2}\right)$$

is an *(n, t)-part of a tree*. We shall say that a normed linear space $(B, \|\ \|)$ has the *finite tree property* iff there exists a $t > 0$ such that for every $n \in N$ there exists an (n, t)-part of a tree where all elements have a norm at most 1.

The following theorem is due to Enflo [3].

THEOREM 3. *A normed linear space can be endowed with an equivalent uniformly convex norm if and only if it does not have the finite tree property.*

PROPOSITION 2. *Let (X, d) be a linear metric space such that every isometry $T \in I(X)$ is linear. Then every $x \in X$ is a 2-extremal point of the set $A_x = \{T(x) \in X\colon T \in I(X)\}$ provided that there exists a uniformly convex norm $\|\cdot\|$ on X which induces the topology of X.*

Proof. Let $\|\cdot\|$ be a uniformly convex norm on X. Let $x \in X$. Since $A_x \in \mathrm{Inv}(X, d)$ and A_x is the smallest isometrically invariant set containing x, we deduce (by Lemma 4) that A_x is bounded. Without loss of generality we may assume that

$$\sup\{\|y\|\colon y \in A_x\} \leqslant 1. \tag{15}$$

Assume that $x = \frac{1}{2}(x_1 + x_2)$ where $x_1, x_2 \in A_x$ and $x_1 \neq x \neq x_2$ (i.e., x is not a 2-extremal point of A_x). Let $d(x_1, x_2) = r > 0$. Since the metric induced by the norm $\|\cdot\|$ is equivalent to d, we infer that there exists a $t > 0$ such that $d(y, z) \geqslant r$ implies $\|y - z\| \geqslant t$. Thus (x_1, x_2) is a $(1, t)$-part of a tree. By the definition of A_x, there exists a $T_1 \in I(X, d)$ such that $x_1 = T_1(x)$. Let $x_1^2 = T_1(x_1)$ and $x_2^2 = T_1(x_2)$. Since T_1 is linear, we have

$$\frac{x_1^2+x_2^2}{2} = \frac{T_1(x_1)+T_1(x_2)}{2} = T\left(\frac{x_1+x_2}{2}\right) = T(x) = x_1.$$

Similarly, let $T_2 \in I(X, d)$ be such that $x_2 = T_2(x)$. Let $x_3^2 = T_2(x_1)$ and $x_4^2 = T_2(x_2)$. As before, we conclude that

$$\frac{x_3^2+x_4^2}{2} = x_2.$$

Since $T_1, T_2 \in I(X, d)$, we have $d(x_1^2, x_2^2) = d(x_3^2, x_4^2) = d(x_1, x_2)$. Hence $\|x_1^2 - x_2^2\| \geqslant t$ and $\|x_3^2 - x_4^2\| \geqslant t$ and therefore the quadruple $(x_1^2, x_2^2, x_3^2, x_4^2)$ is a $(2, t)$-part of a tree. On the other hand, observe that $x_i^2 \in A_x$ for $i = 1, 2, 3, 4$ and, by (15), we deduce that

$$\sup\{\|x_i^2\|\colon i = 1, 2, 3, 4\} \leqslant 1.$$

Now, applying the same argument to x_1^2, x_2^2, x_3^2, and x_4^2 independently, one can define a $(3, t)$-part of a tree with all elements of the norm at most 1, and, in the same manner, one can define by induction an (n, t)-part of a tree with all elements of the norm at most 1, for $n = 4, 5, \ldots$ Thus $(X, \|\ \|)$ has the finite tree property and therefore, by Theorem 3, $\|\ \|$ cannot be uniformly convex, a contradiction. Hence x is a 2-extremal point of A_x, which completes the proof of the proposition.

By Theorem 1 and Proposition 2 we have

THEOREM 4. *Let (X, d) be a linear metric space whose topology can be generated by a uniformly convex norm. Then the following two statements are equivalent:*

(i) *Every $x \in X$ is a 2-extremal point of the set $A_x = \{T(x)\colon T \in I(X, d)\}$.*

(ii) *Every isometry from (X, d) onto another linear metric space (Y, h) which maps the origin onto the origin is linear.*

Note that combining the technique used in this paper with the argument of the proof of Theorem 2 in [4] one can obtain the following result:

THEOREM 5. *Let V be an open, connected subset of a Montel locally convex linear metric space (X, d). Then every isometry T from V onto an open subset U of a linear metric space (Y, h) can be uniquely extended to an affine isometry $\tilde{T}$ between (X, d) and (Y, h).*

Isometric embeddings in Montel locally convex linear metric spaces. The following fact is well known (cf. [6], proof of Theorem 2).

THEOREM 6. *Let (K, d) be a compact metric space and let J be an isometry from K into itself. Then J is surjective.*

The aim of this section is to prove

THEOREM 7. *Let (X, d) be a complete, Montel, locally convex linear metric space. Then every isometry from (X, d) into itself is surjective and affine.*

Remark. The finite-dimensional version of Theorem 7 is a well-known consequence of the Invariant Domain Theorem (cf. [5]).

For a linear metric space (X, d) we denote by $J(X)$ the set of all isometric embeddings of X into itself which fix the origin. Define

$$\mathrm{Inv}^*(X) = \{A \subset X\colon J(A) \subset A \text{ for every } J \in J(X)\}.$$

Using a similar argument as in the proof of Lemma 2, one can prove

LEMMA 7. *Let (X, d) be a linear metric space and let $A \in \mathrm{Inv}^*(X)$. Then*

(i) $T(x+A) \subset T(x)+A$ *for* $x \in X$ *and* $T \in J(X)$,

(ii) A *is symmetric.*

The following two lemmas show that the family $\mathrm{Inv}^*(X)$ is relatively rich.

LEMMA 8. *Let (X, d) be a linear metric space. Then*

(i) *every ball (open or closed) with the centre at the origin belongs to* $\mathrm{Inv}^*(X)$,

(ii) *if* $A_t \in \mathrm{Inv}^*(X)$ *for* $t \in \mathcal{T}$, *where* $\mathcal{T}$ *is an arbitrary set of indices, then* $\bigcap_{t\in\mathcal{T}} A_t \in \mathrm{Inv}^*(X)$,

(iii) *if* $A \in \mathrm{Inv}^*(X)$, *then* $\bar{A} \in \mathrm{Inv}^*(X)$,

(iv) *if* $A_1, A_2, \ldots, A_n \in \mathrm{Inv}^*(X)$, *then* $A_1+A_2+\ldots+A_n \in \mathrm{Inv}^*(X)$,

(v) $\mathrm{Inv}^*(X) \subset \mathrm{Inv}(X)$.

Proof. The proofs of (i)–(iv) are, in fact, the same as in the case of Lemma 3. To prove (v) observe that if $A \in \mathrm{Inv}^*(X)$ then both $T(A) \subset A$ and $T^{-1}(A) \subset A$ for every $T \in I(X)$. But $T^{-1}(A) \subset A$ implies $A \subset T(A)$. Hence $T(A) = A$ for every $T \in I(X)$, which completes the proof of the lemma.

LEMMA 9. *For every bounded subset B in a locally convex linear metric space (X, d), there exists a closed bounded subset $A \in \mathrm{Inv}^*(X)$ which contains B.*

Proof. Define A as in the proof of Lemma 4. By Lemma 8, A has the required properties.

Proof of Theorem 7. Take an arbitrary isometry $T \in J(X)$. We shall show that T is surjective, i.e., that $x \in T(X)$ for every $x \in X$. To this end, fix an arbitrary $x \in X$ and let A be a closed bounded subset belonging to $\mathrm{Inv}^*(X)$ which contains x (the existence of such a subset is ensured by Lemma 9). By the definition of $\mathrm{Inv}^*(X)$, we infer that T restricted to A is an isometry of A into itself. On the other hand, A is bounded and closed, and therefore compact. Hence, by Theorem 6, every isometry from A into itself is surjective. In particular, $T(A) = A$. Thus, $x \in A = T(A) \subset T(X)$ and T is surjective. To complete the proof of the theorem, observe that every isometry from X into itself is a "translation" of an isometry from $J(X)$ and therefore is surjective and (by Theorem 2) affine.

Remarks added in proof. Recently, R. R. Phelps [8] and R. E. Huff and P. D. Morris [7] have proved that a Banach space has the Radon–Nikodym property if and only if every closed bounded subset of it has an extremal point. Thus, using the same argument as in the proof of Theorem 2, one can show the following

THEOREM 9. *Let $(X, \|\ \|)$ be a Banach space with the Radon–Nikodym property and let d be an arbitrary translation invariant metric generating the same topology as $\|\ \|$. Then every isometry from (X, d) onto another linear metric space which sends the origin to the origin is linear.*

Also, using the results mentioned above, one can prove (cf. Theorem 4)

THEOREM 10. *If $(X, \|\ \|)$ and d are as in the assumption of Theorem 9, then every $x \in X$ is an extremal point (even strongly exposed) of the set* $A(x) = \{T(x) \in X\colon T \in I(X, d)\}$.

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