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Received July 12, 1974 (864)

STUDIA MATHEMATICA, T. LV. (1976)

Toeplitz operators for a certain class of function algebras

by

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Abstract. In this paper we describe a joint approximate spectrum of a p-tuple of Toeplitz operators for a certain class of function algebras (approximating in modulus). We also give a characterization of a C^*-algebra generated by Toeplitz operators modulo commutator ideal.

Let A be a function algebra on a compact Hausdorff space X. Suppose we are given a finite, regular Borel measure $\mu > 0$ on X. Denote by $\mathscr{L}^2(\mu)$ the Hilbert space of all complex-valued μ-square integrable functions. One can define the $H^2(\mu)$ space as the closure of A in $\mathscr{L}^2(\mu)$ and $H^\infty(\mu)$ as the set of all functions $\varphi \in L^\infty(\mu)$ such that $\varphi H^2(\mu) \subset H^2(\mu)$. For $\varphi \in L^\infty(\mu)$ we define the Toeplitz operator T_φ on $H^2(\mu)$ by $T_\varphi f = P(\varphi \cdot f)$, where $f \in H^2(\mu)$ and $P\colon \mathscr{L}^2(\mu) \to H^2(\mu)$ is an orthogonal projection. Denoting by L_φ the operator of multiplication by φ in $\mathscr{L}^2(\mu)$, we can write

$$T_\varphi f = PL_\varphi f.$$

Let $L(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. Denote by $\sigma_\Pi(T_1, \ldots, T_s)$ the joint approximate point spectrum for a system $T_1, \ldots, T_s$ of commuting operators in $L(H)$ (see [1] for the definition). In the case where $X = \Gamma$ is the unit circle, A the disc algebra and μ the Lebesgue measure on X, it is well known that

$$\sigma_\Pi(L_{\varphi_1}, \ldots, L_{\varphi_s}) = \sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}), \text{ where } \varphi_i \in H^\infty(\mu) \text{ for } i = 1, \ldots, s.$$

Let M_{L^∞} be the spectrum of $L^\infty(\mu)$. Since

$$\sigma_\Pi(L_{\varphi_1}, \ldots, L_{\varphi_s}) = \{(\hat{\varphi}_1(m), \ldots, \hat{\varphi}_s(m)),\ m \in M_{L^\infty}\},$$

where $\hat{\varphi}_i$ denotes the Gelfand transform of φ_i, we have

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) = \{(\hat{\varphi}_1(m), \ldots, \hat{\varphi}_s(m)),\ m \in M_{L^\infty}\}.$$

The lemma which we shall now prove shows that the above equality is true in a more general setting.

LEMMA 1. *Let A be a function algebra approximating in modulus* (see [5] for the definition)(1). *If $\varphi_i \in H^\infty(\mu)$ for $i = 1, \ldots, s$, then*

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) = \{(\hat\varphi_1(m), \ldots, \hat\varphi_s(m)),\ m \in M_{L^\infty}\}.$$

Proof. Let $\lambda \notin \sigma_\Pi(L_{\varphi_1}, \ldots, L_{\varphi_s})$, i.e. let there exist $\psi_i \in L^\infty(\mu)$ ($i = 1, \ldots, s$) such that

$$\sum_{i=1}^{s} \psi_i(\lambda_i - \varphi_i) = 1.$$

Hence

$$\sum_{i=1}^{s} T_{\psi_i}(\lambda_i J - T_{\varphi_i}) = I \quad (I \text{ the identity in } L(H)),$$

which implies that $\lambda \notin \sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s})$. Thus we have proved the inclusion

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) \subset \{(\hat\varphi_1(m), \ldots, \hat\varphi_s(m)),\ m \in M_{L^\infty}\}.$$

To prove the inverse inclusion we may assume without loss of generality that $\hat\varphi_i(m_0) = 0$, $i = 1, \ldots, s$. It follows that for every $\varepsilon > 0$ there exists a neighbourhood of m_0, U_{m_0} and $|\hat\varphi_i(m)| \leqslant \varepsilon$ for $m \in U_{m_0}$. Let $\hat\mu$ be a Borel measure on M_{L^∞} induced by μ, i.e.,

$$\hat\mu(\hat f) = \int f\, d\mu, \quad \text{for all } f \in L^\infty(\mu).$$

Let $v \geqslant 0$ be a continuous function on M_{L^∞} satisfying two conditions, $\int v\, d\hat\mu = 1$ and $v(m) = 0$ for $m \notin U_{m_0}$. Then, denoting by $\tau\colon L^\infty(\mu) \to C(M_{L^\infty})$ the Gelfand transform, we have

$$\int |\varphi_i|^2 \tau^{-1}(v)\, d\mu = \int |\hat\varphi_i|^2 v\, d\hat\mu \leqslant \varepsilon^2 \quad (i = 1, \ldots, s).$$

Let $M = \max_{1\leqslant i\leqslant s}\{\|\varphi_i\|_\infty\}$ and $g = \tau^{-1}(v)$. By the Łuzin theorem, for every $\eta > 0$ there exists a compact set F such that $g|_F$ is continuous and $\mu(X \setminus F) \leqslant \eta$. We can extend $g|_F$ to a continuous function on X, $\tilde g \geqslant 0$, $\|g|_F\|_\infty = \|\tilde g\|_\infty$ and $\int \tilde g\, d\hat\mu \geqslant 1 - \delta > 0$, for a certain $\delta > 0$. Next we have

$$\Big|\int |\varphi_i|^2 g\, d\mu - \int |\varphi_i|^2 \tilde g\, d\mu\Big| \leqslant \int_{X\setminus F} |\varphi_i|^2 |g - \tilde g|\, d\mu \leqslant 2M^2 \|v\|_\infty \mu(X \setminus F) \leqslant 2M^2 \|v\|_\infty \eta.$$

Since η is arbitrary, this implies that

$$(*) \qquad \int |\varphi_i|^2 \tilde g\, d\mu \leqslant 2\varepsilon \quad \text{for} \quad i = 1, \ldots, s.$$

(1) A function algebra A on X is approximating in modulus if for every positive, continuous function v on X there exists a sequence $h_n \in A$ such that

$$\||v - |h_n|\||_\infty = \sup_{x\in X} |v(x) - |h_n(x)|| \xrightarrow[n\to\infty]{} 0.$$

Let $\{k_n\} \subset A$ and $\||k_n|^2 - \tilde g\|_\infty \to 0$. By inequality $(*)$ we have

$$\|T_{\varphi_i} k_n\|^2 = \int |\varphi_i|^2 |k_n|^2\, d\mu \leqslant 3\varepsilon \quad \text{for} \quad n \geqslant n_0 \text{ and } i = 1, \ldots, s.$$

But $\int \tilde g\, d\mu \geqslant 1 - \delta$ and consequently there exists a $\delta' > 0$ such that

$$\int |k_n|^2 d\mu \geqslant \delta' \text{ for } n \geqslant n_1, \quad \text{and so} \quad 0 \in \sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}). \ ■$$

LEMMA 2. *Let A be as in Lemma 1. If $g_i \in A$ ($i = 1, \ldots, s$), then*

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) = \{(g_1(x), \ldots, g_s(x)),\ x \in X\}.$$

Proof. Since the proof is similar to the proof of Lemma 1, we omit it.

It is an interesting question whether Lemma 1 is true for more general function algebras. The following reasoning shows that it may be true for a more general case. Let A be a function algebra on X. We claim that there are $m \in M_{L^\infty}$ for which $(\hat\varphi_1(m), \ldots, \hat\varphi_s(m))$ belongs to $\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s})$, where $\varphi_i \in H^\infty(\mu)$, $i = 1, \ldots, s$. Indeed, denote by $\mathscr{A}$ the smallest commutative and closed algebra in $L(H^2(\mu))$ which contains T_{φ_i} ($i = 1, \ldots, s$). Let $\mathscr{B}$ be the closed subalgebra of $L^\infty(\mu)$ generated by φ_i ($i = 1, \ldots, s$). Then, by Theorem 1.11 in [6], every point $(\xi(T_{\varphi_1}), \ldots, \xi(T_{\varphi_s}))$ belongs to $\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s})$, where $\xi \in \Gamma(\mathscr{A})$ is the Shilov boundary of $\mathscr{A}$. Next note that $\|T_{\varphi_i}\| = \|\varphi_i\|_\infty$. Indeed, we can assume $\mu(X) = 1$; then for $\varphi \in H^\infty(\mu)$ and $n = 1, 2, \ldots$

$$\|T_\varphi^n 1\|^2 \leqslant \|T_\varphi^n\|^2 \leqslant \|T_\varphi\|^{2n},$$

which implies that $\int (|\varphi|/\|T_\varphi\|)^2 d\mu \leqslant 1$ and so $\|\varphi\|_\infty \leqslant \|T_\varphi\|$. Therefore the algebras $\mathscr{A}$ and $\mathscr{B}$ are isometrically isomorphic. Denote this isomorphism by $\tau\colon \mathscr{A} \to \mathscr{B}$. It induces a homeomorphism $\tau^*\colon \Gamma(\mathscr{A}) \to \Gamma(\mathscr{B})$ of the Shilov boundaries and we have for $\varphi \in H^\infty(\mu)$ the equality

$$\xi(T_\varphi) = \xi(\tau^{-1}(\varphi)) = \tau^*(\xi)(\varphi), \quad \text{where} \quad \xi \in \Gamma(\mathscr{A}).$$

Since $\tau^*(\xi) \in \Gamma(\mathscr{B})$, there exists an extension of $\tau^*(\xi)$ to a certain $\eta \in M_{L^\infty}$, and so we get the equality

$$(\xi(T_{\varphi_1}), \ldots, \xi(T_{\varphi_s})) = (\tau^*(\xi)(\varphi_1), \ldots, \tau^*(\xi)(\varphi_s)) = (\eta(\varphi_1), \ldots, \eta(\varphi_s)) \in \sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}).$$

The proof of the claim is complete.

In the case where $X = \Gamma$ it is well known [2] that a joint approximate point spectrum of the family $\{T_\varphi,\ \varphi \in H^\infty(\mu)\}$ is homeomorphic with M_{L^∞}. If A is approximating in modulus, then by Lemma 1

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) = \{(\hat\varphi_1(m), \ldots, \hat\varphi_s(m)),\ m \in M_{L^\infty}\} \quad \text{for} \quad \varphi_i \in H^\infty(\mu),\ i = 1, \ldots, s.$$

Assume also that the set $\{\varphi\bar{\psi},\ \varphi,\ \psi \in H^\infty(\mu)\}$ is linearly dense in $L^\infty(\mu)$. Then $\sigma_\Pi\{T_\varphi,\ \varphi \in H^\infty(\mu)\}$ is homeomorphic with M_{L^∞}. Indeed, denote by $\mathscr{R}$ the family of all finite sets contained in $H^\infty(\mu)$. For a finite set $(\varphi_1, \ldots, \varphi_s) = a \in \mathscr{R}$, write

$$X_a = \{(\hat{\varphi}_1(m), \ldots, \hat{\varphi}_s(m)),\ m \in M_{L^\infty}\}.$$

Then the mapping

$$r\colon M_{L^\infty} \ni m \to \{f_m\} \in \varprojlim_{a\in\mathscr{R}} \{X_a\} \subset \prod_{a\in\mathscr{R}} X_a, \text{ where } f_m(a) = (\hat{\varphi}_1(m), \ldots, \hat{\varphi}_s(m)).$$

and $\varprojlim_{a\in\mathscr{R}} X_a$ is the inverse limit of $\{X_a\}$, is the homeomorphism of M_{L^∞} fand $\sigma_\Pi\{T_\varphi,\ \varphi \in H^\infty(\mu)\}$. To check this, note that r is continuous by definition and is a bijection, since the set $\{\varphi\bar{\psi},\ \varphi,\ \psi \in H^\infty(\mu)\}$ is linearly dense in $L^\infty(\mu)$. Thus r is a homeomorphism. Now by Proposition 4 of [1] we can identify a C^*-algebra $\mathscr{C}$ generated by $\{T_\varphi, \varphi \in L^\infty\}$ modulo commutator ideal in $\mathscr{C}$. Denoting this commutator ideal by G, we conclude that $\mathscr{C}/G$ is isometrically isomorphic with $C(\sigma_\Pi\{T_\varphi, \varphi \in H^\infty(\mu)\})$. Since $C(\sigma_\Pi\{T_\varphi, \varphi \in H^\infty(\mu)\}) \cong C(M_{L^\infty}) \cong L^\infty(\mu)$ ($\cong$ means isometrically isomorphic), we have $\mathscr{C}/G \cong L^\infty(\mu)$. Summing up, we get

THEOREM 1. *Let A be a function algebra approximating in modulus. Assume that the set $\{\varphi\bar{\psi}, \varphi, \psi \in H^\infty\}$ is linearly dense in $L^\infty(\mu)$. Then we have*

$$\mathscr{C}/G \cong L^\infty(\mu),$$

where $\mathscr{C}$ is the C^*-algebra generated by $\{T_\varphi, \varphi \in L^\infty\}$ and G is the commutator ideal in $\mathscr{C}$.

Now we consider a C^*-algebra generated by the set $\{T_\varphi, \varphi \in A\}$. First of all note that for the function algebra A approximating in modulus the set $\sigma_\Pi\{T_\varphi, \varphi \in A\}$ is homeomorphic with X. Indeed, by Lemma 2 we have the equality

$$\sigma_\Pi(T_{\varphi_1}, \ldots, T_{\varphi_s}) = \{\varphi_1(x), \ldots, \varphi_s(x),\ x \in X\} \quad \text{for} \quad \varphi_i \in A.$$

Denote by $\mathscr{R}$ the family of all finite sets contained in A. For the finite set $(\varphi_1, \ldots, \varphi_s) = a \in \mathscr{R}$ write $X_a = \{\varphi_1(x), \ldots, \varphi_s(x),\ x \in X\}$. Then the mapping

$$h\colon X \ni x \to \{f_x\} \in \prod_{a\in\mathscr{R}} X_a, \quad \text{where} \quad f_x(a) = (\varphi_1(x), \ldots, \varphi_s(x)),$$

is the homeomorphism of X and $\sigma_\Pi\{T_\varphi, \varphi \in A\}$.

To prove this note that h is continuous by definition and is a bijection since A separates the points of X. Therefore h is the homeomorphism. Applying the theorem of Bunce once again, we get Theorem 2.

THEOREM 2. *Let A be a function algebra approximating in modulus. Denote by $\mathscr{C}$ the C^*-algebra generated by $\{T_\varphi, \varphi \in A\}$ and by G the commutator ideal in $\mathscr{C}$. Then there exists an $*$-homomorphism ϱ from $\mathscr{C}$ onto $C(X)$ such that the sequence*

$$(0) \to G \xrightarrow{i} \mathscr{C} \xrightarrow{\varrho} C(X) \to (0)$$

is exact, and $\varrho(T_\varphi) = \varphi$.

EXAMPLE. Let U be an arbitrary open and bounded set in $\boldsymbol{C}$. Denote by $H^\infty(U)$ the Banach algebra of all bounded and holomorphic functions in U. T. Gamelin proved in [4] that $\widehat{H^\infty(U)}$ is a logmodular function algebra on its Shilov boundary. Since every logmodular function algebra is approximating in modulus, Theorem 2 applies in this situation.

We can extend Theorem 2 to the matrix case. To do this we apply Lemma 2.3 in [2]. Let us recall it now.

LEMMA *If $0 \to A \to B \to C \to 0$ is an exact sequence of C^*-algebras, then the sequence*

$$(0) \to A \otimes M_n \to B \otimes M_n \to C \otimes M_n \to (0)$$

is also exact. (M_n denotes the C^*-algebra of all complex matrices.)

We leave to the reader the precise formulation of the extension of Theorem 2.

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Received July 19, 1974 (868)