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Integral solutions for non-linear evolution equations on Banach spaces

by

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Abstract. One extends locally a result of Barbu [1]. Also, a general result, useful in the construction of ε-approximate solutions for some functional equations on Banach spaces, is given.

Introduction. First, in this paper there is given a general result (i.e., Lemma 2.1) useful in the construction of approximate solutions for a large class of functional equations in Banach spaces (evolution equation [4], [5], [7], [8], [10], some integral equations [6], [9], integral inequalities [1]). We shall use the above result for proving the existence and uniqueness of the solution of the following evolution equation

$$(E)\ \frac{du}{dt} + Au \ni B(t)u(t), \qquad u(0) = x,\ x \in X,$$

where X is a general Banach space, A is a non-linear m-accretive set of $X \times X$, the function $(t, u) \to B(t)u$ is continuous from $[0, T] \times X$ into X and $B(t)\colon X \to X$ is a non-linear dissipative operator. Our result (Theorem 2.1) extends locally a result of Barbu [1] used in the proof of a generalization of a result of Webb on continuous perturbation of linear m-accretive operators in Banach spaces [9].

1. Preliminaries. As usual denote by F the duality mapping of X, i.e., $F(x) = \{x^* \in X^*,\ \|x\|^2 = \|x^*\|^2 = (x, x^*)\}$, where (x, x^*) is the value of x^* at x. Set

$$(y, x)_s = \sup\{(y, x^*),\ x^* \in F(x)\}.$$

A subset $A \subset X \times X$ is said to be *accretive* if

(1.1) $\qquad (y_1 - y_2, x_1 - x_2)_s \geqslant 0 \quad$ for every $[x_i, y_i] \in A,\ i = 1, 2$

(i.e. $x_i \in D(A)$ and $y_i \in Ax_i$).

The condition (1.1) is equivalent to

(1.2) *There exists* $x^* \in F(x_1 - x_2)$ *such that* $(y_1 - y_2, x^*) \geqslant 0$ (see e.g. [2]).

The subset $A \subset X \times X$ is said to be *m-accretive* if it is accretive and in addition $R(I + A) = X$.

A is called *dissipative* (*m-dissipative*) if $-A$ is accretive (m-accretive). Recall that if A is m-accretive then the limit (1.3) exists,

$$\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}x = S(t)x, \quad x\in\overline{D(A)}, \tag{1.3}$$

uniformly on compact subsets of $[0, +\infty)$.

Moreover, $S(t)$ is a semi-group of contractions on $\overline{D(A)}$ (see Crandall and Liggett [3]). Here $\bar{B}$ denotes the closure of B. Recall that by a result of Martin [4] it follows that a continuous operator L: $X\to X$ is accretive iff for every $x_1, x_2\in X$ we have

$$(Lx_1-Lx_2, x^*)\geqslant 0 \quad \text{for every } x^*\in F(x_1-x_2).$$

DEFINITION. Let $A\subset X\times X$ be accretive and $f\in L^1(0, T; X)$, $T>0$. A function $u\in C(0, T; X)$ is said to be an *integral solution on* $[0, T]$ of the equation

$$\frac{du}{dt}+Au\ni f \tag{1.4}$$

if the following inequality is satisfied

$$\|u(t)-x\|^2\leqslant\|u(s)-x\|^2+2\int_s^t\big(f(\tau)-y, u(\tau)-x\big)_s d\tau \tag{1.5}$$

for every $[x, y]\in A$, $0\leqslant s\leqslant t\leqslant T$.

By a result of Benilan [2], if A is m-accretive, then for each $x\in D(A)$ there is a unique integral solution on $[0, T]$ of (1.4) such that $u(0) = x$. Actually, if the initial condition $u(0) = x_0$ ($x_0\in D(A)$) is given, the uniqueness of the integral solution of (1.4) follows from the following result (due to Benilan [2] and proved independently by Barbu [1])

LEMMA 1.1. *Let $A\subset X\times X$ be m-accretive and $f, g\in L^1(0, T; X)$. If u and v are integral solutions of $u'+Au\ni f$, $v'+Av\ni g$ (respectively) on $[0, T]$, then the following inequality holds:*

$$\|u(t)-v(t)\|^2\leqslant\|u(s)-v(s)\|^2+2\int_s^t\big(f(\tau)-g(\tau), u(\tau)-v(\tau)\big)_s d\tau \tag{1.6}$$

for $0\leqslant s\leqslant t\leqslant T$.

Since $(a, b)_s\leqslant\|a\|\,\|b\|$, $a, b\in X$, by a classical result on integral inequalities, (1.6) yields

$$\|u(t)-v(t)\|\leqslant\|u(s)-v(s)\|+\int_s^t\|f(\tau)-g(\tau)\|d\tau \quad \text{(see Appendix).} \tag{1.7}$$

2. The main results.

LEMMA 2.1. *Suppose that the following conditions are satisfied.*

(a) *The function $(t, u)\to B(t)u$ from $[0, T]\times X$ into X is continuous.*

(b) *For each $\varepsilon>0$ there are the sequences $t_i^\varepsilon = t_i$, $u_i^\varepsilon = u_i$ with the properties*

(1) *$t_0 = 0$, $t_i\in[0, T]$ for $i = 0, 1, 2, \ldots$, $t_i-t_{i-1} = \delta_i$, and δ_i is the maximal positive number such that*

$$\|B(t)u-B(t_{i-1})u_{i-1}\|\leqslant\varepsilon, \tag{2.1}$$

for all t, u, such that $\|(t, u)-(t_{i-1}, u_{i-1})\|\leqslant f_i(\delta_i)$, where f_i: $R_+\to R_+$ is a continuous strictly increasing function, $f_i(0) = 0$, $R_+ = [0, \infty)$.

(2) *If $t<T$ for all $i = 1, 2, \ldots$, then $\lim_{i\to\infty}u_i = u_0$ exists and $\lim_{i\to\infty}f_i(\delta_i) = 0$.*

Then there is a natural number $n = n(\varepsilon)$ such that $t_n = T$.

Proof. If we assume for contradiction that $t_i<T$ for all $i = 1, 2, \ldots$, then $\lim_{i\to\infty}t_i = t_1$ exists (since t_i is strictly increasing). Since f_i is strictly increasing and continuous, it follows that $f_i(\delta_i)$ is the maximal positive number such that (2.1) holds. Therefore, for each i there are $\bar{t}_i$, y_i such that

$$f_i(\delta_i)<\|(\bar{t}_i, y_i)-(t_{i-1}, u_{i-1})\|\leqslant f_i(\delta_i)+\frac{1}{i}, \tag{2.2}$$

$$\|B(\bar{t}_i)y_i-B(t_{i-1})u_{i-1}\|>\varepsilon. \tag{2.3}$$

By the hypothesis (2) and (2.2) it follows that $\lim_{i\to\infty}u_i = u_0$, $\lim_{i\to\infty}\bar{t}_i = t_1$. Letting $i\to\infty$ in (2.3) and taking into account that $B(t)u$ is continuous, we get the contradiction $0>\varepsilon$. The proof is complete.

Remark 2.1. (1) In the case of Theorem 2.1 [5], $f_i(r) = rC$, where C is a constant independent of i, $\|u_i-u_{i-1}\|\leqslant\delta_iC$, $\varepsilon = 1/n$. The conditions of Lemma 2.1 are obviously satisfied. (If $t_i<T$ for all $i = 1, 2, \ldots$, then $\sum_{i=1}^\infty\delta_i<\infty$, so $\sum_{i=1}^\infty(u_i-u_{i-1})$ is convergent too, that is, $\lim_{i\to\infty}u_i$ exists.) The same arguments can be used for the last part of the proof as in Theorem 1.3.1 in Cartan [10].

(2) In the case of Lemma 2 [1], Theorem 1 [6], Proposition 3.1 [9] we have

$$f_i(r) = rM+\sup_{0\leqslant t\leqslant r}\|S(t)-I)u_{i-1}\|, \quad M>0.$$

Clearly, f_i: $R_+\to R_+$ is continuously and strictly increasing. Taking into account that $S(t)$ is a semi-group of contraction, we can easily prove that if $\lim_{i\to\infty}u_i$ exists, then $\lim_{i\to\infty}f_i(\delta_i) = 0$. On the other hand, in the above cases it is proved that if $t_i < T$ for all i, then $\lim_{i\to\infty} u_i$ exists. Thus the conditions of Lemma 2.1 are satisfied for each of the above cases.

THEOREM 2.1. *Let A be a m-accretive subset of $X\times X$ and in addition*

(i) *The function $(t, u)\to B(t)u$ is continuous from $[0, T)\times X$ into X.*

(ii) *For each $t\in[0, T)$, $B(t)$: $X\to X$ is dissipative.*

Then for each $x_0\in \overline{D(A)}$ there exists a unique integral solution on $[0, T_1] \subset [0, T)$ of the problem

$$u'(t)+Au(t)\ni B(t)u(t), \quad u(0)=x_0, \quad \text{where} \quad u'=\frac{du}{dt}, \tag{2.4}$$

i.e. there is a unique function $u\in C[0, T_1; X]$ such that

$$\|u(t)-x\|^2\leqslant \|u(s)-x\|^2+2\int_s^t \big(B(\tau)u(\tau)-y, u(\tau)-x\big)_s d\tau \tag{2.5}$$

for every $[x, y]\in A$, $0\leqslant s\leqslant t\leqslant T_1$.

Proof. Let $r>0$, $T_1\in(0, T]$ and $M>0$ be such that

$$\|B(t)u\|\leqslant M \quad \text{for every } (t, u)\in[0, T_1]\times S(x_0, r), \tag{2.6}$$

$$MT_1+\{\sup\|S(t)x_0-x_0\|,\ 0\leqslant t\leqslant T_1\}\leqslant r, \tag{2.7}$$

where $S(x_0, r)=S$ is the closed convex ball of radius r about x_0, and $S(t)$ is given by (1.3). Let $n>0$ be a natural number and let $\delta_1=\delta_1^n$ be the largest positive number such that

$$\|B(t)u-B(t_0)u_0\|\leqslant\frac{1}{n} \quad \text{for} \quad \|(t, u)-(t_0, u_0)\|\leqslant\delta_1, \tag{2.8}$$

$$t_0=0, \quad u_0=x_0.$$

Recall that with respect to the norm $\|(t, u)\|=|t|+\|u\|$, $t\in\boldsymbol{R}$, $u\in X$, the space $\boldsymbol{R}\times X$ is a Banach space. Since the function

$$f_1(r)=rM_1+\{\sup\|S(t)u_0-u_0\|,\ 0\leqslant t\leqslant r\} \quad (\text{where } M_1=M+1)$$

is a continuous strictly increasing function from R_+ into R_+, $f_1(0)=0$, it follows that there is a $\delta_1>0$ such that

$$\delta_1=\delta_1M_1+\{\sup\|S(t)u_0-u_0\|,\ 0\leqslant t\leqslant\delta_1\}$$

and moreover, δ_1 is the largest number with such a property.

Set $t_1^n=t_0^n+\delta_1^n$ where $t_0^n=t_0=0$. (When there is no danger of confusion we drop n.) Define $u_1^n=v_1(t_1-t_0)$, where v_1 is the integral solution on $[0, T_1]$ of $v'+Av\ni B(t_0)u_0$, $v(0)=u_0$. Inductively define $t_i^n=t_i=t_{i-1}+\delta_i$, where δ_i is the largest number such that

$$\|B(t)u-B(t_{i-1})u_{i-1}\|\leqslant\frac{1}{n} \quad \text{for every } (t, u) \text{ satisfying} \tag{2.9}$$

$$\|(t, u)-(t_{i-1}, u_{i-1})\|\leqslant\delta_iM_1+\{\sup\|S(t)u_{i-1}-u_{i-1}\|,\ 0\leqslant t\leqslant\delta_i\}.$$

Now we set $u_i=v_i(t_i-t_{i-1})$, where v_i is the integral solution on $[0, T_1]$ of $v'+Av\ni B(t_{i-1})u_{i-1}$, $v_i(0)=u_{i-1}$. Define

$$u_n(t)=v_i(t-t_{i-1}) \quad \text{for} \quad t_{i-1}\leqslant t\leqslant t_i. \tag{2.10}$$

First we prove that $u_n(t)\in S$ for all $t\in Du_n$. Writing

$$g_n(t)=B(t_{i-1})u_{i-1} \quad \text{for} \quad t\in(t_{i-1}, t_i]$$

and taking into account that $S(t)u_0$ is the integral solution on $[0, T_1]$ of $u'+Au\ni 0$, $u(0)=x_0=u_0$ (see [2]), by Lemma 1.1 (more precisely by (1.7)), we derive

$$\|u_n(t)-S(t)u_0\|\leqslant\int_0^t\|g_n(\tau)\|d\tau, \quad t\in Du_n. \tag{2.11}$$

Here we have used the fact that u_n is an integral solution on $[0, T_1]$ of $u'+Au\ni g_n$, $u(0)=x_0$ (see also [2]). Using once more Lemma 1.1 for $u(s)=v(s)=u_{i-1}$, $f=0$, $g=B(t_{i-1})u_{i-1}$, $s=t_{i-1}$, we get

$$\|u_n(t)-S(t-t_{i-1})u_{i-1}\|\leqslant(t-t_{i-1})\|B(t_{i-1})u_{i-1}\|, \quad t_{i-1}\leqslant t\leqslant t_i. \tag{2.12}$$

From (2.12) it follows (for $t\in[t_{i-1}, t_i]$)

$$\|u_n(t)-u_{i-1}\|\leqslant\delta_i\|B(t_{i-1})u_{i-1}\|+\|S(t-t_{i-1})u_{i-1}-u_{i-1}\|. \tag{2.13}$$

Similarly, by (2.11) we derive

$$\|u_n(t)-x_0\|\leqslant\int_0^t\|g_n(\tau)\|d\tau+\|S(t)x_0-x_0\| \quad \text{for all } t\in Du_n. \tag{2.14}$$

Using (2.13) with $t=t_1$ and taking into account that $u_n(t_i)=u_i$, we get at once

$$\|u_1-x_0\|\leqslant t_1\|B(t_0)u_0\|+\|S(t_1)x_0-x_0\|\leqslant T_1M+\|S(t_1)x_0-x_0\|\leqslant r.$$

Thus it is proved that $u_1\in S$.

Suppose now that $u_{j-1}\in S$, so

$$\|B(t_{j-1})u_{j-1}\|\leqslant M \quad \text{for} \quad j=1, 2, \ldots, i.$$

Then by (2.14) it follows

$$\|u_n(t)-x_0\|\leqslant\sum_{j=1}^{i}(t_j-t_{j-1})M+\{\sup\|S(t)x_0-x_0\|,\ 0\leqslant t\leqslant T_1\},$$

that is, $u_n(t)\in S$ for all $0\leqslant t\leqslant t_i$, which shows, in particular, that $u_i=u_n(t_i)\in S$ (thus it follows that $u_n(t)\in S$ for all $t\in Du_n$). Moreover, from (2.13) we derive

$$\big\|\big(t, u_n(t)\big)-(t_{i-1}, u_{i-1})\big\|=(t-t_{i-1})+\|u_n(t)-u_{i-1}\|$$

$$\leqslant\delta_i+\delta_iM+\big\{\sup\big\|\big(S(t)-I\big)u_{i-1}\big\|, \text{ for } t_{i-1}\leqslant t\leqslant t_i\big\},$$

so by (2.9) we get

$$\|B(t)u_n(t)-B(t_{i-1})u_{i-1}\| \leqslant \frac{1}{n} \quad \text{for} \quad t\in[t_{i-1}, t_i]. \tag{2.15}$$

To prove that there is a natural number $N(n) = N$ such that $t_N = T_1$ we apply Lemma 2.1 as follows: Assume that $t_i < T_1$ for all $i = 1, 2, \ldots$ Since in this case

$$f_i(r) = rM_1 + \{\sup\|S(t)u_{i-1}-u_{i-1}\|,\ 0\leqslant t\leqslant r\}$$

satisfies the conditions required by Lemma 2.1, it remains to prove only that $\lim u_i$ as $i\to\infty$, exists. In this way, write $t_{ij} = t_j - t_i$ for $j > i$. For all t and i, j such that $t+t_{ij} < \bar{t}$, where $\bar{t} = \lim\limits_{i\to\infty} t_i$, $u_n(t+t_{ij})$ is an integral solution of $u(t+t_{ij})+Au(t+t_{ij}) \ni g(t+t_{ij})$.

If we apply (1.7) with $t = t_i$, $s = 0$, we get

$$\|u_j-u_i\|^2 \leqslant \|u_n(t_{ij})-u_n(0)\|^2 + 2\int_0^{t_i}\big(g_n(\tau+t_{ij})-g_n(\tau),\, u_n(\tau+t_{ij})-u_n(\tau)\big)_s\,d\tau. \tag{2.16}$$

Since $\|g_n(t)\|\leqslant M$ for all $t\in[0,\bar{t}]$, (2.16) becomes

$$\|u_j-u_i\|^2 \leqslant \|u_n(t_{ij})-u_n(0)\|^2 + 4M\int_0^{t_i}\|u_n(\tau+t_{ij})-u_n(\tau)\|\,d\tau. \tag{2.17}$$

Let $\varepsilon > 0$ be arbitrary and $h = h(\varepsilon) > 0$ be such that

$$\|u_n(t_{ij})-u_n(0)\| \leqslant \frac{\varepsilon}{\sqrt{3}}, \quad t_j - t_i \leqslant \frac{\varepsilon^2}{24M(r+\|x_0\|)}, \quad j\geqslant i\geqslant h(\varepsilon). \tag{2.18}$$

Let $\varepsilon_1 > 0$ be such that $t_h+\varepsilon_1 < \bar{t}$ and let $h_1 \geqslant h$ be such that $t_{ij} < \varepsilon_1$ for $j\geqslant i\geqslant h_1$. Since u_n is uniformly continuous on $[0, t_h+\varepsilon_1]$, there is $h_2\geqslant h_1$ such that

$$\|u_n(\tau+t_{ij})-u_n(\tau)\| \leqslant \frac{\varepsilon^2}{12M\bar{t}} \quad \text{for } j\geqslant i\geqslant h_2 \text{ and for all } \tau[0, t_h]. \tag{2.19}$$

Finally, taking into account $\|u_n(t)\|\leqslant r+\|x_0\|$ for all $t\in[0,\bar{t})$, by (2.17), (2.18) and (2.19), we derive

$$\begin{aligned}\|u_j-u_i\|^2 &\leqslant \tfrac{1}{3}\varepsilon^2 + 4M\int_0^{t_i}\|u_n(\tau+t_{ij})-u_n(\tau)\|\,d\tau + \\ &\quad + 4M\int_{t_i}^{t_h}\|u_n(\tau+t_{ij})-u_n(\tau)\|\,d\tau \\ &\leqslant \tfrac{1}{3}\varepsilon^2 + \frac{\varepsilon^2 4Mt_i}{12M\bar{t}} + 8M(r+\|x_0\|)(t_h-t_i) \\ &\leqslant \tfrac{1}{3}\varepsilon^2+\tfrac{1}{3}\varepsilon^2+\tfrac{1}{3}\varepsilon^2 = \varepsilon^2, \quad j\geqslant i\geqslant h_2.\end{aligned} \tag{2.20}$$

Thus, by (2.20), $\lim u_i$ as i tends to ∞ exists, so by Lemma 2.1 there is a natural number N such that $t_N = T_1$. If m and n are natural numbers, by Lemma 1.1 and (1.7) (inasmuch as $u_n(0) = u_m(0) = x_0$) we get

$$\|u_n(t)-u_m(t)\|^2 \leqslant 2\int_0^t\big(g_n(\tau)-g_m(\tau),\, u_n(\tau)-u_m(\tau)\big)_s\,d\tau. \tag{2.21}$$

Let $i\in\{1,\ldots,N(n)\}$, $j\in\{1,\ldots,N(m)\}$ such that $\tau\in[t_{i-1}^n, t_i^n]\cap[t_{j-1}^m, t_j^m]$. Since $\big(B(\tau)u_n(\tau)-B(\tau)u_m(\tau),\, u_n(\tau)-u_m(\tau)\big)_s\leqslant 0$, we have

$$\begin{aligned}\big(g_n(\tau)-g_m(\tau),\, u_n(\tau)-u_m(\tau)\big)_s &\leqslant \big(B(t_{i-1})u_{i-1}-B(\tau)u_n(\tau),\, u_n(\tau)-u_m(\tau)\big)_s + \\ &\quad + \big(B(\tau)u_m(\tau)-B(t_{j-1})u_{j-1},\, u_n(\tau)-u_m(\tau)\big)_s \\ &\leqslant \Big(\frac{1}{n}+\frac{1}{m}\Big)\|u_n(\tau)-u_m(\tau)\| \leqslant K\Big(\frac{1}{n}+\frac{1}{m}\Big).\end{aligned} \tag{2.22}$$

Here we have used (2.15) and $\|u_n(t)\|\leqslant r+\|x_0\|$, so $K = 2(r+\|x_0\|)$. Therefore, (2.21) and (2.22) show that u_n is uniformly convergent on $[0, T_1]$. Write $u(t) = \lim\limits_{n\to\infty} u_n(t)$. Since u_n is an integral solution of $u'+Au\ni g_n$, we have

$$\|u_n(t)-x\|^2 \leqslant \|u_n(s)-x\|^2 + 2\int_s^t\big(g_n(\tau)-y,\, u_n(\tau)-x\big)_s\,d\tau, \quad 0\leqslant s\leqslant t\leqslant T_1, \tag{2.23}$$

for every $[x,y]\in A$. Let $\tau\in[t_{i-1}, t_i]$. By (2.15) it follows

$$\begin{aligned}\big(g_n(\tau)-y,\, u_n(\tau)-x\big)_s &= \big(B(t_{i-1})u_{i-1}-B(\tau)u_n(\tau)+B(\tau)u_n(\tau)-y,\, u_n(\tau)-x\big)_s \\ &\leqslant \frac{1}{n}\|u_n(\tau)-x\| + \big(B(\tau)u_n(\tau)-y,\, u_n(\tau)-x\big)_s.\end{aligned}$$

Since u_n is uniformly convergent on $[0, T_1]$ (to u), so does $B(\cdot)u_n$ (according to (i)). Letting $n\to\infty$ in (2.23), we get

$$\|u(t)-x\|^2 \leqslant \|u(s)-x\|^2 + 2\int_s^t\big(B(\tau)u(\tau)-y,\, u(\tau)-x\big)_s\,d\tau, \quad 0\leqslant s\leqslant t\leqslant T_1, \tag{2.24}$$

for every $[x,y]\in A$. The condition $u(0) = x_0$ is a consequence of $u_n(0) = x_0$, $n = 1, 2, \ldots$

The uniqueness of u follows as below: Write $u(t) = u(t, x_0)$. If $v(t) = v(t, y_0)$ is an integral solution of the equation (2.4) with $v(0) = y_0$, then according to (1.7) we have

$$\|u(t)-v(t)\|^2 \leqslant \|x_0-y_0\|^2+2\int_s^t \big(B(\tau)u(\tau,x_0)-B(\tau)v(\tau,y_0),\, u(\tau,x_0)-v(\tau,y_0)\big)_s d\tau \leqslant \|x_0-y_0\|^2, \quad \text{for all } t\in[0,T_1],$$

which shows the uniqueness of u. The proof is complete.

Remark 2.2. If $B(t)=B$ is independent of t, we can extend the solution u to the whole $[0,T)$, so if $T=\infty$ we get a result of Barbu [1]. However, in time-dependent case (i.e. our case) the problem of the extension of u to the whole $[0,T)$ remains open.

Appendix. In our paper we have used some clasical results on integral inequalities of the following type:

THEOREM 1. *Let h be a continuous strictly increasing function defined on an open subset $D(h)\subset(0,\infty)$ and $g\in L^1_{\mathrm{loc}}(0,T;R_+)$. If f is a continuous function on $[0,T]$ with $Rf\subset Dh$ satisfying the integral inequality:*

$$h\big(f(t)\big)\leqslant M+\int_s^t g(\tau)f(\tau)\,d\tau,\qquad 0\leqslant s\leqslant t\leqslant T,\quad M\in \boldsymbol{R}, \tag{1}$$

then the following estimation holds

$$f(t)\leqslant h^{-1}\Big[G^{-1}\Big(G(M)+\int_s^t g(\tau)\,d\tau\Big)\Big] \tag{2}$$

for all $t\in[s,T]$ such that the right-hand side of (2) is defined.

Here G denotes a primitive of $1/h^{-1}$ and h^{-1} is the inverse of h.

Proof. Set $y(t)=M+\int_s^t g(\tau)f(\tau)\,d\tau$. Therefore

$$y'(t)=g(t)f(t)\qquad \text{a.e. on } [s,T]. \tag{3}$$

From (1) we have $f(t)\leqslant h^{-1}\big(y(t)\big)$ so (3) becomes

$$y'(t)\leqslant g(t)h^{-1}\big(y(t)\big)\qquad \text{a.e. on } [s,T],\ y(s)=M. \tag{4}$$

By standard arguments, (4) yields

$$y(t)\leqslant G^{-1}\Big[G(M)+\int_s^t g(\tau)\,d\tau\Big]. \tag{5}$$

Using once again $f(t)\leqslant h^{-1}\big(y(t)\big)$, from (5) we get (2). Taking into account that $R(G)$ is an open subset of $(0,\infty)$, it follows that the right-hand side of (2) is defined at least for t sufficiently close to s. Thus the proof is complete.

COROLLARY 1. *If f is a continuous function from $[0,T]$ into $\boldsymbol{R}$,*

$$\tfrac12 f^2(t)\leqslant M+\int_s^t g(\tau)f(\tau)\,d\tau,\qquad 0\leqslant s\leqslant t\leqslant T, \tag{6}$$

where $g\in L^1_{\mathrm{loc}}(0,T;R_+)$, then we have the following estimation

$$f(t)\leqslant\sqrt{2M}+\int_s^t g(\tau)\,d\tau\quad \text{for}\quad t\in[s,T]. \tag{7}$$

Proof. Take $h=x^2/2$, $x>0$, and then, applying Theorem 1, we get (7). Indeed, in this case $h^{-1}(x)=\sqrt{2x}$. Obviously, we can choose $G(x)=\sqrt{2x}$, $x>0$, since $G'(x)=1/\sqrt{2x}=1/h^{-1}(x)$, for $x>0$. Thus $h^{-1}\cdot G^{-1}=G\cdot G^{-1}=I$ so (2) implies (7). ∎

Remark 1. For $h=x$, $x>0$, we choose $G(x)=\ln x$, so by (2) we derive $f(t)\leqslant M\exp\big(\int_s^t g(\tau)\,d\tau\big)$, that is, the classical result of Gronwall.

Remark 2. Similarly to Theorem 1, we can give an estimation for the function f below

$$h\big(f(t)\big)\leqslant M+\int_s^t g(\tau)L\big(f(\tau)\big)\,d\tau,\qquad 0\leqslant s\leqslant t\leqslant T. \tag{8}$$

We omit to formulate here for (8) the theorems corresponding to Theorem 1, as well as the corresponding references.

References

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