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Weak compact generating in duality

by

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Abstract. If \( X, Y \) are Banach spaces generated by a weakly compact set (WCG) and \( Y \subset X^\ast \) is norming on \( X \), then a projectional resolution of identity on \( X \) is con-
structed such that all projections are \( \sigma(X, Y) \) continuous and dual projectional form
resolution of identity on \( Y \). In this case there exists an equivalent \( \sigma(X, Y) \)-lower
semicontinuous locally uniformly rotund norm on \( X \) the dual of which is rotund on \( X^\ast \)
and locally uniformly rotund on \( Y \). Also the existence of \( \delta \)-smooth partitions of
unity on \( X \) is proved. Some results of [14] are generalized, namely it is shown
that any WCG space has a large quotient with projectional basis and that if \( X^\ast, X^{**} \)
are WCG, then \( X \) and \( X^{**} \) contain large reflexive subspaces. If \( X \) is a subspace of
WCG space, then some sufficient conditions for \( X \) to be WCG are given: \( X \) is Fréchet
smooth or \( X^\ast \) is WCG.

I. Introduction. Among nonseparable Banach spaces there is a class of
spaces which behave well with respect to many geometrical and topologi-
cal properties. This is a class of weakly compactly generated Banach
spaces, introduced by H. Corson and studied by D. Amir, J. Lindenstrauss
and others. A Banach space \( X \) is weakly compactly generated (WCG)
if there is a weakly compact set \( K \subset X \) such that \( X \) is the closed linear
hull of \( K \). These spaces form a very suitable roof over separable and reflexi-

space and include for example \( c_0(l^p), L_1(l^1) \) for finite \( \mu, L_\infty(l^p) \) — direct
sum of WCG spaces, \( p > 1, C(K) \), where \( K \) is weakly compact
subset of a Banach space, etc. They possess a nice projectional resolution
of identity ([1]), Markushevich bases, admit some nice norms ([1], [23],
[10], [11]), and behave well as to some convex extremal problems ([15], [4],
[13]). On the other hand, it was recently shown by H. Rosenthal (cf. [20])
that the WCG property is not hereditary and W. Johnson and J. Linden-
strauss ([14]) constructed a non-WCG Fréchet smooth space whose dual
is WCG.

In this paper we study weak compact generating in duality. In Section
III we prove some results on projectional resolution of identity compatible
with duality for WCG spaces. Also we prove a representation theorem for
bounded subspaces (in the weak topology) of Fréchet smooth WCG spaces
(Proposition 5).

The lemmas from Section III are used in Section IV to extend and
strengthen the following renorming theorem of M.I. Kadec (cf. e.g. [16],
...
Theorem 1.9: If $X, Y \subseteq X'$ are separable spaces, $Y$ norming on $X$, then there is an equivalent norm on $X$ which is $w(X, Y)$-lower semicontinuous and locally uniformly rotund. We prove, for example, that this equivalent norm may be also Gâteaux differentiable in the case if $X, Y$ are WCG.

More exactly, we prove: If $X, Y \subseteq X'$ are WCG spaces, $Y$ norming on $X$, then there is an equivalent norm $\| \cdot \|$ on $X$ with the properties (i)-(vi) stated in Theorem 1. This is applied to show the existence of Gâteaux smooth partitions of unity in WCG spaces $X$ for which a norming WCG $Y \subseteq X'$ exists.

In Section V we use the projectional resolutions of identity to prove that if $X$ has Fréchet differentiable norm (or $X'$ is WCG) and if $X$ is a subspace of a WCG space, then $X$ is WCG. The same result was independently obtained by D. Friedland [9] and also by W. Johnson and J. Lindenstrauss [14].

In Section VI we show that if $X'$ and $X''$ are WCG, then any subspace of $X$ and $X''$ contains a reflexive subspace of the same density, extending (necessarily by a different proof) the results of W. Johnson and H. Rosenthal [13]. In the proof we use a concept of projectional bases introduced by C. Bessaga [5] for which some needed facts (which may be of an independent interest) are proved in Section VI. Let us remark that from the results of C. Bessaga and the result mentioned above immediately follows that if $X'$ and $X''$ are WCG, then $X, X', X''$ are all homeomorphic to a Hilbert space.

Appendix contains an application of the above methods to the problem of the existence of special Markushevich bases and quasicomplements.

II. Notations and definitions. We will work in nonseparable real Banach spaces. By a subspace of a Banach space we mean a norm closed subspace. Unless stated otherwise, if $X \subseteq X'$, then $Y (X')$ denotes the Berezin operator of $Y$ to $X (X')$. If $(X, Y)$ is a dual pair of vector spaces, then $w(X, Y)$ is the weak topology on $X$ given by the duality $(X, Y)$. If $(X', Y')$ is a $w'(X', Y')$ topology denoted by $w'$-topology. If $X$ is a Banach space, then $X \subseteq X'$ is a closed linear subspace of $X'$. Also, we put $sp M = W(X, X')$. If $X$ is a subspace of $X'$, then $\inf_{x \in X} \sup_{x \in X'} f(x)$ is called $\| f \|$-norming if $\delta < \sup_{x \in X} f(x) \iff \| x \| < 1$. If $Y$ is $\delta$-norming for some $\delta > 0$, then we say $X$ is norming of $Y$. A B-space $X$ is locally uniformly rotund (LUR) if whenever $a_0, a_1 \in X, \| a_0 \| = \| a_1 \| = 1$, then $\lim_{n \to \infty} \| a_n \| = 2$. If $X$ is rotund if whenever $a, y \in X, \| a \| = \| y \| = 1$, then $\| a \| = \| y \|$.

A topological space $T$ is called an Eberlein compact if it is homeomorphic to a weakly compact subset of Banach space (in its $w$-topology), $\text{den} T$ denotes the density of $T$, i.e. the smallest cardinal number of a dense subset in $T$. The restriction of a map $\varphi$ onto a subset $A$ is denoted by $\varphi|_A$. A system $\{ a_n \} \subseteq X, \{ f_n \} \subseteq X'$ is a Markushevich basis of $X$ if $f_n(a_m) = \delta_{mn}$ (the Kronecker delta), $sp\{ a_n \} = X$ and $sp\{ f_n \} = \{ f \}$ (the coefficient space of $X$) is total on $X$, which means that $\bigcap_{n=1}^{\infty} f_n^{-1}(0) = \{ 0 \}$.

A Markushevich basis $\{ a_n, f_n \}$ is shrinking if $sp\{ f_n \} = X'$ (cf. [13], [18]). A $B$-space $X$ is relatively weakly norming (cf. [16], [14]) if any subspace $Y \subseteq X$ contains a reflexive subspace $Z \subseteq Y$ with $\text{den} Z = \text{den} Y$.

III. Dual projections in WCG spaces.

Lemma 1. Let $(X, \| \cdot \|)$ be a normed linear space, $Y \subseteq X$, a closed subspace of $X$, $B$ finite dimensional. Let $\| \cdot \|$ be another norm on $X$ such that either $\| x \| \leq \| y \|$ for all $x, y \in X$ or $\| y \| \geq \| x \|$ for all $x, y \in X$.

Then there is a $\| \cdot \|$-continuous projection $P$ of $X$ onto $B$ such that $P Y = Y$ and $P$ is $\| \cdot \|$-continuous on $X$.

Proof. Let $\| y \| \leq \| x \|$ for all $x, y \in X$. Let $\{ a_n \}$ be a basis of $Y \subseteq X$, completed by $\{ a_{n+1}, \ldots, a_n \}$ to a basis of $B$. Let $\{ x_1, \ldots, x_p \}$. We prove that there is a $\| \cdot \|$-continuous projection $P$ of $X$ onto $sp\{ a_n \}$ such that $P x_i = 0$ for $i \neq 1, P X = Y$ and $P Y = Y$ is $\| \cdot \|$-continuous. For let $f_1$ be a linear functional defined on $sp\{ a_n \}$:

$$f_1 \left( \sum_{n=1}^{p} a_n x_n \right) = a_1.$$ 

Let $f_1$ be a $\| \cdot \|$-continuous extension of $f_1$ to $Y$. Then $f_1$ is a fortiori $\| \cdot \|$-continuous on $Y$. Let $Z = sp\{ Y, \{ a_n \}_{n=1}^{p} \}$. Then $Z$ is $\| \cdot \|$-closed and $\text{den} Z = \text{den} sp\{ a_n \}_{n=1}^{p}$. Let $Q$ be another norm-continuous projections of $Z$ onto $Y$ such that $Q^{-1}(0) = sp\{ a_n \}_{n=1}^{p}$. Define the functional $f_2$ on $Z$ by $f_2 = Q f_1$. Furthermore, extend $f_2$ $\| \cdot \|$-continuously on $X$ to $f$ and define on $X$ a projection $P f = f(a_1)$. If $i \in \{ 1, \ldots, n \}$, then $a_i = \text{sp} \{ Y, \{ a_n \}_{n=1}^{i-1} \}$. Thus there is a $\| \cdot \|$-continuous linear functional $f$ on $X$ such that $f(a_i) = 0$ and $f(a_1) = 1$. Then define projection $P_i$ on $X$: $P_i f = f(a_i) a_i$. Now take $P = 1 + \ldots + P_a$ to have the desired projection. Similarly in case $\| y \| \geq \| x \|$ for $y, x \in X$.

The following lemma is a slight generalization of Lemma 2 in [10], namely we are given another norm $\| \cdot \|_4$ on the subspace $Y$.

Lemma 2. Let $(X, \| \cdot \|_4)$ be a normed linear space with another norm $\| \cdot \|_4$ on $X$. Let $N = X$ and $Y \subseteq X$ be linear subspaces of $X$ not necessarily closed. Further let $\| \cdot \|$, resp. $\| \cdot \|_4$ be norms of $N$, resp. $Y$. Let $\| \cdot \| \leq \| \cdot \|_4$ on $X$ and let $\| \cdot \|_4$ one of the following conditions hold:
LEMMA 3. Let \( X, Y \) be WCG spaces, \( Y = X^* \). Let \( P: X \to X \) be a bounded linear projection of \( X \) such that \( P^* Y \subset Y \). Then \( \text{dens} P^* Y \subset \text{dens} P X \) and if moreover \( Y \) is total on \( X \), then \( \text{dens} P X = \text{dens} P^* Y \).

Proof. Let \( K, K_0 \) be absolutely convex weakly compact sets generating \( X, Y \), respectively. Let \( M \subset P X \) be a dense subset of \( P X \). Then, as \( J \). Lindenstrauss in Proposition 3.2 of [15], using the restrictions of \( M \) to \( P^* K_0 \) and the Stone-Weierstrass theorem, we derive that \( \text{dens} \subset P^* K_0 \subset \text{card} M \) (where \( \text{dens} \subset P^* K_0 \) is \( \omega \)-topology) and there thus is a set \( \bar{M} \subset P^* K_0 \) such that \( \bar{M} \) is \( \omega \)-dense in \( P^* K_0 \) and card \( M \). Thus \( \text{dens} P^* Y \subset \text{dens} P X \). The reversed inequality is proved similarly.

LEMMA 4. Let \( X, Z \subset X^* \) be \( B \)-spaces and let \( Z = \text{sp} K \), where \( K \) is an absolutely convex weakly compact set in \( Z \). Let \( P \) be a norm bounded linear operator of \( X \) into \( Z \), such that \( P K \subset Z_0 \) (where \( Z_0 \) is the polar of \( K \) in \( X \)). Then \( P \) is \( w(Z, X) \) - \( w(Z, Z) \)-continuous.

Proof. For \( P^* \in X^* \in X \) we have \( P^* K = P^* (K_0)^* \subset (K_0)^* \subset X \) and thus from boundedness of \( P \), also \( P^* \subset Z \).

PROPOSITION 2. Let \( Y \) be a closed subspace of a WCG \( B \)-space \( (X, \| \cdot \|) \) and let \( Z \subset X^* \) be a WCG subspace total on \( Y \). Denote by \( \mu \) the first ordinal of cardinality \( \text{dens} X \). Then there is a long sequence \( \{ P_i \} \in \mu \) of linear projections of \( X \) such that

1. \( P_0 = 0 \), \( P_0^* = \text{identity} \) on \( X \), \( P_0 X = Y \), \( P_0^* Y \subset Y \) for \( a > 0 \);
2. \( P_0 P_1 = P_1 P_0 \) if \( a < b \);
3. (for every \( \mu X \) is \( a \)-\( P_\mu \) \( a \)-\( \text{norm continuous function on ordinals} \))
4. \( \text{dens} P_\mu X \subset \mu \), \( \text{dens} (P_\mu Y)^* Y \subset \mu \) for \( a > 0 \);
5. (for every \( \mu X \) is \( a \)-\( P_\mu \) \( a \)-\( \text{norm continuous function on ordinals} \))

Proof. We may suppose that \( \| x \| \leq \| y \| \) for \( x, y \in X \). Let \( K \) denote an absolutely convex weakly compact set generating \( X, Z \) and contained in the \( \| \cdot \| \)-unit ball of \( X \). We put in Lemma 2

1. \( \| x \| = \| x \| = \| x \| \) for \( \sigma X \), \( \| x \| = \inf \{ \xi > 0 : \sigma x \epsilon K \} \) for \( \sigma \text{acK} \) and \( \text{acK} = N \) and \( \| x \| = \| x \| \) for \( x \in X \).

Then we use the methods of [1] to derive the existence of projections (i), (ii), (iii). Let \( P_\mu X \subset \mu \) for \( a > \infty \).

Now, since \( \| P_\mu X \| \leq 1 \) and \( \| x \| \) is Fréchet smooth, (v) follows from the results of \([11], \text{Lemma 3})\). Also, \( \text{dens} (P_\mu Y)^* Y \subset \text{dens} (P_\mu Y)^* Y \subset \text{dens} P_\mu X \) since the norm \( \| \cdot \| \) is Fréchet differentiable (cf. e.g. [15], the end of Section 5).

In the sequel, we will use the following lemmas. The first is a variant of Proposition 2.3 of [15].
in the weak topology of $Z$. From this, the boundedness of $P_a^*$'s and from the argument of [1] (see Lemma 2 of [11]), (iv) follows. Furthermore Lemma 3 is used to derive (v).

**Proposition 3.** Let $(X, |||)\) be a WCG $B$-space, $Y \subset X$ a closed subspace of $(X, |||)$ and let one of the following two conditions hold:

(a) $(X, |||)$ is WCG,
(b) $X^*$ is WCG.

Let $|||$ be an equivalent norm on $Y$ and denote by $\mu$ the first ordinal of cardinality $\text{dens} Y$. Then there is a long sequence $\{P_a; 0 \leq a < \mu\}$ of linear projections of $X$ such that

(i) $P_a = 0$, $P_a^* = \text{identity on } Y$, $P_a^* Y \subset Y$, $|P_a| = |P_a^* Y| = 1$ if $a > 0$;

(ii) $P_a P_b = P_b P_a = P_a$ if $a < b$;

(iii) $\text{dens} P_a \leq a$, $\text{dens} P_a^* \leq a$ for $a \geq \alpha$;

(iv) for every $x \in X$ the function $a \mapsto P_a x$ is norm continuous on $\alpha$;

(v) for every $y \in Y$ the function $a \mapsto P_a y$ is norm continuous on $\alpha$.

Proof of (a). This requires a different approach than that of the statements above. We may suppose that $|y| > |x|$ for $y \in X$. Let $K, K_1$ be absolutely convex weakly compact sets generating $X, X_1$, respectively. We put in Lemma 2(ii) $X = X_1, |x| = |y|$ on $X_1, |x| = \sup |f(k)|$ for $f \in X^*, X = Y, N = \text{sp} K_1, |f|_1 = \text{inf} \{a > 0; f \in \text{AK}_0\}$ on $N, |f|_4 = |f||$ on $Y$. We then work on $X^*$ in the weak topology (details for this approach are in [12]). The limiting points of operators $T_a$ and projections appearing in Amir-Lindenstrauss construction exist in the $w^*$ sense by the $w^*$-compactness of the $\alpha$-unit ball of $X^*$. The fact that the $w^*$-cluster point $0$ of the sequence $(T_a)$ from the proof of Lemma 4 of [1] is a projection follows from the $w^*$-continuity of $\alpha$ (see Lemma 4). As in Lemma 6 of [1] we choose a dense subset $(y_a; a < \mu)$ of $N$. Using the fact that all operators appearing in the construction, are bounded and preserve $K_1$, and thus $N$ and $Y$ (Lemma 4), we see that the cluster points may be taken with respect to the $w^*$-topology on $X$ and simultaneously with respect to the weak topology on $Y$. We use also Lemma 3 and the fact that if $X$ is WCG, then $\text{dens} P_a X = \text{dens} P_a X$ (Proposition 2.3 of [13]). From these remarks, and on Amir-Lindenstrauss construction (see [1], [10], [11]) the proof of part (a) follows.

Proof of (b). Let $K, K_1$ be absolutely convex weakly compact sets generating $X, X_1$, respectively. We use again Lemma 2(ii), putting:

(i) $|x| = (X_1, |||), |x_1| = \sup |f(k)|$ for $f \in X^*, X = \text{sp} K_1, N = Y, |f|_1 = |f||$ for $f \in X$ and $|f|_4 = \text{inf} \{a > 0; f \in \text{AK}_0\}$ for $f \in \text{sp} K_1$. We then work on $X^*$ and the cluster points of projections and linear operators in Amir-Lindenstrauss construction ([1]) are in weak topology on $X^*$ because they preserve $K_1$. Further we proceed similarly as in (a). As in Lemma 6 of [1] we choose a dense subset $(y_a; a < \mu)$ of $Y = N$.

At the end of this section we show two propositions on $M$-bases:

**Proposition 4.** If $X$ is a WCG $B$-space and $Y \subset X$ a WCG total subspace of $X$, then there is an $M$-basis $(\{a, f_i\}; a \in I)$ of $X$ such that $\overline{\text{span}} \{f_i; a \in I\} = Y$.

Proof. By transfinite induction on $\text{dens} X = \text{dens} Y$ (Lemma 3). If $\text{dens} X = \text{dens} Y = N_1$, then we have Theorem III.1 of [12]. If $\text{dens} X = N$ and $\mu$ is the first ordinal of cardinality $N$, then by Proposition 2 (or by Proposition 3) there is a long sequence of projections $\{P_a; 0 \leq a < \mu\}$ on $X, |P_a| = 1, \text{dens} P_a X < N, P_a Y \subset Y$ and $\text{a} \in F Y$ norm continuous for all $y \in Y$. By the induction hypothesis there is an $M$-basis $(\{a, f_i\}; a \in I)$ of $(P_{a+1} - P_a) X$ such that

$\overline{\text{span}} \{f_i; a \in I\} = (P_{a+1} - P_a) Y$ for $0 \leq a < \mu$.

Evidently $(\{a, f_i\}; a \in I), 0 \leq a < \mu$ is an $M$-basis of $X$ such that $\overline{\text{span}} \{f_i; a \in I\}, 0 \leq a < \mu = Y$.

**Proposition 5.** Let $X$ be a WCG $f$-space. Then every bounded subset $B \subset X$ is isomorphic to a subspace of $c_0(I)$ (for certain $I$) by an affine homeomorphism with respect to weak topologies.

Proof. By Theorem 1 of [11], there is a shrinking $M$-basis $(\{a, f_i\}; a \in I)$ of $X$. Suppose that $|f|_1 \leq 1$ for all $a \in I$. Then for every $a \in I$ is $T(a) = |f(a)| < c_0(I)$ and thus $T$ is continuous linear imbedding of $X$ into $c_0(I)$. Now let $(a, x) < B$ be a net and $x \in B$. Then we have

$f(a, x) = f(a, x) = f(a, x) = f(a, x)$ for every $f \in X$, $f = f(a, x) = f(a, x)$ for every $a \in I$.

because $(a, x) \in B$ is bounded and $\overline{\text{span}} \{f_i; a \in I\} = X^*$. But

$f(a, x) = (T(x), f(a, x) = (T(x), f(a, x)$ for $(x, a) \in T(x)$ in the weak topology on $c_0(I)$ because $(T(x), T(x))$ is bounded in $c_0(I)$.

**IV. A renorming theorem.** The main result of this section is:

**Theorem 1.** If $X, Y \subset X^*$ are WCG Banach spaces, $Y$ norming on $X$, then there is an equivalent norm $|||\|$ on $X$ with the following properties:

(i) $|||\|$ is $w(X, Y)$ lower semicontinuous;

(ii) $|||\|$ is locally uniformly rotund;

(iii) on the unit sphere $\{a, x; |||a||| = 1\}$ the norm and the $w(X, Y)$ topologies coincide;

(iv) the dual norm $|||\|$ on $Y$ is locally uniformly rotund;
(v) on the unit sphere \((y, x, |y|^2 = 1)\) the norm and the \(w(X,Y)\) topologies coincide;
(vi) the dual norm \(\|\cdot^*\|\) on \(X^*\) is rotund.

Before proving Theorem 1 we state some corollaries of it.

**Corollary 1.** Under the assumptions of Theorem 1, \(X\) has a Gâteaux smooth partitions of unity (subordinated to any open covering) of \(X\).

**Proof.** By the results of H. Toruńczyk ([22], Theorem 1) it suffices to prove the following lemma.

**Lemma 5.** Under the assumptions of Theorem 1, there is a homeomorphic imbedding \(u\) of \(X\) into \(c_0(Γ)\) for some Γ such that \(p_c u\) is Gâteaux differentiable, where \(p_c : c_0(Γ)\) denotes the functional \(\langle x_κ \mapsto x_κ \rangle\).

**Proof of Lemma 5.** First, using Proposition 4, we see that there is a \(M\)-basis \(f_κ = X, f_ν = Y, \alpha \in Ι\), \(\{f_κ\}\) bounded, such that \(\sup_{f_κ} f_κ = Y\). Let \(f_κ \in Π\) and define \(u : X \mapsto c_0(Γ)\) as:

\[
P_κ u(x) = \begin{cases} \|x\| & \text{for } \alpha = 1, \\ f_κ(x) & \text{for } \alpha \in Ι, \end{cases}
\]

where \(\|\cdot\|\) is a norm from Theorem 1.

Then to prove that \(w^*\) is continuous, let \(lim_{n \to \infty} \|u(x_n) - u(x)\| = 0\).

Then

\[\lim_{n \to \infty} \|u(x_n) - u(x)\| = 0\]

Thus by the property (iii) of the norm \(\|\cdot\|\) from Theorem 1,

\[\lim_{n \to \infty} \|x_n - x\| = 0\]

**Corollary 2 ([10]).** If \(X, X^*\) are both WOG, then there is an equivalent norm on \(X^*\) which is LUR, the dual of which on \(X^*\) is LUR and bidual on \(X^{**}\) is rotund.

**Proof.** Put in Theorem 1: \(Y = X, X = X^*\).

Remark. Assumptions of Theorem 1 do not cover all spaces with nice renorming properties. For example, there is no WOG total space \(Y \subset c_0(Γ)\), Γ uncountable, since \(l_2(Γ)\) has only separable WOG subspaces by the Phillips theorem (cf. e.g. [15], Section 2) and total \(Y\) cannot be \(w\) separable (Proposition 2.2 of [15]).

First we need the following two observations:

**Lemma 6 ([10]).** If \(X, Y \subset X^*\) are Banach spaces, \(Y\) norming on \(X\), then \(Y\) is 1-norming on \(X, \|\cdot\|\), where \(\|\cdot\|\) is an equivalent norm on \(X\) given by

\[\|x\| = \sup \{f(x); f \in Y, \|f\| \leq 1\}\]

**Proof.** Denote by \(K\) the unit ball of \(X^*\) in its original norm. To see that \(Y\) is 1-norming on \(X\) it suffices to show that \(\{K \cap Y\}_{\|\cdot\|}\) and \(Y\) is \(w\) dense in \(\{K \cap Y\}_{\|\cdot\|}\). But \(\{K \cap Y\}_{\|\cdot\|}\) and \(Y\) are \(K\) and \(Y\) is \(w\) dense in \(\{K \cap Y\}_{\|\cdot\|}\) by the bipolar theorem.

**Lemma 7.** Let \((X, \|\cdot\|)\) be a Banach space, \(v\) a linear Hausdorff topology on \(X\) such that \(\|\cdot\|\) is \(v\)-lower semicontinuous. If \(L\) is a finite-dimensional subspace of \(X\), then

\[g(x, L) = \inf \{||x - l||; l \in L\}\]

is \(v\)-lower semicontinuous on norm bounded sets.

**Proof.** We will show that \(\liminf g(x, \cdot) \geq g(x)\) if \(\liminf = \sup\) in the \(v\)-topology. First, from \(\liminf g = x\) in \(v\) follows that there is a subnet \(\{x_i\}\) of the net \(\{x_n\}\) and a norm bounded net \(\{y_n\} \subset L\) such that

\[\lim g(x_n) = \liminf g(x_n) = \lim g(y_n, y) = 0\]

Thus by the property (iii) of the norm \(\|\cdot\|\) from Theorem 1,

\[\lim_{n \to \infty} \|x_n - x\| = 0\]

by the \(v\)-lower semicontinuity of the norm \(\|\cdot\|\).

Also we need the following

**Lemma 8.** Let \(X \subset X^*\) be WOG B-space, \(Y\) total on \(X\), then there is a bounded linear one-to-one embedding of \(X\) into \(c_0(Γ)\) for some Γ which is \(w(X, Y) - w(c_0(Γ), l_2(Γ))\) continuous.

**Proof.** We have \(X \subset X^*\) imbedded continuously in \(w(X, Y) = w^*\) sense. By Proposition 2 of [1], there is \(w^* - w\) continuous imbedding of \(Y^*\) into some \(c_0(Γ)\).

**Lemma 9.** Under the assumptions of Theorem 1, there is an equivalent norm \(\|\cdot\|_\gamma\) on \(X\) with the properties (i)–(iii) of Theorem 1 and there is an equivalent norm \(\|\cdot\|_\nu\) on \(X\) with the properties (i), (iv), (v).

**Proof.** First we introduce an equivalent norm \(\|\cdot\|_\nu\) defined in Lemma 6. Then in Proposition 2 we put \(Y = X - (X, \|\cdot\|)\), \(Z = Y\). Using this decomposition of \(X\), Lemma 7 and the \(w(X, Y)\) continuous imbedding of \(X\) into \(c_0(Γ)\) from Lemma 6, we see that Turoński's LUR-renorming construction on \(X\) ([23]) can be built up in the \(w(X, Y)\) sense. The property (ii) follows easily from (i) and (ii) if \(x_n, x, x, \|x_n\| - \|x\|_\gamma = 1,\)

\[\lim_{n \to \infty} \|x_n - x\| = 0\]

Thus, (ii), \(\lim_{n \to \infty} \|x_n - x\| = 0\). For the second part of the statement we construct similarly a \(v\)-lower continuous LUR norm on \(Y\). Then we use its dual norm on \(X\).
LEMMA 10. Under the assumptions of Theorem 1, there is an equivalent norm $\| \cdot \|$ on $X$ with the properties (i)–(v).

Proof. We use the following variant of an averaging procedure of E. Asplund ([2], [3]): Starting with $f_0 = \frac{1}{2} \| \cdot \|$, $g_0 = \frac{1}{2} \| \cdot \|^2$, and supposing that $g_n \leq f_n \leq (1 + \frac{1}{2})g_n$, we define

$$f_{n+1} = \frac{1}{2} (f_n + g_n), \quad g_{n+1} = \frac{1}{2} (f_n + g_n), \quad n \geq 0,$$

where $f_n^*$ denotes the dual function of $f_n$ on $Y$ in the sense of Fenchel and $\langle f_n^*, g_n \rangle$ means the dual function on $X$. Then, similarly as in [3], we have

$$g_n \leq f_n, \quad g_n \leq g_{n+1}, \quad f_n \geq f_{n+1}, \quad n \geq 0, \quad g_{n+1} \leq (1 + \frac{1}{2})g_{n+2}.$$

From this follows ([3], [2]), that its common limit $k$ is LUR, the dual of $k$ on $Y$ is also LUR. Furthermore, $k$ is $\omega(X, X^*)$-lower semicontinuous, since $f_n, g_n$ are such and $k$ is the supremum of $g_n$.

LEMMA 11. If $X$ is WOG and $Y$ is $X^*$-1-norming, then there is an equivalent norm $\| \cdot \|$ on $X$ which is $\omega(X, X^*)$-lower semicontinuous and whose dual norm on $X^*$ is rotated.

Proof. We will work on $X$ with the norm $\| \cdot \|$ introduced in Lemma 6. Let $Y$ be a bounded linear one-to-one $\omega(X^*, X) - \omega(\bar{G}(I), \bar{I}(I))$ continuous mapping of $X^*$ into $\bar{G}(I)$ (see Lemma 8). Put for a $\omega(X^*)$-continuous mapping $p(\omega^*), a \omega(X^*)$

$$p(\omega^*) \leq \frac{1}{2} \| \omega^* \|^2 + \frac{1}{2} \| T \omega^* \|^2,$$

where $\| T \omega \|$ means the Day’s LUR norm on $\bar{G}(I)$ (see e.g. [23]). Denote by $K_1$ the polar of the unit ball of Day’s norm in $\bar{I}(I)$ and by $q$ the extended-valued Minkowski functional of the $\omega(X, X)$ compact set $\bar{I}K_1 < X$. Then for the Fenchel dual function of $\frac{1}{2}q^2$ in $X^*$ we have

$$\langle \frac{1}{2}q^2 \rangle^* (\omega^*) = \sup_{\omega \omega^*} (\| \omega^* \|^2 - \frac{1}{2}q^2(\omega)) = \sup_{\omega \omega^*} (\| \omega^* \|^2 - \frac{1}{2}q^2(\omega)) = \sup_{\omega \omega^*} (\| \omega \|^2 - \frac{1}{2}q^2(\omega)) = \frac{1}{2} \| T \omega^* \|^2.$$

Now if we put $r = \inf$-convolution of $\frac{1}{2}q^2$, then $r$ is $\omega(X, X)$-lower semicontinuous since $\frac{1}{2}q^2$ is inf-compact in $\omega(X, X)$ (see [17], p. 22) and $\frac{1}{2}q^2$ is $\omega(X, X)$-lower semicontinuous ([17], p. 23). Furthermore, an easy calculation shows that $r^* = p$ on $X^*$ (see [3], p. 23), so $\sqrt{r}$ is the desired norm on $X$.

Proof of Theorem 1. We use again an averaging procedure of E. Asplund ([3], [2]) introduced in the proof of Lemma 11, i.e. in the duality $(X, Y)$, for the norms $\| \cdot \|, \| \cdot \|^2$ on $X$. The fact that the dual function of $k$ on $X$ from the proof of Lemma 10 is rotated follows from the inequalities

$$f_{n+1} \leq g_{n+1} \leq f_n (1 + \frac{1}{2})g_{n+1},$$

(whence the stars mean the dual functions on $X^*$) and from the fact that

$$g_n \leq (1 + \frac{1}{2})g_{n+1}, \quad T \omega(Y)$$

is a $\omega(X^*, X)$-lower semicontinuous convex function on $X^*$. To sketch the proof of this here, assume it holds for numbers $\leq n$. Then

$$\langle (\frac{1}{2}q^2 + \frac{1}{2}k_n) \omega \rangle^2 = \langle (\frac{1}{2}q^2 + \frac{1}{2}k_n) \omega \rangle^2 = \inf\langle \frac{1}{2}q^2 + \frac{1}{2}k_n \rangle^2, \frac{1}{2} \omega^2 \rangle^2 = \frac{1}{2} \omega^2 \langle \frac{1}{2}q^2 + \frac{1}{2}k_n \rangle^2 = \frac{1}{2} \omega^2 \langle \frac{1}{2}q^2 + \frac{1}{2}k_n \rangle^2 = \frac{1}{2} \omega^2 \langle \frac{1}{2}q^2 + \frac{1}{2}k_n \rangle^2.$$

Next we state the main result of this section.

THEOREM 2. A Banach space $X$ has a shrinking Markusevich basis, then $X$ is WOG.

Proof. It is easy to see (cf. [33], [11]) that if $\{x_n\} \subset X$, then $\{x_n\}$ is a Markusevich basis of $X$ with $\{x_n\}$, $\omega(X, X)$-weakly compact set generating $X$.

Now we state the main result of this section.

THEOREM 2. A Banach space $X$ has a shrinking Markusevich basis if one of the following conditions holds:

(i) $X$ has an equivalent Fréchet norm and $X$ is a subspace of a WOG B-space $Z$;

(ii) $X^*$ is WOG and $X$ is a subspace of a WOG B-space $Z$.

Proof. Similarly as in [11] and using Proposition 1 in (i) and Proposition 2 in (ii).

COROLLARY. A Banach space $X$ is WOG if one of the following conditions holds:

(i) $X$ is an f-space and $X$ is a subspace of a WOG B-space $Z$;

(ii) $X$ is an f-space and the unit ball of $X^*$ with $\omega$ topology is an Eberlein compact (for the definition see Section 11);

(iii) $X^*$ is WOG and the unit ball of $X$ with $\omega$ topology is an Eberlein compact.

Proof. It follows from Theorem 1, Lemma 5 and the result of D. Amir and J. Lindenstrauss that if $X$ is an Eberlein compact, then $C(X)$ is WOG ([11]).
(ii) \( S_\alpha S_\beta = S_\beta S_\alpha = S_\alpha \) if \( \alpha < \beta \);
(iii) for every \( x \in X \) the function \( a \mapsto S_\alpha x \) norm continuous on ordinals;
(iv) \( \dim(\overline{S_{\alpha+1} - S_\alpha}X) = 1 \) for \( \alpha < \xi \).

Then any system \( \{\epsilon_\alpha, f_\alpha \alpha \in \xi \} \) such that
\[
\epsilon_\alpha \epsilon_X, \quad f_\alpha \epsilon X^*, \quad f_\alpha(x) = (S_{\alpha+1} - S_\alpha)x
\]
is said to be a biorthogonal system associated with \( \{S_\alpha \alpha < \xi \} \).

We call any system \( \{\epsilon_\alpha \alpha < \xi \} \) of the elements of \( X \) a basis of \( X \) of the type \( \xi \) if there is a projectional basis \( \{S_\alpha \alpha < \xi \} \) of \( X \) of the type \( \xi \) such that \( \epsilon_\alpha (S_{\alpha+1} - S_\alpha)X \).

It is easy to see that a basis \( \{\epsilon_\alpha \alpha < \xi \} \) determines its projectional basis uniquely: if \( x \in \mathbb{X} \), \( x = \sum \lambda_\alpha \epsilon_\alpha \). \( \mathbb{K} \) finite, then \( S_\alpha x = \sum \lambda_\alpha \epsilon_{\alpha} \).

A system \( \{\epsilon_\alpha \alpha < \xi \} \) is called a (long) basis sequence if it forms a basis for \( \mathbb{F}[\epsilon_\alpha \alpha < \xi \} \) A system of projections \( \{S_\alpha \alpha < \xi \} \) of \( Y \subset X \) is said to be a \( \omega \) projectional basis of \( Y \) if (i) and (ii) of the definition above hold for \( \{S_\alpha \alpha < \xi \} \) are \( \omega \)-bounded on \( Y \) and \( S_\alpha y \) is a \( \omega \)-continuous function on ordinals for every fixed \( y \in Y \).

Dealing with these notions, we have two definitions of bounded completeness. First is a classical one analogous to that for Schauder bases:

A projectional basis \( \{S_\alpha \alpha < \xi \} \) of \( X \) is said to be \( S \)-boundedly complete if whenever \( \gamma \leq \xi \) is a limiting ordinal and \( \{y_\beta \beta < \gamma \} \), where \( y_\beta = S_\beta y \); for \( \gamma > \alpha > \beta \), \( y_\beta \) is a norm bounded net, then \( y_\beta \) exists.

The second is that given for Markushevich bases in [13], [36]: A Markushevich basis \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) of a Banach space \( X \) is said to be \( M \)-boundedly complete if whenever \( \{x_\alpha \} \) exists in \( X \) such that \( \lim_{\alpha \to \gamma} f_\alpha(x_\alpha) \) exists for each \( \alpha \) then there is an \( \epsilon \in X \) such that \( f_\alpha(x_\alpha) = \lim_{\alpha \to \gamma} f_\alpha(x_\alpha) \) for each \( \alpha \in I \).

A long basis sequence \( \{\epsilon_\alpha \alpha < \xi \} \) is called shrinking (boundedly complete) if \( \{\epsilon_\alpha \alpha < \xi \} \) is a shrinking (boundedly complete) basis of \( \mathbb{F}[\epsilon_\alpha \alpha < \xi \} \).

Evidently \( M \)-basis \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) of \( X \) is \( M \)-boundedly complete if and only if every bounded subset of \( X \) is relatively compact in the \( \omega(X, \mathbb{F}[f_\alpha]) \) topology. The "only if" part uses Tychonoff's theorem.

Remarks. 1. It is easy to see that the \( S \)-bounded completeness of a projectional basis \( \{S_\alpha \alpha < \xi \} \) implies whenever \( \beta \) is a limiting ordinal, and \( \{f_\beta \beta \in I \} \) is a net of \( \alpha \)-nets of \( \beta \), then \( \lim f_\beta = f_\beta \), and \( y_\beta \) is a bounded net such that \( y_\beta = S_\beta y_\beta \), then \( y_\beta \) is a norm convergent net.

2. Similarly as in [13], if \( T : X \to Y \), \( T \) is a \( \omega \) closed subspace \( Y \subset X \) is the quotient map, then \( T^* : X^* \to Y^* \) is a \( \omega \) isomorphism and norm isometry and \( \{S_\alpha \alpha \in \xi \} \) is the \( \omega \)-projectional basis of \( Y \). \( \{T_\alpha \alpha \in \xi \} \) is a projectional basis of \( X \).

Furthermore, analogously to [13], we call \( \{\epsilon_\alpha \alpha \in \xi \} \), \( \epsilon_\alpha \epsilon X^* \) a long \( \omega \)-basis sequence if there is a \( \omega \)-projectional basis \( \{S_\alpha \alpha \in \xi \} \) of \( X \) such that \( \epsilon_\alpha (S_{\alpha+1} - S_\alpha)X \).

To compare the two definitions of bounded completeness we state the following:

**Proposition 6.** If \( \{S_\alpha \alpha \in \xi \} \) is a projectional basis of a Banach space \( X \), and \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) is its biorthogonal system, then

(i) If \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) is \( M \)-boundedly complete, then \( \{S_\alpha \alpha \in \xi \} \) is \( S \)-boundedly complete;

(ii) If \( \mathbb{F}[\epsilon_\alpha \alpha \in \xi \} \) is norming on \( X \) and \( \{S_\alpha \alpha \in \xi \} \) is \( S \)-boundedly complete, then \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) is \( M \)-boundedly complete;

(iii) If \( \{S_\alpha \alpha \in \xi \} \) is a \( \omega \) basis of a \( \omega \) closed subspace \( Y \subset X \), and \( \{f_\alpha \alpha \in \xi \} \) is norming on \( Y \), then \( \mathbb{F}[\epsilon_\alpha \alpha \in \xi \} \) is norming on \( Y \).

**Proof.** (i) Let \( \gamma \leq \xi \) be a limiting ordinal and \( \{y_\alpha \alpha \in \gamma \} \) be a bounded net such that \( y_\alpha = S_\gamma y_\alpha \), for \( \alpha \leq \beta \). We are to prove that \( \{y_\alpha \alpha \in \gamma \} \) is a norm convergent net. For it observe that if \( \delta > \gamma \), then \( f_\delta(y_\alpha) = f_\delta(S_\delta y_\alpha) = 0 \). If \( \delta < \gamma \), then \( f_\delta(y_\alpha) \) is equal to some number \( \lambda_\delta \) for \( \delta > \beta \). Hence, \( \lim f_\delta(y_\alpha) \) exists for any \( \delta < \xi \). Thus, by the \( M \)-boundedly completeness of \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \), there is a \( \epsilon \) in \( X \) such that \( f_\alpha(y_\alpha) = 0 \) for \( \alpha > \gamma \) and \( f_\alpha(y_\alpha) = \lambda_\alpha \) for \( \delta < \gamma \).

We have for every fixed \( \alpha < \gamma \):

\[
f_\alpha(S_\gamma y) = f_\alpha(y_\alpha) = \lambda_\alpha = f_\alpha(S_\alpha y_\alpha) \quad \text{for} \quad \delta < \alpha
\]

and

\[
f_\alpha(S_\gamma y) = 0 = f_\alpha(y_\alpha) \quad \text{for} \quad \delta > \alpha.
\]

Thus \( S_\gamma y = y_\gamma \) by the totality of \( \{f_\alpha \alpha \in \xi \} \) on \( X \). Furthermore, since for \( \delta > \gamma \), \( S_\delta y = S_\delta y \) by the transfinite induction, we have that

\[
\lim_{\alpha \to \gamma} S_\alpha y = S_\delta y = S_\delta y = y.
\]

Thus \( \lim y_\alpha = y \).

(ii) Suppose that the unit ball of \( X \) is \( \omega(X, Y) \) closed (Lemma 6), where \( Y = \mathbb{F}[f_\alpha] \). Let \( \{\epsilon_\alpha, f_\alpha \alpha \in I \} \) be a bounded net in \( X \) with \( \lim f_\alpha(\epsilon_\alpha) = \lambda_\alpha \), for \( \alpha < \xi \). We are to prove that there is an \( \epsilon \) in \( X \) such that \( f_\alpha(\epsilon) = \lambda_\alpha \), for \( \alpha < \xi \). First we show by induction on \( \alpha \) that \( \lim S_\alpha y \) equals to some \( y_\beta \) for any \( \alpha < \xi \) in the \( \mathbb{F}(X, Y) \) sense, and \( f_\alpha(y_\beta) = 0 \) if \( \alpha > \beta \) and \( f_\alpha(y_\beta) = \lambda_\alpha \), if \( \gamma < \beta \). This is true for \( \alpha = 0,1 \) also and supposing it for \( \beta - 1 \), it holds for \( \beta \) if \( \beta \) is not limiting. Suppose it holds for \( \alpha < \beta, \beta \) limiting.
Let $a$ be such that $(P^*_{-a})f)(a) > 1 - \epsilon$. Evidently
\[ P^*_a f \in \mathcal{sp}(f) \quad \text{and} \quad |P^*_a f| \leq |f| = 1. \]

Thus Proposition 6 says that in the separable case a basis $M$-boundedly complete iff it is $S$-boundedly complete (c.f. [12], Theorem II.3.11).

We will need the following two observations.

**Lemma 15.** Let $\{x_i; a < \xi\}$ be a linearly independent sequence.

Define on $\mathcal{sp}(x_i)$ the projections $S_{x_i}$
\[ S_{x_i} \{ \sum_{i=1}^n \lambda_i x_i \} = \sum_{i=1}^n \lambda_i x_i. \]

Then $\{x_i; \alpha < \xi\}$ is basic iff
\[ \sup \{|S_{x_i}; a < \xi\} < \infty. \]

**Proof.** $S_{x_i}$ is obviously continuous on $\mathcal{sp}(x_i)$ and thus continuous on $\mathcal{sp}(x_i)$ since $\sup \{|S_{x_i}; a < \xi\} < \infty$.

**Lemma 16.** Let $\{P_i; a < \xi\}$ be a long sequence of linear projections of a Banach space $X$ such that
\[ \sup \{|P_i; a < \xi\} < \infty \]
and
\[ P_i P_j = P_j P_i = P_i \quad \text{if} \quad a < b \quad \text{and} \quad P_i \neq P_{a+1}. \]

If $0 \neq x_1 \in (P_{a+1} - P_a \ast X$, then $\{x_i; \alpha < \xi\}$ is a basic sequence where $\{P_i | Z; \alpha < \xi\}$ is the associated projectional basis of $Z = \mathcal{sp}(x_i; \alpha < \xi)$. If moreover the function $a \mapsto P_a \ast x$ is continuous on ordinals and
\[ 0 \neq y_1 \in (P_{a+1} - P_a \ast X, \]
then $\{y_1; \alpha < \xi\}$ is a $w^*$ basis sequence with its biorthogonal functionals $f \mapsto (P_{a+1} - P_a \ast \mathcal{sp}(y_1)$ and with the associated $w^*$ projectional basis $\{P_i | Y; a < \xi\}$ of $Y = w^* \mathcal{sp}(y_1)$. Furthermore
\[ \text{dens} \mathcal{sp}(x_i) = \text{dens} \mathcal{sp}(y_1) = \xi. \]

**Proof.** Put $S_i = P_i \mathcal{sp}(x_i)$. Then for $x = \sum \lambda_i x_i \mathcal{sp}(x_i)$ we have that $S_i x = \sum \lambda_i x_i$ and thus $\{S_i; \alpha < \xi\}$ is a basic sequence, according to Lemma 15. If moreover the function $a \mapsto P_a \ast x$ is continuous and $y_1 \neq (P_{a+1} - P_a \ast \mathcal{sp}(y_1)$ we put $M = P_a \ast Y$. Then $S_i Y = \mathcal{sp}(y_1)$ and by $S_i \ast y_1 \ast \mathcal{sp}(y_1)$ and denote by $S_i \ast y_1$ the dual of $S_i$ in $Z$ (we use the $w^*$ continuity of $P_i \ast$). Then if $x_1 \in (S_{a+1} - S_a) \ast Z$ is so that $e_n (y_1) = 1$, then $(S_i, y_1)$ is a biorthogonal system and if $x_1 \ast x_2$, $x_2 \ast X$, then $\{(P_{a+1} - P_a \ast x_2, y_2)\}$, forms a biorthogonal system. The sta-
tement on the density of $\mathcal{S}(x)$ follows from the fact that
\[
\frac{x_n}{\|x_n\|} - \frac{x_n}{\|x_n\|} = \frac{1}{2} \left( |P_{x} - P_{x}| \frac{x_n}{\|x_n\|} - \frac{x_n}{\|x_n\|} \right) = \frac{1}{2} \frac{x_n}{\|x_n\|} - \frac{1}{2}
\]
if $a \neq 1$.

Remark. A similar approach to that in Lemma 14 is contained in [19]. We will now use the following variant of the result of Johnson and Rosenthal ([13, Proposition III.1]) which has also a similar proof by using Proposition 6 with the remark under it. We enclose it here for the completeness.

**Proposition 7.** Assume that $\{y_n; a < \xi\} \subseteq X^*$ and $T: X \rightarrow X/(y_n)_+$ be the quotient map. Then
(a) $\{y_n; a < \xi\}$ is $w*$ basic iff $X/(y_n)_+$ has a basis $\{x_n; a < \xi\}$ with associated biorthogonal functionals $\{x_n^*\}$ such that $T^* x_n^* = y_n, a < \xi$. Thus if $\{y_n; a < \xi\}$ is $w*$ basic, then $\{y_n; a < \xi\}$ is basic.
(b) The following are equivalent:
(i) $\{y_n; a < \xi\}$ is an $S (= M$ here$)$-boundedly complete long $w*$ basic sequence;
(ii) $\{y_n; a < \xi\}$ is $w*$ basic and $\mathcal{S}(y_n) = w*$-sp$(y_n)$;
(iii) $X/(y_n)_+$ has a shrinking basis $\{x_n; a < \xi\}$ with associated biorthogonal functionals $\{x_n^*\}$ such that $T^* x_n^* = y_n$.

Here and also in the sequel the term $\{y_n\}$ is a shrinking $w*$ long basic sequence means that $\{y_n\}$ is a $w*$ long basic sequence which is shrinking as a basis of $\mathcal{S}(y_n)$ (see (a)). Similarly for the case of bounded completeness.

In the sequel we present some results which are nonseparable analogues to some results of [13] and have necessarily different proofs. In these results in [13] is often a typical assumption that some space, say $Z$, has separable dual $Z^*$. We consider two generalizations of this assumption in nonseparable case: $Z^*$ is WCG or $Z$ is f. They both coincide in the separable case. Thus our propositions have two alternatives.

**Proposition 8.** Let $X$ be a WCG Banach space, $Y \subseteq X^*$, $\mu$ be the first ordinal of cardinality $\dim X$. Assume that
(i) $X^*$ is WCG or
(ii) $X$ has an equivalent Fréchet smooth norm and $X$ is WCG.
Then there is a boundedly complete $w*$ basic long sequence $\{y_n; a < \mu\}$ $\subseteq Y$, $\{y_n; a < \mu\}$ may be orthogonal if $Y$ is nonseparable.

Proof. If $\dim X = N_\omega$, then there is a proposition $P$: $X \rightarrow X^*, |P| = 1$ such that $P_X$ is separable and $P^* X^* \rightarrow Y (11)$. Now we may use Theorem III. 2 of [13] for $P_X$, $Y \subseteq Y^* = \{x^*\}$, since in both cases $\dim P^* X^* = N_\omega$ ([15]). If $\dim Y > N_\omega$, let $|I|$ be the norm of $X$ and $|I|$ be an equivalent norm on $X$ such that its dual on $X^*$ is LUR (cf. [10], Proposition 9 for case (i) and (11), Theorem 1 for case (ii)). Then in both cases there is a long sequence of linear projections $\{P_x; x \subseteq \mu\}$ of $X$ such that $|I| = |I| = 1$ for $a > 0, P_a X = Y, P_a Y = Y$ for $y \in Y, P_a X Y = P_a X Y$ on $Y$ and the functions $a \rightarrow P_a x, a \rightarrow P_a x$ are norm continuous for all $x \in X$, $y \in Y$ (by Proposition 3(a), b)).

If we take $0 \neq y \in P_a (P_{a+1} - P_a) Y$, we have a $w*$ basic long sequence $\{y_n; a < \xi\} = \{y_n; a < \xi\}$ which is orthogonal ($|I| \leq 1$) and exactly as in the proof of Theorem III. 2 of [13] we show that $\mathcal{S}(y_n) = w*$-sp$(y_n)$, then $\lim S_n y = y$ in the $w*$ sense and since $\|y\| \geq \|S_n y\|$, we have by the $w*$ lower semicontinuity of $\|\|y\|$ on $X^*$ that $\lim S_n y = y$ and by the LUR of $I$, $\lim S_n y - y = 0$. Now by induction, $S_n y \in \mathcal{S}(y_n)$ and the result follows by using Proposition 7(b).
that \( \{ y_n; a < \xi \} \) is shrinking (cf., e.g., Lemma 3 of [11]). The rest of the proposition follows from Proposition 7(a) and [12], Theorem II.5.

Before proceeding we will need the following definition (cf. [16], [13]).

**Definition.** A Banach space \( X \) is called *somewhat reflexive* if any closed subspace \( Y \subset X \) contains a reflexive subspace of the same density.

**Proposition 11** (cf. [13], Theorem IV.2). Let \( X, Y \subset X^* \) be Banach spaces. Then \( Y \) is somewhat reflexive if

(i) \( X, X^* \) are WCG which both admit equivalent Fréchet smooth norms, or

(ii) \( X, X^* \) are WCG and \( Y \) admits an equivalent Fréchet smooth norm.

**Proof.** First, let us recall that if \( X, X^* \) are WCG, then \( X \) admits an equivalent Fréchet smooth norm ([10]). Now let \( Z \subset Y \). If \( Z \) is separable, we proceed exactly as in the first part of the proof of Proposition 8 and use Theorem IV.2 of [13]. Suppose \( \text{dens} Z > N_1 \), \( Z \) is WCG as a subspace of WCG space which admits an equivalent Fréchet smooth norm ([11] or Corollary 1 of Theorem 2). Hence we may use Proposition 11(ii).

**Corollary.** If \( X, X^* \) are WCG, then \( X, X^* \) are somewhat reflexive.

**Proof.** If \( X \) is then WCG space by Theorem 4 of [11], \( X \) admits an equivalent Fréchet smooth norm ([10]).

**Corollary.** If \( X, X^* \) are WCG and \( \text{dens} X = N \), then \( X, X^* \) are all homeomorphic to the Hilbert space \( L^2(N) \).

**Proof.** Since \( X, X^* \) are WCG, \( \text{dens} X = \text{dens} X^* = \text{dens} X^{**} \) (Proposition 2.2 of [15]). Furthermore, \( X, X^*, X^{**} \) contain reflexive subspaces of density character \( N \) (\( X^{**} \) that from \( X \)). Further we use the following results of C. Bessaga and A. Pełczyński (cf. [6]): The first says that all reflexive Banach spaces \( X \) with density character \( N \) are homeomorphic to \( l_2(N) \), and the second, the well-known Bessaga–Pełczyński lemma, says that if a Banach space \( X \) of density character \( N \) contains a subspace homeomorphic to \( l_2(N) \), then \( X \) is homeomorphic to \( l_2(N) \).

**VII. Appendix.** Here we apply the above results to some problems discussed in [10].

**Remark.** Similarly as in Proposition 3, using Lemma 3 of [11] we see that if \( X \) is a WCG Banach space which admits an equivalent Fréchet smooth norm \( \| \cdot \| \) and \( Y \subset X^* \) is a WCG Banach space, then there is a long sequence of linear projections \( (P_r; 0 \leq s < \xi, \xi) \), where \( \xi \) is the first ordinal of \( \text{dens} X \) such that \( \| P_r \| = 1 \) for \( a > 0 \). \( P_x P_x = P_x, \) if \( a < \beta, \text{dens} P_x^2 = \text{dens} (P_x X)^* \leq \alpha \) for \( a < \xi, \beta, P_x Y \subset Y \), and the functions \( a \rightarrow P_x, a \rightarrow P_x^2 \) are norm continuous on ordinals. (The assumption on \( Y \) to be WCG is necessary – cf. [11].)

From this and the results of [10] follows that Propositions 5, 6, 7 of [10] remain valid if the assumptions \( X, X^* \) are WCG replaced by \( X \) is WCG and admits an equivalent Fréchet smooth norm and \( Y \subset X^* \) is a WCG Banach space.

Therefore we say that a subspace \( Z \subset Y \) is a quasi-complement of a subspace \( Y \) if \( \text{dens} Z \geq X \) and \( Z \cap Y = \{ 0 \} \), Proposition 7 of [10] (where some results of [1] are used) gives:

**Theorem.** If \( X \) is a WCG Banach space which admits an equivalent Fréchet smooth norm and \( Y \subset X^* \) is a WCG subspace of \( X^* \), then \( Y \) has a \( w^* \)-closed quasi-complement.

**References**


Localisation des sommes de Riesz sur un groupe de Lie compact

par

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Résumé. Grâce à une étude précise des noyaux des sommes de Riesz, on obtient des résultats de localisation pour les développements de Peter-Weyl sur un groupe de Lie compact.

Ce travail est la suite de l'article [3], dont il reprend les notations: \( \Theta \) est un groupe de Lie réel, compact, connexe et simplement connexe (demi-simpliciel), de dimension \( s \), et de rang \( l \); si \( f \) est une fonction sommable sur \( \Theta \), on pose

\[
S_{\delta}^n f = \sum_{\alpha \in \Lambda} \left( 1 - \frac{2 \lambda + \beta}{H^2} \right)^{k} d_{\delta} \exp f,
\]

où \( \delta \geq 0 \) et \( R > 0 \) (sommes de Riesz d’indice \( \delta \)). L’opérateur \( S_{\delta}^n \) s’interprète comme une convolution avec une fonction centrale \( s_{\delta} \), dont un développement a été obtenu dans [3], lorsque \( \delta > (l-1)/2 \).

\[
s_{\delta}(\exp H) = C \int_{D(\exp H)} \left( \prod_{\alpha \in \Lambda} (\alpha + \gamma) \right) \mathcal{F}_{\alpha} + \epsilon(R|H|\gamma) \mathcal{F}_{\alpha} \delta_{\alpha}^{\epsilon} f,
\]

où \( \mathcal{F}_{\alpha} \) est la fonction de Bessel d’indice \( \alpha \). On se propose ici d’améliorer les estimations obtenues précédemment et d’obtenir des résultats de localisation.

Lemme. Soit \( \delta > (l-1)/2 \), et soit \( \varepsilon > 0 \). Il existe une constante \( C > 0 \), telle que

\[
\alpha \in \Theta \quad \text{et} \quad d(\alpha, C) > \varepsilon \Rightarrow |s_{\delta}(\alpha)| \leq C R^{-1} \frac{l}{2} - \varepsilon.
\]

Rappelons que \( \Theta \) est un domaine fondamental d’un tore maximal centré à l’origine; et soit \( B_\varepsilon \) la boule ouverte de centre \( 0 \) et de rayon \( \varepsilon \). Soit \( H, e^\Theta \subset C B_\varepsilon \). Soit \( J \) l’ensemble des racines positives qui viennent en

(1) La série converge absolument pour tout \( H \), et le membre de droite de l’égalité défini a priori pour \( H \) régulier se prolonge par continuité.