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# Examples of separable spaces which do not contain $l_1$ and whose duals are non-separable

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**Abstract.** Two examples of separable Banach spaces which do not contain $l$ and whose duals are non-separable are presented.

**1. Introduction.** We are concerned here with two (negative) solutions to a problem in Banach space theory which was open for some time. The problem is the following

(P) *Assume that $X$ is a separable Banach space such that $X^*$ is non-separable. Must $X$ contain a subspace isomorphic to $l_1$?*

The two solutions to (P) which we present here were obtained independently and at about the same time. One counterexample to (P) which we denote by $JT$ and call the *James Tree* was obtained by R. C. James [7]. The second space which we denote by $JF$ and call the *James function space* was obtained by one of the present authors. The space $JF$ has been studied for some time by M. Zippin and it was he who suggested it to one of the authors in connection with some problems related to (P).

Both spaces $JT$ and $JF$ are closely related to the famous example $J$ of James [5] of a sequence space which satisfies $\dim J^{**}/QJ = 1$ ($Q$ denotes the canonical embedding of a space into its second dual). The space $JF$ is the natural function space analogue of $J$. The space $JT$ is obtained from $J$ by replacing the index set on which the space $J$ is defined by a suitable infinite tree.

The space $JF$ is easy to define and it is a space which is also of interest outside the framework of Banach space theory. Before we define $JF$, let us recall (one of) the definition(s) of $J$. The space $J$ consists of those sequences $y = (y_1, y_2, \ldots)$ of reals such that

$$\|y\| = \sup \Big( \sum_{i=0}^{k-1} \Big( \sum_{j=n_i+1}^{n_{i+1}} y_j \Big)^2 \Big)^{1/2} < \infty \tag{1.1}$$

where the sup is taken over all increasing finite sequences of integers $0 = n_0 < n_1 < \ldots < n_k$. The function space analogue $JF$ of $J$ is defined as the completion of the linear span of characteristic functions of subintervals of [0, 1] with respect to the norm

$$\|f\| = \sup\Big(\sum_{i=0}^{k-1}\Big(\int_{t_i}^{t_{i+1}} f(t)\,dt\Big)^2\Big)^{1/2} \tag{1.2}$$

where the sup is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ of [0, 1]. A somewhat more elegant way to consider $JF$ is obtained by working with $g(t) = \int_0^t f(u)\,du$. In this way $JF$ becomes the completion of the space of piecewise linear functions $g$ on [0, 1] with respect to the "square variation" norm, i.e.

$$\|g\| = \sup\Big(\sum_{i=0}^{k-1}\big(g(t_{i+1}) - g(t_i)\big)^2\Big)^{1/2} \tag{1.3}$$

where the sup is again over all partitions of [0, 1] (in order to get a norm in (1.3) we have to normalize $g$, e.g. by requiring $g(0) = 0$). The definition of $JT$ will be given later (in the beginning of Section 2).

We devote one section to the study of each of the spaces $JT$ and $JF$. There is no dependence relation between these sections and the reader interested only in one of these spaces may read only the appropriate section. Also, in our presentation of the properties of $JT$ we do not assume that the reader is familar with James's work on this space (i.e. [7]). Though $JF$ is perhaps a more "natural" space than $JT$ it turns out that it is more difficult to analyze its structure. From a Banach space theoretic point of view the information known thus far on these two spaces makes $JT$ a more interesting (as well as more transparent) counterexample to (P).

In proving that $JT$ is a counterexample to (P) James proved that (1) $JT^*$ is non-separable (this is easy) and the rather deep fact that (2) every infinite dimensional subspace of $JT$ contains a subspace isomorphic to $l_2$. Thus, in particular, (3) $JT$ does not contain a subspace isomorphic to $c_0$ or $l_1$. We do not prove (2) here. Instead, we give an explicit description of the conjugates of $JT$. It follows in particular that (4) $JT^{**}$ is equal to $QJT \oplus X$ where $X$ is isometric to $l_2(\Gamma)$ with $\Gamma$ a set of the cardinality of the continuum. The proof of (4) is much simpler than James's proof of (2). It is clear that (4) implies (3). Moreover, the information on the conjugates of $JT$ enables the deduction of further properties of $JT$ which show that $JT$ answers negatively some other open problems besides (P). We show in particular that (5) $QJT$ is $w^*$ sequentially dense in $JT^*$ and (6) every bounded sequence in $JT$ has a weakly Cauchy subsequence. (It has been asked whether a separable space which has the properties of (5) or (6) must have also a separable dual. This question goes back to Banach [1], p. 243.) We show also that (7) there is a weakly measurable function with values in $JT^*$ which is not equivalent to a strongly measurable function. For definitions of these notions see below in Section 2; it was a conjecture in the theory of vector measures (this conjecture was communicated to us by D. R. Lewis) that a space which admits a weakly measurable function not equivalent to a strongly measurable one must contain $l_\infty$.



Concerning the space $JF$ we show in Section 3 the following: (i) $JF^*$ is non-separable (this is trivial), (ii) $JF$ contains a subspace isomorphic to $c_0$ (this is an immediate consequence of a result of Giesy and James [2] concerning $J$), and (iii) $JF$ contains no subspace isomorphic to $l_1$ (the proof of this fact is the most delicate part of the present paper).

It can be expected that further investigations into the structure of $JF$ and its conjugates will reveal some new interesting facts.

Let us mention that in an attempt to prove that the answer to (P) is positive the second named author undertook a study of the structure of the separable spaces whose duals are non-separable. The examples presented here and especially the space $JT$ illuminate the results of [14] and show in a sense that they cannot be improved. We shall mention the main result of [14] in the end of Section 2.

**Acknowledgement.** We are indebted to several mathematicians who contributed much to the material presented in this paper. Our greatest indebtedness is to R. C. James who made available to us a preprint of [7] and whose ideas underlie all parts of the present paper. As mentioned already above we owe to M. Zippin the suggestion to consider $JF$. We want also to thank W. J. Davis, T. Figiel, W. B. Johnson and H. P. Rosenthal for some helpful comments and suggestions in connection with our study of $JT$. The second named author would like to thank James Hagler for many conversations about these questions.

## 2. The James tree $JT$.

The space $JT$ is defined as a space of functions defined on an infinite tree $T$. Let

$$T = \{(n, i);\ n = 0, 1, 2, \ldots,\ 0 \leqslant i < 2^n\}.$$

We partially order $T$ by putting $(m, j) \geqslant (n, i)$ if $m > n$ and there exist integers $i_0 = i, i_1, \ldots, i_k = j$ with $k = m-n$ and $i_1 \in \{2i, 2i+1\}$, $i_2 \in \{2i_1, 2i_1+1\}, \ldots, i_k \in \{2i_{k-1}, 2i_{k-1}+1\}$. Such a set of indices $\{(n, i), (n+1, i_1), \ldots, (m, j)\}$ is called a *segment*. By a *branch* we mean a maximal segment, i.e. a set of indices of the form

$$\{(0, 0), (1, i_1), (2, i_2), \ldots, (n, i_n), \ldots\}$$

with $i_k \in \{2i_{k-1}, 2i_{k-1}+1\}$ for every $k$. The space $JT$ consists of all functions

$x$ from $T$ to the reals so that

$$\|x\| = \sup\Big(\sum_{j=1}^{k}\Big(\sum_{(n,i)\in\mathscr{S}_j} x(n,i)\Big)^2\Big)^{1/2} < \infty \tag{2.1}$$

where the supremum is taken over all choices of pairwise disjoint segments $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_k$.

Let $e_{n,i}$ be the unit vectors in $JT$, i.e. the elements defined by $e_{n,i}(m,j) = \delta_{n,m}\delta_{i,j}$. It is straightforward to show that $JT$ is a Banach space and that $\{e_{n,i}\}_{n=0}^{\infty}{}_{i=0}^{2^n}$ (enumerated $e_{0,0}, e_{1,0}, e_{1,1}, \ldots, e_{n,0}, e_{n,1}, \ldots, e_{n,2^n-1}, e_{n+1,0}, \ldots$) is a boundedly complete basis of $JT$. Hence (cf. [2]) $JT$ is canonically isometric to $B^*$ where $B$ is the closed linear span in $JT^*$ of the biorthogonal functionals $\{f_{n,i}\}$ to the $\{e_{n,i}\}$ (i.e. $f_{n,i}(e_{m,j}) = \delta_{n,m}\delta_{i,j}$). Clearly, also $\|f_{n,i}\| = \|e_{n,i}\| = 1$ for all $n$ and $i$. There are many projections of norm one on $JT$. We shall define here some such projections which will be of use in the future. For

$$x = \sum_{n=0}^{\infty}\sum_{i=0}^{2^n-1} t_{n,i}e_{n,i} \in JT$$

define

$$P_m x = \sum_{n=m}^{\infty}\sum_{i=0}^{2^n-1} t_{n,i}e_{n,i}, \quad m = 0, 1, 2, \ldots, \tag{2.2}$$

$$P_{m,j}x = \sum_{n=m}^{\infty}\sum_{(n,i)\geqslant(m,j)} t_{n,i}e_{n,i}, \quad m = 0, 1, 2, \ldots, \ 0 \leqslant j < 2^m, \tag{2.3}$$

and, finally, for every branch $\boldsymbol{B}$ of $T$

$$P_{\boldsymbol{B}}x = \sum_{(n,i)\in\boldsymbol{B}} t_{n,i}e_{n,i}. \tag{2.4}$$

The verification that (2.2), (2.3) and (2.4) all define projections of norm 1 on $JT$ is immediate.

Let $\sigma_1, \ldots, \sigma_q$ denote finite subsets of $T$ such that any segment $\mathscr{S}$ of $T$ intersects at most one $\sigma_p$. Then, as easily checked, for every choice of scalars $t_{n,i}$ we have

$$\Big\|\sum_{(n,i)\in\bigcup_{p=1}^{q}\sigma_p} t_{n,i}e_{n,i}\Big\|^2 = \sum_{p=1}^{q}\Big\|\sum_{(n,i)\in\sigma_p} t_{n,i}e_{n,i}\Big\|^2. \tag{2.5}$$

In particular, for every $x\in JT$ and every integer $m$

$$\|P_m x\|^2 = \sum_{j=0}^{2^m-1}\|P_{m,j}x\|^2. \tag{2.6}$$



For every pair $(m,j)$ the space $P_{m,j}JT$ is isometric to $JT$ and for every branch $\boldsymbol{B}$ of $T$ the space $P_{\boldsymbol{B}}JT$ is isometric to the classical James space $J$ (see (1.1)). Let us recall that the unit vectors $\{e_i\}$ clearly form a boundedly complete basis for $J$. The space $J^*$ contains besides the span $\tilde{J}$ of the biorthogonal functionals $\{f_i\}$ to $\{e_i\}$, also the functional $g$ defined by $g(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} y_i$. Since $\dim J^*/\tilde{J} = 1$, it follows that $J^* = \tilde{J}\oplus\{\operatorname{span} g\}$. Hence, for every $j\in J^*$, $\lim_{i\to\infty} f(e_i)$ exists and this limit is 0 if and only if $f\in\tilde{J}$.

We consider now the dual $JT^*$ of $JT$ and define a map $S\colon JT^*\to l_2(\Gamma)$ where $\Gamma$ is the set of all branches of $T$, i.e. a set of the cardinality of the continuum. The definition of $S$ is

$$Sx^*(\boldsymbol{B}) = \lim_{(n,i)\in\boldsymbol{B}} x^*(e_{n,i}). \tag{2.7}$$

The following theorem is our main result on the structure of $JT^*$. All other facts concerning $JT$ which we are going to establish will be easy consequences of this theorem.

THEOREM 1. *The operator $S$ defined in* (2.7) *is a quotient map from $JT^*$ onto $l_2(\Gamma)$. The kernel of $S$ is equal to $B = \operatorname{span}\{f_{n,i}\}_{n=0}^{\infty}{}_{i=0}^{2^n-1}$. In particular, $B^{**}/QB$ is isometric to $l_2(\Gamma)$.*

Proof. We check first that $S$ is a well defined operator of norm $\leqslant 1$. The limit in (2.7) exists by the preceding remarks concerning the properties of $J^*$. Let now $\{\boldsymbol{B}_p\}_{p=1}^{q}$ be distinct branches of $T$. There is an integer $m$ such that the sets $\boldsymbol{B}_p\cap\{(n,i);\ n\geqslant m\}$, $p = 1, \ldots, q$, are pairwise disjoint. It follows that if $(n_p, i_p)\in\boldsymbol{B}_p$ with $n_p\geqslant m$ and if $\{t_p\}_{p=1}^{q}$ are scalars then

$$\Big\|\sum_{p=1}^{q} t_p e_{n_p,i_p}\Big\|^2 = \sum_{p=1}^{q} t_p^2$$

(this is a special case of (2.5)). Hence, for every $x^*\in JT^*$,

$$\sum_{p=1}^{q} x^*(e_{n_p,i_p})^2 \leqslant \|x^*\|^2.$$

The same remains true as $(n_p, i_p)$ tends to $\infty$ in $\boldsymbol{B}_p$, $p = 1, \ldots, q$, and hence $S$ is bounded and of norm $\leqslant 1$.

It is also easy to prove that $S$ is a quotient map. Indeed, let $\{\boldsymbol{B}_p\}_{p=1}^{q}$ and $m$ be as above. Let $\{t_p\}_{p=1}^{q}$ be also given. Define $x^*\in JT^*$ by

$$x^*\Big(\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1} t_{n,j}e_{n,j}\Big) = \sum_{p=1}^{q} t_p \sum_{\substack{(n,j)\in\boldsymbol{B}_p\\ n>m}} t_{n,j}.$$

It is easy to verify (directly from (2.1)) that $\|x^*\| = 1$. Also by (2.7) $Sx^*(\boldsymbol{B}_p) = t_p$ and $Sx^*(\boldsymbol{B}) = 0$ if $\boldsymbol{B}$ is not any of the $\boldsymbol{B}_p$, $p = 1, \ldots, q$. Thus $S$ is a quotient map.

Let $Y$ be the kernel of $S$, i.e.

$$(2.8)\qquad Y = \{x^* \in JT^*;\ \lim_{\substack{(n,i)\in \boldsymbol{B}\\ n\to\infty}} x^*(e_{n,i}) = 0 \text{ for all } \boldsymbol{B}\}.$$

It is evident that $B \subseteq Y$. The main point in the proof of Theorem 1 is to verify the reverse inclusion, i.e. that $Y$ is equal to $B$. We prove first a lemma.

LEMMA 1. *For* $x^* \in Y$

$$\lim_{n\to\infty}\big(\max_{0\leqslant i<2^n} \|P^*_{n,i}x^*\|\big) = 0.$$

Proof. Suppose there is an $a > 0$ and a sequence $(n_k, i_k)$ such that

$$(2.9)\qquad \|P^*_{n_k,i_k}x^*\| > a, \qquad k = 1, 2, \ldots$$

We show first that among the $(n_k, i_k)$ there exists only a limited number of mutually incomparable elements (with respect to the partial order of $T$). Indeed, assume that $(n_k, i_k)$, $k = 1, \ldots, j$, are all mutually incomparable. By (2.9) there is, for every $k$, an $x_k \in P_{n_k,i_k}JT$ with $\|x_k\| = 1$ and $x^*(x_k) \geqslant a$. By (2.5)

$$\Big\|\sum_{k=1}^{j} x_k\Big\| = j^{1/2}$$

and hence

$$ja \leqslant x^*\Big(\sum_{k=1}^{j} x_k\Big) \leqslant \|x^*\| j^{1/2},$$

i.e. $j < (\|x^*\|/a)^2$.

It follows from the preceding argument that there is no loss of generality to assume that the sequence $(n_k, i_k)$ satisfying (2.9) is totally ordered and thus determines a unique branch $\boldsymbol{B}$ of $T$. By passing to a subsequence if necessary we can further assume without loss of generality that $n_{k+1} > n_k$ and

$$(2.10)\qquad \|P^*_{n_k,i_k}x^* - P^*_{n_{k+1},i_{k+1}}x^*\| > a \quad \text{for all } k.$$

(Observe that, for every $y^* \in JT^*$ and any choice of $0 \leqslant i_n < 2^n$, $\lim_{n\to\infty}\|y^* - P^*_{n,i}y^*\| = \|y^*\|$.) Since $x^* \in Y$, it follows from our discussion of $J$ preceding the statement of Theorem 1 that $P^*_{\boldsymbol{B}}x^* \in \overline{\operatorname{span}}\{f_{n,i}; (n,i)\in\boldsymbol{B}\}$. Thus for sufficiently large $k$ (and therefore without loss of generality of every $k$)

$$(2.11)\qquad \|(P^*_{n_k} - P^*_{n_{k+1}})P^*_{\boldsymbol{B}}x^*\| < \tfrac{1}{2}a.$$



Put now for $k = 1, 2, \ldots$

$$(2.12)\qquad U^*_k = P^*_{n_k,i_k} - P^*_{n_{k+1},i_{k+1}} + (P^*_{n_k} - P^*_{n_{k+1}})P^*_{\boldsymbol{B}}.$$

It is easily checked that each $U^*_k$ is the dual of a projection $U_k$ on $JT$ such that the support $\sigma_k$ of the elements in $U_kJT$ is given by

$$\sigma_k = \{(n,i);\ (n,i) \leqslant (n_k, i_k),\ (n,i) \nleqslant (n_{k+1}, i_{k+1}),\ (n,i)\notin\boldsymbol{B}\}.$$

It is easily verified that these $\{\sigma_k\}_{k=1}^{\infty}$ verify the assumption of (2.5) (i.e. each segment intersects at most one of them) and hence for every $j$

$$(2.13)\qquad \Big\|\sum_{k=1}^{j} U^*_k x^*\Big\|^2 = \sum_{k=1}^{j}\|U^*_k x^*\|^2.$$

However, for every $k$, $\|U^*_k x^*\| > \frac{1}{2}a$ by (2.10) and (2.11) while by (2.12) $\big\|\sum_{k=1}^{j} U^*_k\big\| \leqslant 4$ for all $j$. This contradicts (2.13) for $j > 64\|x^*\|^2/a^2$, and thus concludes the proof of the lemma.

We return to the proof of Theorem 1 itself. Assume that $B$ is a proper subspace of $Y$. Let $\delta > 0$ be such that

$$(2.14)\qquad 3,5 < 4(1-\delta)^2$$

and pick $x^* \in Y$ such that

$$(2.15)\qquad d(x^*, B) > 1-\delta, \qquad \|x^*\| = 1.$$

Let $m$ be an integer so that

$$(2.16)\qquad \|x^* - P^*_m x^*\| > 1-\delta,$$

and let $\varepsilon > 0$ be such that

$$(2.17)\qquad 2^{m+2}\varepsilon^2 < (1-\delta)^2.$$

By Lemma 1 there is a $q > m$ such that

$$(2.18)\qquad \|P^*_{q,j}x^*\| \leqslant \varepsilon, \qquad 0 \leqslant j < 2^q.$$

By (2.15) $\|P^*_q x^*\| \geqslant 1-\delta$ and hence by (2.6)

$$\sum_{j=0}^{2^q-1}\|P^*_{q,j}x^*\|^2 > (1-\delta)^2.$$

It follows that for $0 \leqslant j < 2^q$ there exist $x_j \in JT$ with $\|x_j\| = 1$, $P_{q,j}x_j = x_j$ and

$$(2.19)\qquad C^2 = \sum_{j=0}^{2^q-1}|P^*_{q,j}x^*(x_j)|^2 = \sum_{j=0}^{2^q-1}|x^*(x_j)|^2 > (1-\delta)^2.$$

Define next

$$x = \sum_{j=0}^{2^q-1} x^*(x_j)x_j/C. \tag{2.20}$$

Note that by (2.6), (2.18), (2.19), and (2.20) we have

$$\|x\| = 1, \quad x^*(x) \geqslant 1-\delta, \quad \|P_{q,j}x\| = |x^*(x_j)|/C \leqslant \varepsilon/(1-\delta), \quad 0 \leqslant j < 2^q. \tag{2.21}$$

It will be convenient to assume also (and we clearly can do this without loss of generality) that $P_p(x) = 0$ for some $p > q$. By (2.16) there is a $y \in JT$ with

$$\|y\| = 1, \quad P_{m+1}y = 0, \quad x^*(y) > 1-\delta, \tag{2.22}$$

and thus, in particular,

$$x^*(x+y) > 2(1-\delta). \tag{2.23}$$

Our next aim is to obtain an estimate for $\|x+y\|$ and use this via (2.14) to obtain a contradiction to (2.23) and thus prove the theorem. Let

$$y+x = \sum_{n=0}^{p}\sum_{i=0}^{2^n-1} t_{n,i}e_{n,i};$$

then by our construction $t_{n,i} = 0$ for $m < n < q$. By the definition of the norm in $JT$ there are pairwise disjoint segments $\mathscr{S}_k, \mathscr{R}_{k'}, \mathscr{T}_{k''}$ ($1 \leqslant k \leqslant k(s)$, $1 \leqslant k' \leqslant k'(r)$, $1 \leqslant k'' \leqslant k''(t)$) such that each $\mathscr{R}_{k'}$ contains no element $(n, i)$ with $n \geqslant q$, each $\mathscr{T}_{k''}$ does not contain an element $(n, i)$ with $n \leqslant m$, and each $\mathscr{S}_k$ contains elements of the form $(m, k)$ and $(q, j)$ so that

$$\|x+y\|^2 = \sum_{k=1}^{k(s)}\Big(\sum_{(n,i)\in\mathscr{S}_k} t_{n,i}\Big)^2 + \sum_{k'=1}^{k'(r)}\Big(\sum_{(n,i)\in\mathscr{R}_{k'}} t_{n,i}\Big)^2 + \sum_{k''=1}^{k''(t)}\Big(\sum_{(n,i)\in\mathscr{T}_{k''}} t_{n,i}\Big)^2 = s+r+t. \tag{2.24}$$

First note that

$$t \leqslant \|x\|^2 = 1. \tag{2.25}$$

Next since $(\alpha+\beta)^2 \leqslant 2(\alpha^2+\beta^2)$ for all $\alpha$ and $\beta$, we get that

$$s \leqslant 2\Big(\sum_{k=1}^{k(s)}\Big(\sum_{\substack{(n,i)\in\mathscr{S}_k\\ n<q}} t_{n,i}\Big)^2 + \sum_{k=1}^{k(s)}\Big(\sum_{\substack{(n,i)\in\mathscr{S}_k\\ n\geqslant q}} t_{n,i}\Big)^2\Big) = 2(s'+s''). \tag{2.26}$$

Note that

$$2s'+r \leqslant 2(s'+r) \leqslant 2\|y\|^2 = 2. \tag{2.27}$$



Observe also that the number of the $\mathscr{S}_k$, i.e. $k(s)$, is less than $2^m$. Let $(q, j_k) \in \mathscr{S}_k$, $1 \leqslant k \leqslant k(s)$. By (2.21)

$$s'' = \sum_{k=1}^{k(s)}\Big(\sum_{(n,i)\geqslant(q,j_k)} t_{n,i}\Big)^2 \leqslant \sum_{k=1}^{k(s)} \|P_{q,j_k}x\|^2 \leqslant 2^m\varepsilon^2/(1-\delta)^2. \tag{2.28}$$

Combining (2.17), (2.24)—(2.28), we get that

$$\|x+y\|^2 \leqslant t+2(s'+r)+2s'' \leqslant 1+2+0{,}5 = 3{,}5. \tag{2.29}$$

This however contradicts (2.14) and (2.23) and concludes the proof of the theorem.

We pass now to some corollaries of the theorem. We would first like to recall the obvious fact that if $B$ is the space appearing in the statement of Theorem 1 then $B^*$ is isometric to $JT$.

COROLLARY 1. *For every integer* $k > 1$, $B^{(2k)} \approx B^{**} \oplus l_2(\Gamma)$ *and* $B^{(2k-1)} \approx B^* \oplus l_2(\Gamma)$. *Thus none of the conjugates of* $B$ *contains a subspace isomorphic to* $c_0$ *or* $l_1$. *The conjugates of odd order of* $B$ *are all weakly compactly generated* (WCG *in short*) *while those of even order* (*except* $B$ *itself*) *are not* WCG.

Proof. By the Theorem, $B^{**}/QB$ is isometric to $l_2(\Gamma)$. By standard facts concerning duality (cf. e.g. [2]) $B^{***} = QB^{\perp} \oplus Q_1B^*$ (where $Q_1$: $B^* \to B^{***}$ denotes the canonical embedding). Hence $B^{***} \approx l_2(\Gamma) \oplus B^*$. The first assertion of the corollary follows now by trivial induction on $k$ using the fact that $l_2(\Gamma) \oplus l_2(\Gamma) \approx l_2(\Gamma)$. The second assertion follows easily from the first one (use e.g. the fact that the cardinality of all the conjugates of $B$ is less than that of $l_\infty^*$). As for the third assertion recall that a Banach space is said to be WCG if it is the closed linear span of a weakly compact set. Separable and reflexive spaces are trivially WCG and hence the same is true for $B^{(2k-1)}$ for all integers $k$. Also, trivially, a non-separable conjugate of a separable space is non-WCG and a complemented subspace of a WCG space is WCG hence $B^{(2k)}$ for $k = 1, 2, \ldots$ is non-WCG.

Remark. The even conjugates of $B$ are thus examples of non-WCG spaces whose duals are WCG. This answer question 3 in survey paper [10] on WCG spaces. Another (much simpler) answer to this question was recently given in [9].

COROLLARY 2. *$JT$ is $w^*$-sequentially dense in* $(JT)^{**}$.

Proof. For every branch $\boldsymbol{B}$ of $T$, let $f_{\boldsymbol{B}} \in JT^*$ be defined by

$$f_{\boldsymbol{B}}\Big(\sum_{n=0}^{\infty}\sum_{i=0}^{2^n-1} t_{n,i}e_{n,i}\Big) = \sum_{(n,i)\in\boldsymbol{B}} t_{n,i}. \tag{2.30}$$

For each such $\boldsymbol{B}$ define $F_{\boldsymbol{B}} \in JT^{**}$ by $F_{\boldsymbol{B}}(f_{n,i}) = 0$ for all $n$ and $i$, $F_{\boldsymbol{B}}(f_{\boldsymbol{B}}) = 1$ and $F_{\boldsymbol{B}}(f_{\boldsymbol{B}'}) = 0$ if $\boldsymbol{B} \neq \boldsymbol{B}'$. The representation of $B^{***} = JT^{**}$ given in the beginning of the proof of Corollary 1 means explicitly that each $x^{**}\in JT^{**}$ has a unique representation of the form

$$x^{**} = \sum_{j=1}^{\infty} s_j F_{B_j} + Qx$$

where $x\in JT$, the $B_j$ are distinct branches of $T$ and $\sum s_j^2 < \infty$ (actually to ensure uniqueness we have also to require that $s_j \neq 0$ for all $j$, so the sum on $j$ may also be finite or even empty). Moreover, $\|x\| \leqslant \|x^{**}\|$ and $(\sum s_j^2)^{1/2} \leqslant \|x^{**}\|$. Choose now a sequence $n_1 < n_2 < \ldots$ of integers so that $\{B_j\}_{j=1}^k$ do not intersect on or below the $n_k$th level. Pick $i_{k(j)}$ so that $(n_k, i_{k(j)})\in B_j$ and put $x_k = \sum_{j=1}^{k} s_j e_{n_k i_{k(j)}}$. Then $\|x_k\| \leqslant \|x^{**}\|$ for all $k$. It is obvious that $x^*(x+x_k) \to x^{**}(x^*)$ for $x^* = f_{n,i}$ for some $n$ and $i$ or for $x^* = f_B$ for some $B$. Hence $Q(x+x_k) \xrightarrow{\omega^*} x^{**}$.

Remark. The sequence $x+x_k$ satisfies $\|x+x_k\| \leqslant 2\|x^{**}\|$. It follows, however, from the Corollary and an observation of McWilliams [11] that there is a sequence $\{z_k\}_{k=1}^{\infty}$ in $X$ such that $\|z_k\| = \|x^{**}\|$ and $Qz_k \xrightarrow{\omega} x^{**}$.

COROLLARY 3. *Every bounded sequence in $JT$ has a weakly Cauchy subsequence.*

Proof. Let $\{y_m\}_{m=1}^{\infty} \subset JT = B^*$ with $\|y_m\| \leqslant 1$ for all $m$. There is a subsequence of $\{y_m\}_{m=1}^{\infty}$ (which we may assume without loss of generality to be the sequence itself) which converges $\omega^*$ to some $y\in JT$, i.e. $\lim_{m\to\infty} (y_m - y)(b) = 0$ for all $b\in B$. Put $x_m = y_m - y$. Since all the coordinates of $x_m$ tend to 0 as $m\to\infty$, it follows from (2.5) that for every choice of distinct branches $\{B_j\}_{j=1}^k$ of $T$

$$\limsup_{m\to\infty} \sum_{j=1}^{k} |f_{B_j}(x_m)|^2 \leqslant \|x_m\|^2 \leqslant 4. \tag{2.31}$$

By (2.31) there is a subsequence $\{m_i^1\}_{i=1}^{\infty}$ of the integers and an integer $0 \leqslant k_1 \leqslant 16$ such that, for $k_1$ distinct branches $B_j$, $\lim_{i\to\infty} f_{B_j}(x_{m_i^1})$ exists and is of absolute value $\geqslant \frac{1}{2}$ while for all other branches $\limsup_i |f_B(x_{m_i^1})| < \frac{1}{2}$. By using again (2.31) we can find subsequences $\{m_i^2\}_{i=1}^{\infty}$ of $\{m_i^1\}_{i=1}^{\infty}$ and a $k_2$ with $k_1 \leqslant k_2 \leqslant 2^6$ such that, for $k_2$ distinct branches $B_j$, $\lim_{i\to\infty} f_{B_j}(x_{m_i^2})$ exists while for all other branches the absolute value of the corresponding sequence has limsup at most $1/2^2$. Continuing in the same manner and passing to the diagonal sequence, we get finally a subsequence $\{m_i\}$ of the integers so that $\lim_i f_B(x_{m_i})$ exists for all branches $B$ of $T$. Since by Theorem 1 the elements of the form $f_B$ together with $QB$ span $JT^*$, it follows that $\lim_{i\to\infty} x^*(y_{m_i})$ exists for every $x^*\in JT^*$ and this proves the Corollary.



H. P. Rosenthal pointed out to us that Corollary 3 is actually a special case of a general result.

PROPOSITION (H. P. Rosenthal). *Let $X$ be a separable Banach space such that $X^{**}/QX$ is reflexive. Then every bounded sequence in $X^*$ has a weakly Cauchy subsequence.*

Proof. As in the first part of the proof of Corollary 3 it is easily seen that without loss of generality we may assume that the given sequence $\{x_n^*\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} x_n^*(x) = 0$ for all $x\in X$. Let $Q_1: X^* \to X^{***}$ be the canonical embedding. Since $x_n^* \xrightarrow{\omega^*} 0$, it follows immediately that the set $K$ of limiting points of $\{Q_1 x_n^*\}$ in $X^{***}$ in the $\omega^*$-topology (i.e. the topology induced by $X^{**}$) is a $\omega^*$-bounded and closed subset of $QX^{\perp}$. By assumption, $QX^{\perp}$ is reflexive and hence $K$ in its $\omega^*$-topology is an Eberlein compact (in the terminology of [10]). By Theorem 3.8 of [10], $K$ has a $G_\delta$ point, $x^{***}$ say. In other words, there is a sequence of $\omega^*$-open sets $\{O_k\}_{k=1}^{\infty}$ in $X^{***}$ such that

$$x^{***} = K \cap \bigcap_{k=1}^{\infty} O_k. \tag{2.32}$$

By the definition of $K$ there is for every $k$ an integer $n_k$ such that $Q_1 x_{n_k}\in O_k$. By (2.32) the sequence $\{Q_1 x_{n_k}\}_{k=1}^{\infty}$ tends $\omega^*$ to $x^{***}$ and thus $\{x_{n_k}\}_{k=1}^{\infty}$ is a $\omega$-Cauchy sequence in $X^*$.

Our final two corollaries to Theorem 1 are related to vector measures. Before we state them, however, we would like to recall the main result of [14] and observe the form it takes in $JT$. The main theorem in [14] is a certain representation theorem for any separable space $X$ whose dual is non-separable. For $JT$ this representation can be verified directly and very easily. This representation will be of importance in the two remaining corollaries. Thus, though we do not need here the result proved in [14] we mention it since it shows that, like the case of Corollary 3, the remaining corollaries can also be extended to a more general setting.

Let $\Delta = \{0, 1\}^{\aleph_0}$ be the usual Cantor set. Let $\{h_{n,i}\}_{n=0}^{\infty}{}_{i=0}^{2^n-1}$ be the usual Haar system on $\Delta$. The function $h_{(n,i)}$ is defined as the characteristic function of the clopen subset $\Delta_{n,i}$ of $\Delta$ defined by

$$\Delta_{n,i} = \{\theta = (\theta_0, \theta_1, \theta_2, \ldots)\in\Delta;\ i = \theta_{n-1} + 2\theta_{n-2} + \ldots + 2^{n-1}\theta_0\}. \tag{2.33}$$

Observe that

$$h_{n,i} = h_{n+1,2i} + h_{n+1,2i+1}; \quad n = 0, 1, \ldots;\ i = 0, \ldots, 2^n - 1. \tag{2.34}$$

The main result of [14] is the following: Let $X$ be a separable space. Then $X^*$ is non-separable if and only if for every $\varepsilon > 0$ there exists an operator $R: X \to C(\Delta)$ (= the space of continuous functions on $\Delta$ with the sup norm) of norm 1, and a set $\{x_{n,i}\} \subset X$ such that $\|x_{n,i}\| \leqslant 1+\varepsilon$

for all $n$ and $i$ and

$$\sum_{n=0}^{\infty}\sum_{i=0}^{2^n-1}\|Rx_{n,i}-h_{n,i}\|<\varepsilon.$$

The non-trivial part of this result is clearly the "only if" part. For the space $JT$ there is a natural and simple candidate for $R$ which has the required property even with $\varepsilon = 0$. It is clear that

$$(2.35)\qquad Re_{n,i}=h_{n,i},\quad n=0,1,2,\ldots;\ i=0,1,\ldots,2^n-1,$$

defines an operator of norm 1 from $JT$ into $C(\Delta)$. Let us observe that this operator $R$ defines a 1-1 correspondence between $\Delta$ and the set $\Gamma$ of branches of $T$. Indeed, for every $\theta\in\Delta$ there corresponds a unique $\boldsymbol{B}\in\Gamma$ such that $Rx(\theta)=f_{\boldsymbol{B}}(x)$.

Let us recall that a Banach space $X$ has the *Radon–Nikodym Property* (RNP in short) if for any finite measure space $(S, \Sigma, \mu)$ any $\mu$-continuous $X$ valued measure $m$ on $\Sigma$ of finite total variation is the indefinite integral with respect to $\mu$ of an $X$ valued Bochner measurable function on $S$. There are several equivalent formulations (which look entirely different) of the RNP. We refer to [14] for background and further references.

COROLLARY 4. *The spaces $B^{(2k+1)}$, $k=0,1,2,\ldots$, all have the* RNP, *while the spaces $B^{(2k)}$, $k=0,1,2,\ldots$, fail to have the* RNP. *In particular, $B$ is not a subspace of a separable conjugate space.*

Proof. Since separable conjugate spaces and reflexive spaces have the RNP (cf. [14]), the first assertion of Corollary 4 follows immediately from Corollary 1. In order to prove the second (and third) assertion of Corollary 4 it is enough to show that $B$ does not have the RNP. Let $R$ be the operator given by (2.35) and let $\mu$ be the Haar measure on $\Delta$ (defined by $\int h_{n,i}d\mu=2^{-n}$ for all $n$ and $i$). We can consider the $h_{n,i}$ also as elements in $L_1(\mu)$ and thus in a canonical way as elements of $C(\Delta)^*$. Let $y^*_{n,i}=2^nR^*h_{n,i}\in JT^*=B^{**}$. By definition, $|y^*_{n,i}(e_{m,j})|\leqslant 2^{n-m}$ and hence, by Theorem 1, ($y^*_{n,i}\in Y$; cf. (2.8)) or by a simple direct verification, $y^*_{n,i}\in QB$ for all $n$ and $i$. We have that $y^*_{n,i}=(y^*_{n+1,2i}+y^*_{n+1,2i+1})/2$ (by (2.34)), $\|y^*_{n,i}\|\leqslant 1$ and, as easily checked, $\|y^*_{n,i}-y^*_{m,j}\|\geqslant 1/2$ unless $(n,i)=(m,j)$. It is known that a space which has a set like $\{y^*_{n,i}\}$ does not have RNP (again see [14] and its references; the set $\{y^*_{n,i}\}$ is a typical example of a "non-dentable" set).

Our last corollary answers a question from the theory of vector measures. Let $K$ be a compact Hausdorff space and let $\mu$ be a measure on $K$. We say that a function $\sigma\colon K\to X$ for some Banach space $X$ is *weakly measurable* (with respect to $\mu$) if for every $x^*\in X^*$ the function $x^*\circ\sigma$ belongs to $L_\infty(K,\mu)$. We say that two such functions $\sigma$ and $\tau$ are *equivalent* if $x^*\circ\sigma=x^*\circ\tau$ $\mu$-almost everywhere for every $x^*\in X^*$. It is well known



that for separable $X$, every weakly measurable function is equivalent to a Bochner (i.e. strongly) measurable function. It is also known that this is no longer the case if $X\supset l_\infty$. The question was whether the existence of a weakly measurable function which is not equivalent to a strongly measurable one implies that $X\supset l_\infty$. Our next corollary shows that the answer is negative.

COROLLARY 5. *There is a weakly measurable function $\sigma$ from the Cantor set $\Delta$ (endowed with the Haar measure $\mu$) into $JT^*$ which is not equivalent to any strongly measurable function.*

Proof. The function $\sigma$ is simply the correspondence between $\Delta$ and $\Gamma$ discussed above, i.e.

$$\sigma(\theta)=R^*(\delta_\theta),\qquad \theta\in\Delta,$$

where $\delta_\theta$ denotes the Dirac measure. Clearly, $\sigma(\theta)(e_{n,i})=h_{n,i}\in C(\Delta)$ and hence $\sigma(\theta)(x)\in C(\Delta)$ for every $x\in JT$. By Corollary 2 for every $x^{**}\in JT^{**}$ there is a sequence $\{x_k\}$ with $Qx_k\xrightarrow{\omega^*}x^{**}$ and hence

$$x^{**}\circ\sigma(\theta)=\lim_{k\to\infty}\sigma(\theta)(x_k)\in L_\infty(\Delta,\mu).$$

Hence $\sigma$ is weakly measurable. Assume that there is a $\tau\colon\Delta\to JT^*$ which is strongly measurable and equivalent to $\sigma$. By Lusins' theorem there is a closed subset $K$ of $\Delta$ with $\mu(K)>0$ such that

(2.36) *The restriction of $\tau$ to $K$ is norm continuous*

and

$$(2.37)\qquad Qe_{n,i}\circ\tau\in C(K),\quad n=0,1,2,\ldots,\ i=0,\ldots,2^n-1.$$

By (2.36) and (2.37) the set $\{Qe_{n,i}\circ\tau\}$ is equicontinuous and bounded and thus a relatively norm compact set in $C(K)$. Since $\tau$ is equivalent to $\sigma$, it follows also from (2.37) that

$$Qe_{n,i}\circ\tau(\theta)=Qe_{n,i}\circ\sigma(\theta)=h_{n,i}(\theta)$$

for every $\theta$ in $K$. It is however easy to verify that the restriction of the Haar system is not relatively compact on any set of positive measure. This contradiction proves Corollary 5.

Our final corollary to Theorem 1 is a weaker version of the result proved in [7]. Our proof is easier (though rather contrived) than the argument given in [7].

COROLLARY 6. *Let $Y$ be any infinite dimensioned closed subspace of $JT$. Then $Y$ has an infinite dimensional reflexive subspace.*

Proof. From the results of [8] it suffices to show that $Y$ has a subspace $Z$ such that $Z^{**}$ is separable. It follows from Theorem 1 that if $Z^*$ is separable then $Z^{**}$ is separable. Let $R\colon JT\to C(\Delta)$ be the operator defined by (2.35). Let $l_1(\Delta)$ be the subspace of $C(\Delta)^*$ consisting of the purely atomic measures. It is easy to check that $SR^*$ defines the canonical diagonal operator from $l_1(\Delta)$ to $l_2(\Delta)$ ($S$ is the operator defined in (2.7)), where $l_2(\Delta)$ is naturally identified with $JT^*/B$. Also note that if $\mu$ is a purely non-atomic element of $C(\Delta)^*$ then $R^*(\mu)$ is in $B$.

Let $Y$ be a subspaces of $JT$ such that $Y^*$ is non-separable and let $I\colon Y\to JT$ denote the containment operator. Suppose there exists a $\delta > 0$ such that $\delta\|y\| \leqslant \|Ry\| \leqslant \|y\|$ for all $y$ in $Y$. Let $Q\colon Y^*\to Y^*/\overline{I^*(B)}$ denote the quotient operator and $V\colon JT^*/B\to Y^*/\overline{I^*(B)}$ the induced operator; that is, $QI^* = VS$. Since $I^*R^*$ is onto $QI^*R^*$ and $VSR^*$ are onto. Thus we have that $VSR^*(l_1(\Delta)) = Y^*/\overline{I^*(B)}$. Since $SR^*$, when restricted to $l_1(\Delta)$, is the canonical diagonal operator from $l_1(\Delta)$ to $l_2(\Delta)$ and $Y^*/\overline{I^*(B)}$ is infinite dimensional, it is impossible that $VSR^*(l_1(\Delta)) = Y^*/\overline{I^*(B)}$. Thus $RI$ is not an isomorphism. To complete the proof, let $Y$ be an infinite dimensional subspace of $JT$. Let $\{b_k\}_{k=1}^{\infty}$ be a normalized basic sequence in $Y$ (see [2]). We may assume that $[b_k]^*$ is non-separable. Then there exists an integer $N$, and an element $z_1$ in $[b_k]$, $1\leqslant k\leqslant N$, $\|z_1\| = 1$, such that $\|Rz_1\| < 1/2$. Since $[b_k]^*$, $k > N$, is non-separable, there exists an $N_2 > N$, and an element $z_2$ in $[b_k]$, $N_1 < k \leqslant N_2$, $\|z_2\| = 1$, such that $\|Rz_2\| < 1/4$. Continuing in this manner, we obtain a normalized block-basic sequence $\{z_j\}_{j=1}^{\infty}$ contained in $[b_k]_{k=1}^{\infty}$ such that $\sum_{j=1}^{\infty}\|Rz_j\| \leqslant 1$. Let $Z = [z_j]$ and let $I\colon Z\to JT$ denote the containment operator. Then $RI$ is a compact operator so $I^*R^*$ has separable range. Since $R^*(C(\Delta)^*)+B$ is dense in $JT^*$, $I^*R^*(C(\Delta)^*)+I^*(B)$ is dense in $Z^*$. Both $I^*R^*(C(\Delta)^*)$ and $I^*(B)$ are separable. So $Z^*$ is separable.

After we obtained the results presented here a number of related results have come to our attention.

First, Davis, Johnson, Figiel, and Pełczyński observed that their general construction used for factoring weakly compact operators [3] can be used also to produce examples of a type similar to those given here. Their approach makes the proofs simpler but the examples they obtained are more difficult to describe explicitly. They outlined their approach in the short appendix to [4].

Second, H. Rosenthal [13] has shown that if $X$ is a Banach space that contains a bounded sequence which has no weakly Cauchy subsequence then $X$ has a subspace isomorphic to $l_1$. Also, H. Rosenthal and E. O'dell [12] have shown that if $X$ is a separable Banach space and there exists an element of $X^{**}$ that is not the weak* sequential limit of elements of $X$, then $X$ has a subspace isomorphic to $l_1$. Thus Corollaries 2 and 3 also hold for the space $JF$.



**3. The James Function space $JF$.** The space $JF$ was defined already in the introduction. We find it more convenient to use it in the form given in (1.2). Let us make some preliminary observations concerning this space. For every $f\in L_1(0,1)$ and every partition $\{t_i\}_{i=0}^{k}$ of $[0,1]$

$$\Big(\sum_{i=0}^{k-1}\Big(\int_{t_i}^{t_{i+1}} f(t)\,dt\Big)^2\Big)^{\frac{1}{2}} \leqslant \sum_{i=0}^{k-1}\Big|\int_{t_i}^{t_{i+1}} f(t)\,dt\Big| \leqslant \int_0^1 |f(t)|\,dt,$$

and hence $\|f\| \leqslant \|f\|_{L_1}$. Moreover, if $f\geqslant 0$ we get by taking the partition $0 = t_0 < t_1 = 1$ that $\|f\| = \|f\|_{L_1}$. It follows that $JF$ is separable (the characteristic functions $X_I$ of intervals $I$ with rational endpoints span $JF$) and that $JF \subset L_1(0,1)$. For every interval $I$ the function $X_I$ can be considered also as an element of $JF^*$ in an obvious manner $\big(X_I(f) = \int_I f(t)dt\big)$. From (1.2) it is evident that $\|X_I\|_{JF^*}\leqslant 1$ and by applying $X_I$ on $f = X_I$ we get that actually $\|X_I\|_{JF^*} = 1$ for every interval $I$. In particular,

$$\|X_{[0,s]} - X_{[0,t]}\|_{JF^*} = 1 \quad \text{if} \quad t\neq s$$

and thus $JF^*$ is non-separable.

Consider the system $\{r_n(t)\}_{n=1}^{\infty}$ of Rademacher functions on $[0,1]$. Recall that

$$r_n(t) = (-1)^k \quad \text{for} \quad t\in\big(k2^{-n}, (k+1)2^{-n}\big),\ k = 0, 1, \ldots, 2^n-1.$$

It is easy to compute $\|r_n\|$. The partition for which the supremum in (1.2) is attained is the one determined by the points $\{k2^{-n}\}_{k=0,1,\ldots,2^n-1}$ and thus $\|r_n\| = 2^{-n/2}$.

Computations similar to those made by Giesy and James ([4], Lemma 1) show that if $\{n_i\}_{i=1}^{\infty}$ is a sequence of integers tending fast enough to $\infty$ then the sequence $\{r_{n_i}/\|r_{n_i}\|\}_{i=1}^{\infty}$ is equivalent to the unit vector basis in $c_0$. The passage from Lemma 1 of [4] to the situation here is not, however, immediate. The difficulty of using in the present context Lemma 1 of [4] stems from the fact that (with the notation of [4]) $\varepsilon_2$ there is not allowed to become arbitrary close to $\varepsilon$, but only to $2\varepsilon$. Thus, if we use this lemma iteratively we get that $\varepsilon_k \geqslant 2^k\varepsilon$, and it follows that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is an unbounded sequence. What we need, therefore, is a version of Lemma 1 of [4] where $\varepsilon_2$ is allowed to be close to $\varepsilon$, provided $N$ is large enough. We verified that Lemma 1 is also true in this form. Our proof of this is longer than the argument presented in [4]. (Moreover, it seems to us that the argument in [4] is incomplete since in the formula preceding formula (6) there it is not clear to us that without loss of generality $y(r_1) \leqslant y(p_i)$ and $y(p_{i+1}) \leqslant y(r_\lambda)$). Since we shall not need this in the sequel, we do not present the details here.

We state now our result concerning the space $JF$.

THEOREM 2. *The space $JF$ has no subspace isomorphic to $l_1$.*

**Proof.** We shall present the proof in detail for a space which is actually a slight variant of $JF$. We shall assume that the exponent 2 in (1.2) is replaced by some fixed $p$ with $1 < p < 2$. Thus we shall use in the proof the following expression for the norm

$$\|f\| = \sup\Big(\sum_{i=0}^{k-1}\Big|\int_{t_i}^{t_{i+1}} f(t)\,dt\Big|^p\Big)^{\frac{1}{p}} \tag{3.1}$$

where the sup is over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ of the unit interval. In our approach it turns out that the case $1 < p < 2$ is somewhat simpler to handle than that of $p = 2$. Since the computations are long enough already for $1 < p < 2$ we choose to present this case in detail. The assertion of the theorem is, however, valid as stated (i.e. for $p = 2$) and even for any $1 < p < \infty$. At the end of the proof we shall indicate briefly the additional argument which is needed for the case $p = 2$.

In order to make the proof easier we present it first without giving the details in three places which involve somewhat lengthy computations. The details are presented after the end of the main part of the proof (cf. the proofs of Lemmas 2, 3, and 4).

It was proved by James [6] that if $X$ is a Banach space containing a subspace isomorphic to $l_1$ then for every $\varepsilon > 0$ there is a subspace $Y = Y(\varepsilon)$ of $X$ whose Banach–Mazur distance from $l_1$ is $\leqslant 1+\varepsilon$, i.e. for which there is an onto operator $T\colon Y \to l_1$ with $\|y\| \leqslant \|Ty\| \leqslant (1+\varepsilon)\|y\|$ for all $y \in Y$. It follows from this result that if $X$ contains a subspace isomorphic to $l_1$ and $\varepsilon > 0$ is given, then there is a $y \in X$ with $\|y\| = 1$ such that for every finite set $\{x_i^*\}_{i=1}^n$ in $X^*$ there is a $z \in X$ satisfying

$$\|z\| = 1,\ x_i^*(z) = 0, \quad i = 1, 2, \ldots, n, \quad \|y \pm z\| \geqslant 2 - 4\varepsilon.$$

Indeed, choose $Y = Y(\varepsilon)$ as above and pick $y \in Y$ such that $\|y\| = 1$ and $Ty = \lambda e_1$ for a suitable scalar $\lambda$, where $e_1$ denotes the first unit vector in $l_1$. Now if $\{x_i^*\}_{i=1}^n$ are given, a suitable vector $z$ is obtained by picking any vector of norm 1 in the infinite-dimensional space

$$\overline{\operatorname{span}}\{T^{-1}e_i\}_{i=2}^{\infty} \cap \{x;\ x_i^*(x) = 0,\ i = 1, \ldots, n\}.$$

Let us note also that obviously $y$ (and also $z$) can be chosen so as to belong to any preassigned dense linear subspace of $X$.

Returning to the space $JF$ we note therefore that in order to prove Theorem 2 it is enough to prove that *the following statement leads to a contradiction.*



(*) For every $\varepsilon > 0$ there is a simple function $u$ in $JF$ with $\|u\| = 1$ so that for any integer $n$ there exists a $v \in JF$ satisfying

(i) $\|v\| = 1$,

(ii) $\|u+v\| > 2-\varepsilon$, $\|u-v\| > 2-\varepsilon$,

(iii) $\int_{j/n}^{(j+1)/n} v(t)\,dt = 0$, $j = 0, 1, 2, \ldots, n-1$.

Assume that a $u$ satisfying (*) exists for $\varepsilon > 0$ small enough (the exact requirement on $\varepsilon$ will be determined later). Let

$$K = \sup_{0\leqslant t\leqslant 1}|u(t)|, \qquad n = [(100K/\varepsilon^p)^{2p/(p-1)}]+1 \tag{3.2}$$

where $[\lambda]$ denotes the largest integer $\leqslant \lambda$. Let $v$ be a function satisfying (i), (ii), and (iii) of (*) for this value of $n$. By the first inequality in (ii) there is a partition $\{t_i\}_{i=0}^k$ of $[0, 1]$ so that

$$\Big(\sum_{i=0}^{k-1}\Big|\int_{t_i}^{t_{i+1}}\big(u(t)+v(t)\big)\,dt\Big|^p\Big)^{1/p} > 2-\varepsilon. \tag{3.3}$$

We come now to the first assertion whose proof will be given only later on.

LEMMA 2. *We may assume without loss of generality that the partition used in* (3.3) *satisfies $t_{i+1} - t_i > 1/\sqrt{n}$ for all $i$.*

One obvious consequence of Lemma 2 is that $k$ (the number of intervals in the partition) is less than $\sqrt{n}$. For every $1 \leqslant i \leqslant k-1$, let $l(i)$ be the integer so that

$$l(i)/n \leqslant t_i < \big(l(i)+1\big)/n. \tag{3.4}$$

It follows from Lemma 2 that for $i_1 \neq i_2$ we have $l(i_1) \neq l(i_2)$. Put next for $i = 0, 1, \ldots, k-1$

$$x_i = \int_{t_i}^{t_{i+1}} u(t)\,dt,\quad y_i = \int_{t_i}^{t_{i+1}} v(t)\,dt, \tag{3.5}$$

and for $i = 1, 2, \ldots, k-1$

$$z_i = \int_{l(i)/n}^{t_i} v(t)\,dt. \tag{3.6}$$

In view of property (iii) of $v$ it follows that

$$-z_i = \int_{t_i}^{(l(i)+1)/n} v(t)\,dt \tag{3.7}$$

and (if we put $z_0 = z_k = 0$)

$$y_i = z_{i+1} - z_i, \quad i = 0, 1, \ldots, k-1. \tag{3.8}$$

From (3.3) and the triangle inequality in $l_p$ we get that

$$\Big(\sum_{i=0}^{k-1} |x_i|^p\Big)^{1/p} > 1-\varepsilon, \quad \Big(\sum_{i=0}^{k-1} |y_i|^p\Big)^{1/p} > 1-\varepsilon. \tag{3.9}$$

We state now the second assertion whose proof is postponed till after the end of the main part of the proof. The assertion states that "as a rule" the signs of $x_i$, $-z_i$, and $z_{i+1}$ are the same.

Lemma 3. *There is a $\delta = \delta_1(\varepsilon)$ (with $\delta_1(\varepsilon)\to 0$ as $\varepsilon\to 0$) so that*

$$\Big(\sum{}' |z_i|^p\Big)^{1/p} \leqslant \delta_1(\varepsilon), \quad \Big(\sum{}'' |z_{i+1}|^p\Big)^{1/p} \leqslant \delta_1(\varepsilon)$$

*where the sum $\sum'$ (resp. $\sum''$) extends over those indices $i$ for which $x_i z_i > 0$ (resp. $x_i z_{i+1} < 0$).*

Using the second inequality in (ii) of (*), we can repeat for $-u+v$ the same analysis made thus far for $u+v$. In other words, there is a partition $\{s_j\}_{j=0}^{h}$ of $[0, 1]$ so that

$$\Big(\sum_{j=0}^{h-1} \Big|\int_{s_j}^{s_{j+1}} \big(-u(t)+v(t)\big)\,dt\Big|^p\Big)^{1/p} > 2-\varepsilon \tag{3.10}$$

and so that $s_{j+1} - s_j > 1/\sqrt{n}$ for every $j$ (Lemma 2). Let $m(j)$, $j = 1, \ldots, h-1$, be such that

$$m(j)/n \leqslant s_j < \big(m(j)+1\big)/n, \tag{3.11}$$

and define

$$\hat{x}_j = \int_{s_j}^{s_{j+1}} -u(t)\,dt, \quad \hat{y}_j = \int_{s_j}^{s_{j+1}} v(t)\,dt. \tag{3.12}$$

We also define in an obvious way the numbers $\hat{z}_j$. Lemma 3 shows also that "as a rule" the signs of $\hat{x}_j$, $-\hat{z}_j$, and $\hat{z}_{j+1}$ are the same.

It may happen that $l(i) = m(j)$ for certain $i$ and $j$. Our third assertion, to be proved later, says that if this happens then "in general" the signs of $z_i$ are different from that of $\hat{z}_j$.

Lemma 4. *There is a $\delta = \delta_2(\varepsilon)$ (with $\delta_2(\varepsilon)\to 0$ as $\varepsilon\to 0$) such that*

$$\Big(\sum{}''' \min(|z_i|^p, |z_{j(i)}|^p)\Big)^{1/p} \leqslant \delta_2(\varepsilon)$$



*where $\sum'''$ is extended over all integers $i$ such that there is a $j = j(i)$ with $l(i) = m\big(j(i)\big)$ and $z_i\hat{z}_{j(i)} > 0$.*

The proof of Lemma 4 is based on Lemma 3 and the fact that $x_i$ is defined as the integral of $u$ on some interval while the $\hat{x}_j$ are the integrals of $-u$.

Consider now the partition of $[0, 1]$ obtained by taking as division points all the points of the form $t_i$, $s_j$, $l(i)/n$ and $m(j)/n$ with the following exception: if $i$ and $j$ are such that $l(i) = m(j)$ and $z_i\hat{z}_j > 0$ (i.e. $i$ enters into the sum $\sum'''$ of Lemma 4) we omit either $t_i$ or $s_{j(i)}$; $t_i$ (resp. $s_{j(i)}$) is omitted if $|\hat{z}_i| \leqslant |z_{j(i)}|$ (resp. $|z_i| > |\hat{z}_{j(i)}|$). We use this partition in order to estimate from below $\|v\|^p$. If $i$ is such that $l(i)$ is not equal to any $m(j)$ each of the intervals $[l(i)/n, t_i]$ and $\big[t_i, \big(l(i)+1\big)/n\big]$ will contribute to the sum appearing in (3.1) an amount equal to $|z_i|^p$ (by (3.6) and (3.7)), and thus the combined contribution of the intervals contained in $[l(i)/n, \big(l(i)+1\big)/n]$ is $2|z_i|^p$. Similarly, if $j$ is such that $m(j)$ is not equal to any $l(i)$ the total contribution of the intervals of the partition contained in $[m(j)/n, \big(m(j)+1\big)/n]$ to the same sum is $2|\hat{z}_i|^p$. If $l(i) = m(j)$ for some $j = j(i)$ and $z_i\hat{z}_{j(i)} \leqslant 0$, then in our partition, $[l(i)/n, \big(l(i)+1\big)/n]$ is divided into three intervals whose combined contribution to the sum in (3.1) (for $\|v^p\|$) is

$$|z_i|^p + |z_{j(i)}|^p + |z_i - z_{j(i)}|^p \geqslant 2(|z_i|^p + |z_{j(i)}|^p).$$

Finally, if $l(i) = m\big(j(i)\big)$ and $z_i\hat{z}_{j(i)} > 0$ then in our partition $[l(i)/n, \big(l(i)+1\big)/n]$ is divided into only two subintervals whose combined contribution is $2\max(|z_i|^p, |\hat{z}_{j(i)}|^p)$. Summing up we get that (using the notation of Lemma 4)

$$\begin{aligned} 1 &\geqslant \|v\|^p \\ &\geqslant 2\Big(\sum_{i=1}^{k-1} |z_i|^p + \sum_{j=1}^{h-1} |\hat{z}_j|^p\Big) - \sum{}''' \min(|z_i|^p, |\hat{z}_{j(i)}|^p) \\ &\geqslant 2\Big(\sum_{i=1}^{k-1} |z_i|^p + \sum_{j=1}^{h-1} |\hat{z}_j|^p\Big) - \delta_2(\varepsilon). \end{aligned} \tag{3.13}$$

On the other hand, by (3.8), (3.9) and the triangle inequality we get that

$$\begin{aligned} (1-\varepsilon)^p &\leqslant |z_1|^p + |z_2 - z_1|^p + \ldots + |z_{k-1} - z_{k-2}|^p + |z_{k-1}|^p \\ &\leqslant 2^p \sum_{i=1}^{k-1} |z_i|^p. \end{aligned} \tag{3.14}$$

Similarly,

$$(1-\varepsilon)^p \leqslant 2^p \sum_{j=1}^{h-1} |\hat{z}_j|^p. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we get that

$$1 \geqslant (1-\varepsilon)^p 2^{2-p} - \delta_2(\varepsilon) \tag{3.16}$$

and this is a contradiction for $\varepsilon$ sufficiently small since $p < 2$.

It remains to prove the three assertions made during the preceding argument.

Proof of Lemma 2. Let $\varepsilon > 0$ be given. By the uniform convexity of $l_p$ there is a $\delta > 0$ such that if $x, y \in l_p$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and $\|x+y\| > 2-\delta$ then $\|x-y\| < \varepsilon^p/100$. We can clearly assume that also $\delta < \varepsilon^p/100$. Choose now a $u \in JF$ with $u$ simple and $\|u\| = 1$ which satisfies $(*)$ taking as the "$\varepsilon$" there the number $\delta$. Define now $K$ and $n$ as in (3.2) and choose a $v$ satisfying (i), (ii), and (iii) in $(*)$. Let $\{\tau_\gamma\}_{\gamma=0}^m$ be a partition of $[0, 1]$ so that

$$\Big(\sum_{\gamma=0}^{m-1} |\xi_\gamma + \eta_\gamma|^p\Big)^{1/p} > 2-\delta \tag{3.17}$$

where

$$\xi_\gamma = \int_{\tau_\gamma}^{\tau_{\gamma+1}} u(t)\,dt, \qquad \eta_\gamma = \int_{\tau_\gamma}^{\tau_{\gamma+1}} v(t)\,dt. \tag{3.18}$$

By our choice of $\delta$ and the fact that $\|u\|, \|v\| \leqslant 1$ we get that

$$\Big(\sum_{\gamma=0}^{m-1} |\xi_\gamma - \eta_\gamma|^p\Big)^{1/p} \leqslant \varepsilon^p/100. \tag{3.19}$$

We replace now the partition $\{\tau_\gamma\}_{\gamma=0}^m$ by a partition $\{t_i\}_{i=0}^k$ obtained by deleting some points in the given partition so as to ensure that $t_{i+1} - t_i > 1/\sqrt{n}$ for all $i$. More precisely, $t_1$ is taken as the first of the $\tau_\gamma$ which is larger than $1/\sqrt{n}$. Next $t_2$ is taken as the first of the $\tau_\gamma$ which is larger than $t_1 + 1/\sqrt{n}$ and so on. If in this procedure we get a $t_i$ which is larger than $1-1/\sqrt{n}$ we omit it and replace it by 1.

Define now $x_i$ and $y_i$ by (3.5). Clearly, each $x_i$ (resp. $y_i$) is a sum of those $\xi_\gamma$ (resp. $\eta_\gamma$) which correspond to indices $\gamma$ for which $t_i \leqslant \tau_\gamma < t_{i+1}$. Denote by $\sigma(\gamma)$ the length $\tau_{\gamma+1} - \tau_\gamma$ of the interval $[\tau_\gamma, \tau_{\gamma+1}]$. By the construction of the $\{t_i\}$ there is for every $i$ at most one integer $\gamma = \gamma(i)$ such that $t_i \leqslant \tau_\gamma < t_{i+1}$ and $\sigma(\gamma) > 1/\sqrt{n}$. Hence we can write

$$x_i = a_i + b_{i,1} + b_{i,2} + \ldots + b_{i,k_i} \tag{3.20}$$

where $a_i = \xi_{\gamma(i)}$ if a $\gamma(i)$ as above exists (otherwise $a_i = 0$) and each $b_{i,r}$ is equal to $\xi_{\gamma(i,r)}$ where $\gamma(i,1), \ldots, \gamma(i,k_i)$ is an enumeration of those indices $\gamma$ for which $t_i \leqslant \tau_\gamma < t_{i+1}$ and $\sigma(\gamma) \leqslant 1/\sqrt{n}$. With a similar notation we can write



$$y_i = c_i + d_{i,1} + d_{i,2} + \ldots + d_{i,k_i}. \tag{3.21}$$

Clearly, $|b_{i,r}| \leqslant K\sigma(\gamma_{i,r})$ for all $i$ and $r$ and hence by (3.2)

$$\begin{aligned}\sum_{i,r} |b_{i,r}|^p &\leqslant K^p \sum_{i,r} \sigma(\gamma_{i,r})^p \leqslant K^p n^{(1-p)/2} \sum_{i,r} \sigma(\gamma_{i,r}) \\ &\leqslant K^p n^{(1-p)/2} \leqslant (\varepsilon^p/100)^p.\end{aligned} \tag{3.22}$$

From (3.19) and (3.22) it follows that

$$\begin{aligned}\Big(\sum_{i,r} |d_{i,r}|^p\Big)^{1/p} &\leqslant \Big(\sum_{i,r} |b_{i,r}|^p\Big)^{1/p} + \Big(\sum_{i,r} |d_{i,r} - b_{i,r}|^p\Big)^{1/p} \\ &\leqslant \varepsilon^p/100 + \varepsilon^p/100 = \varepsilon^p/50.\end{aligned} \tag{3.23}$$

Hence, by (3.17), (3.22), and (3.23),

$$\Big(\sum_{i=0}^{k-1} |a_i + c_i|^p\Big)^{1/p} \geqslant 2 - \delta - \varepsilon^p/50 - \varepsilon^p/100 \geqslant 2 - \varepsilon^p/25. \tag{3.24}$$

Consider now the partition of [0,1] obtained by taking as division points all the points $t_i$ as well as the points $\tau_{\gamma(i)}$ and $\tau_{\gamma(i)+1}$ for all those $i$ for which $\gamma(i)$ exists. In this partition each interval $[t_i, t_{i+1}]$ is divided into at most three parts (actually it is easily seen that with the possible exception of the last interval each $[t_i, t_{i+1}]$ is divided into at most two parts). On the subinterval $[\tau_{\gamma(i)}, \tau_{\gamma(i)+1}]$ of $[t_i, t_{i+1}]$ the integral of $u+v$ is equal to $a_i + c_i$. Denote the integrals of $u+v$ on the other two subintervals by $B_{i,1}$ and $B_{i,2}$ (our convention is that whenever there is no interval of a certain type the integral over it is considered as 0). Clearly,

$$x_i + y_i = a_i + c_i + B_{i,1} + B_{i,2}, \qquad i = 0, 1, \ldots, k-1. \tag{3.25}$$

Since $\|u+v\| \leqslant 2$, we get that

$$2^p \geqslant \sum_{i=0}^{k-1} |a_i + c_i|^p + \sum_{i=0}^{k-1} |B_{i,1}|^p + \sum_{i=0}^{k-1} |B_{i,2}|^p. \tag{3.26}$$

Hence, by (3.24) and (3.26),

$$\Big(\sum_{i=0}^{k-1} (|B_{i,1}|^p + |B_{i,2}|^p)\Big)^{1/p} \leqslant \big(2^p - (2-\varepsilon^p/25)^p\big)^{1/p} \leqslant 2\varepsilon/5. \tag{3.27}$$

Finally, by (3.24), (3.25) and (3.27) we get that

$$\begin{aligned}\Big(\sum_{i=0}^{k-1} |x_i + y_i|^p\Big)^{1/p} &\geqslant \Big(\sum_{i=0}^{k-1} |a_i + c_i|^p\Big)^{1/p} - \Big(\sum_{i=0}^{k-1} |B_{i,1}|^p\Big)^{1/p} - \Big(\sum_{i=0}^{k-1} |B_{i,2}|^p\Big)^{1/p} \\ &\geqslant 2 - \varepsilon^p/25 - 2\varepsilon/5 - 2\varepsilon/5 \geqslant 2 - \varepsilon\end{aligned} \tag{3.28}$$

and this concludes the proof of Lemma 2 (since $t_{i+1}-t_i > 1/\sqrt{n}$ for all $i$ by the construction of the $\{t_i\}$).

Proof of Lemma 3. By the uniform convexity of the $l_p$ there is a $\delta_i(\varepsilon)$ (here and below whenever we consider a function of the form $\delta_i(\varepsilon)$ we assume that $\delta_i(\varepsilon)\to 0$ as $\varepsilon\to 0$), such that

$$\sum_{i=0}^{k-1} |x_i-y_i|^p = \sum_{i=0}^{k-1} |x_i-z_{i+1}+z_i|^p \leqslant \delta_3(\varepsilon). \tag{3.29}$$

In the sequel we shall need the following trivial inequality

$$|\alpha+\beta|^p \leqslant |\alpha|^p+4|\beta| \quad \text{if} \quad |\alpha|, |\beta| \leqslant 1 \text{ and } p\leqslant 2. \tag{3.30}$$

Since the proof of both inequalities in Lemma 3 is the same, we prove here only one of them. Let $\Gamma = \{i\colon x_i z_{i+1} < 0\}$ and consider the partition of $[0,1]$ determined by $\{t_i\}_{i=0}^k$ and $\{l(i+1)/n\}_{i\in\Gamma}$. For $i\in\Gamma$ the interval $[t_i, t_{i+1}]$ of the original partition is replaced by the two intervals $[t_i, l(i+1)/n]$ and $[l(i+1)/n, t_{i+1}]$. Corresponding to this division of the interval we get a representation of $x_i$ and $y_i$ $(i\in\Gamma)$ as a sum of two terms $y_i = -z_i+z_{i+1}$ (by (3.8)) and $x_i = x_i'+x_i''$, say, where

$$|x_i''| = \Big|\int_{l(i+1)/n}^{t_{i+1}} u(t)\,dt\Big| \leqslant K/n. \tag{3.31}$$

Since $\|u+v\|\leqslant 2$, we get, using the partition described above, that

$$2^p \geqslant \sum_{i\notin\Gamma} |x_i+y_i|^p + \sum_{i\in\Gamma} |x_i-z_i-x_i''|^p + \sum_{i\in\Gamma} |x_i''+z_{i+1}|^p. \tag{3.32}$$

By (3.30), (3.31), (3.32), and the fact that $k<\sqrt{n}$ (by Lemma 2) we get that

$$\begin{aligned}2^p &\geqslant \sum_{i\notin\Gamma} |x_i+y_i|^p + \sum_{i\in\Gamma} |x_i-z_i|^p + \sum_{i\in\Gamma} |z_{i+1}|^p - 8\sum_{i\in\Gamma} |x_i''| \\ &\geqslant \sum_{i\notin\Gamma} |x_i+y_i|^p + \sum_{i\in\Gamma} |x_i-z_i|^p + \sum_{i\in\Gamma} |z_{i+1}|^p - 8K/\sqrt{n}.\end{aligned} \tag{3.33}$$

By the definition of $\Gamma$ we get for $i\in\Gamma$

$$|x_i+y_i| = |x_i+z_{i+1}-z_i| \leqslant \max(|x_i-z_i|, |x_i-z_{i+1}+z_i|).$$

Hence by (3.3), (3.29), and (3.33)

$$\begin{aligned}2^p &\geqslant \sum_{i=0}^{k-1} |x_i+y_i|^p + \sum_{i\in\Gamma} |z_{i+1}|^p - \delta_3(\varepsilon) - 8K/\sqrt{n} \\ &\geqslant (2-\varepsilon)^p + \sum_{i\in\Gamma} |z_{i+1}|^p - \delta_3(\varepsilon) - 8K/\sqrt{n}.\end{aligned} \tag{3.34}$$

The desired assertion follows now from (3.2) and (3.34).



Proof of Lemma 4. Let

$$\Gamma_1 = \{i;\ \exists j \text{ such that } l(i) = m(j) \text{ and } z_i\hat{z}_{j(i)} > 0\}.$$

We have to show that for a suitable $\delta_2(\varepsilon)$

$$\sum_{i\in\Gamma_1} \min(|z_i|^p, |\hat{z}_{j(i)}|^p) < \delta_2(\varepsilon).$$

In view of Lemma 3 applied to the partitions $\{t_i\}$ and $\{s_j\}$ it is enough to prove that for a suitable $\delta_4(\varepsilon)$

$$\sum_{i\in\Gamma_2} \min(|z_i|^p, |\hat{z}_{j(i)}|^p) < \delta_4(\varepsilon)$$

where

$$\Gamma_2 = \{i;\ i\in\Gamma_1,\ z_i x_{i-1} > 0,\ \hat{z}_{j(i)}\hat{x}_{j(i)-1} > 0\}.$$

It is clear that

$$i\in\Gamma_2 \Rightarrow x_{i-1}\hat{x}_{j(i)-1} > 0. \tag{3.35}$$

We divide $\Gamma_2$ into two sets $\Gamma_3$ and $\Gamma_4$ as follows

$$\Gamma_3 = \{i;\ i\in\Gamma_2,\ t_{i-1}\geqslant s_{j(i)-1}\},$$

$$\Gamma_4 = \{i;\ i\in\Gamma_2,\ t_{i-1} < s_{j(i)-1}\}.$$

By reasons of symmetry it is enough to prove that for some $\delta_5(\varepsilon)$

$$\sum_{i\in\Gamma_3} |z_i|^p < \delta_5(\varepsilon). \tag{3.36}$$

Consider now the partition of $[0,1]$ obtained by taking as division points the points $\{s_j\}_{j=0}^k$ and $\{t_{i-1}\}_{i\in\Gamma_3}$. Since $|t_i - s_{j(i)}| < 1/n$, it follows that

$$\int_{t_{i-1}}^{s_{j(i)}} u(t)\,dt = \int_{t_{i-1}}^{t_i} u(t)\,dt + \int_{t_i}^{s_{j(i)}} u(t)\,dt = x_{i-1}+w_i \tag{3.37}$$

with $|w_i|\leqslant K/n$. Also

$$\int_{s_{j(i)-1}}^{t_{i-1}} u(t)\,dt = -\int_{s_{j(i)-1}}^{s_{j(i)}} -u(t)\,dt - \int_{t_{i-1}}^{s_{j(i)}} u(t)\,dt = -\hat{x}_{j(i)-1} - x_{i-1} - w_i. \tag{3.38}$$

Since $\|u\|\leqslant 1$, we get by using the partition above that

$$1 \geqslant \sum_j{}' |\hat{x}_j|^p + \sum_{i\in\Gamma_3} |x_{i-1}+w_i|^p + \sum_{i\in\Gamma_3} |x_{i-1}+\hat{x}_{j(i)-1}+w_i|^p \tag{3.39}$$

where $\sum'$ ranges over all integers $j$, $0\leqslant j\leqslant h-1$ such that $j\neq j(i)-1$

with $i \in \Gamma_3$. By (3.2), (3.30), and (3.35) we deduce from (3.39) that

$$\begin{aligned} 1 &\geqslant \sum_j{}' |\hat{x}_j|^p + \sum_{i\in\Gamma_3} |x_{i-1}|^p + \sum_{i\in\Gamma_3} |x_{i-1}+\hat{x}_{j(i)}|^p - 8K/\sqrt{n} \\ &\geqslant \sum_j{}' |\hat{x}_j|^p + \sum_{i\in\Gamma_3} |x_i|^p + \sum_{i\in\Gamma_3} |x_{i-1}|^p + \sum_{i\in\Gamma_3} |\hat{x}_{j(i)}|^p - 8K/\sqrt{n} \\ &\geqslant \sum_{j=0}^{h-1} |\hat{x}_j|^p + 2\sum_{i\in\Gamma_3} |x_{i-1}|^p - 8K/\sqrt{n}, \end{aligned}$$

and hence

(3.40) $$\sum_{i\in\Gamma_3} |x_{i-1}|^p \leqslant \delta_6(\varepsilon).$$

From (3.29) and (3.40) we deduce that

(3.41) $$\sum_{i\in\Gamma_3} |z_i - z_{i-1}|^p \leqslant \delta_7(\varepsilon).$$

Let $\Gamma_3 = \Gamma_3' \cup \Gamma_3''$ where

$$\Gamma_3' = \{i;\ i\in\Gamma_3, z_i z_{i-1} > 0\}, \qquad \Gamma_3'' = \{i;\ i\in\Gamma_3,\ z_i z_{i-1} \leqslant 0\}.$$

By (3.41)

(3.42) $$\sum_{i\in\Gamma_3''} |z_i|^p \leqslant \sum_{i\in\Gamma_3''} |z_i - z_{i-1}|^p \leqslant \delta_7(\varepsilon).$$

For $i\in\Gamma_3'$ we have $z_{i-1}x_{i-1} \geqslant 0$ and hence, by Lemma 3,

(3.43) $$\sum_{i\in\Gamma_3'} |z_{i-1}|^p \leqslant \delta_1(\varepsilon).$$

From (3.41), (3.42), and (3.43) we deduce easily that (3.36) holds. This concludes the proof of the lemma and thus of Theorem 2 (if $1 < p < 2$).

The fact that $p < 2$ was used in the proof only at one place, namely at the very end (in deducing a contradiction from (3.16)). Basically the condition $p < 2$ was of importance there because we worked with only two partitions. In order to prove the theorem also for $p = 2$ we have to note that the same reasoning which shows that it is enough to prove that (*) fails shows also that it is enough to prove that statement (**) fails where

(**) For every $\varepsilon > 0$ there are simple functions $u, w \in JF$ with $\|u\| = \|w\| = 1$ so that for every integer $n$ there is a $v \in JF$ with $\|v\| = 1$, $\|\pm u \pm w + v\| \geqslant 3 - \varepsilon$ for all choices of signs and the integral of $v$ on every interval of the form $[j/n, (j+1)/n]$ is 0.

Working with (**), we naturally get four partitions of $[0, 1]$ (corresponding to the various choices of signs in $\pm u \pm w \pm v$) and an argument very similar to that used above proves the theorem as stated (i.e. for $p = 2$).



**References**

[1] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
[2] M. M. Day, *Normed linear spaces*, Berlin, 1973.
[3] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Functional Anal. 17 (1974), pp. 311–323.
[4] D. P. Giesy and R. C. James, *Uniform non-$l^1$ and B-convex Banach spaces*, Studia Math. 48 (1973), pp. 61–69.
[5] R. C. James *A non-reflexive Banach space isometric with its second conjugate*, Proc. Nat. Acad. Sci. USA 37 (1951), pp. 174–177.
[6] — *Uniformly non square Banach spaces*, Ann. of Math. 80 (1964), pp. 542–550.
[7] — *A conjecture about $l_1$ subspaces*, to appear.
[8] W. B. Johnson and H. B. Rosenthal, *On ω*-basic sequences and their applications to the study of Banach spaces*, Studia Math. 43 (1972), pp. 77–92.
[9] W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. 17 (1974), pp. 219–230.
[10] J. Lindenstrauss, *Weakly compact sets — their topological properties and the Banach spaces they generate*, Annals of Math. Studies 69 (1972), pp. 235–273.
[11] R. D. McWilliams, *A note on weak sequential convergence*, Pacific J. Math. 12 (1962), pp. 333–335.
[12] E. O'dell and H. Rosenthal, *A double dual characterization of separable Banach spaces containing $l_1$*, to appear.
[13] H. Rosenthal, *A characterization of Banach spaces containing $l^1$*, to appear.
[14] C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc., 206 (1975), pp. 213–223.

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