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Toeplitz operators related to certain domains in C^n

by

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Abstract. Venugopalkrisna in [9] investigated the Toeplitz operators in strongly pseudoconvex domain $D \subset C^n$, $n > 1$. Among other he proved that Toeplitz operator with continuous symbol φ (smooth in D) is Fredholm if ∂D is smooth and if φ does not vanish on ∂D. On the other hand, Coburn identified C^*-algebra generated by Toeplitz operators on odd spheres, modulo compact operators [2]. We shall identify the C^*-algebra generated by Toeplitz operators in strongly pseudoconvex domain D modulo the compact operators. We shall also prove some simple properties of Toeplitz operators on the n-dimensional torus Γ^n.

1. Let $L(H)$ be the algebra of all linear bounded operators in a complex Hilbert space H and let $\mathscr{K}(H)$ be the ideal of all compact operators in H.

DEFINITION 1.1. For any bounded set D in C^n, denote by $\mathscr{L}^2(D)$ the *space of functions f: $D \to C$ which are square integrable with respect to the Lebesgue measure dV in C^n.*

DEFINITION 2.1. Denote by $H^2(D)$ the *space of all* $f \in \mathscr{L}^2(D)$, *which are holomorphic in D.*

We shall denote by P: $\mathscr{L}^2(D) \to H^2(D)$ the *orthogonal projection onto the subspace* $H^2(D)$.

The definition of Toeplitz operator associated with a function $\varphi \in L^\infty(dV)$ (bounded, measurable in D) reads as follows:

DEFINITION 3.1. Let $\varphi \in L^\infty(dV)$. The *Toeplitz operator* T_φ: $H^2(D) \to H^2(D)$ is defined by

$$T_\varphi f = P(\varphi \cdot f).$$

Let B be the closed unit ball in C^n and let μ be the usual surface measure on $\partial B = S^{2n-1}$. Then one can define the *Hardy space* $H^2(\mu)$ on ∂B as a closed subspace of all functions in $\mathscr{L}^2(\mu)$ which are holomorphic in the int B [2].

The definition of a Toeplitz operator on $H^2(\mu)$ is just the same as Definition 3.1. Let $\mathscr{C}$ be a C^*-algebra generated by Toeplitz operators T_φ ($\varphi \in C(\partial B)$) on $H^2(\mu)$. Then it was proved by Coburn in [2] that the C^*-algebra $\mathscr{C}/\mathscr{K}(H^2(\mu))$ is isometrically isomorphic with $C(\partial B)$. We shall prove an analogous theorem for the Toeplitz operators in strongly pseudoconvex domain in C^n.

In what follows we will consider the Toeplitz operators in strongly pseudoconvex domain D in C^n, with smooth boundary; see [9] for the definition.

Now we prove the following theorem.

THEOREM 1.1. *Let D be a strongly pseudoconvex domain in C^n for $n > 1$, (or an arbitrary bounded domain in C (1) and assume that ∂D is smooth. Denote by $\mathscr{C}$ the C^*-algebra generated by T_φ $(\varphi \in C(\bar{D}))$ on $H^2(D)$; then*

(i) *$\mathscr{C}/\mathscr{K}(H^2(D))$ is isometrically isomorphic with $C(\partial D)$, for $n = 1$;*

(ii) *$\mathscr{C}/\mathscr{K}(H^2(D))$ is isometrically isomorphic with $C(\sigma)$, $\sigma \subseteq \partial D$, for $n > 1$.*

Proof. We shall prove the theorem using an elegant theorem of Bunce [1]. The idea of using the theorem of Bunce in this context was first noted by Douglas in his book [5]. The theorem of Bunce states: *If $\{A_\tau\}_{\tau \in T}$ is commuting family of subnormal operators on H, G is the C^*-algebra generated by this family, and J is the commutator ideal for G, then G/J is isometrically isomorphic with $C(\sigma_\pi(A_\tau)_{\tau\in T})$, where $\sigma_\pi(\{A_\tau\}_{\tau \in T})$ is a joint approximate point spectrum for $\{A_\tau\}_{\tau\in T}$.* To apply the Bunce theorem, let us note that T_{z_i} $(i = 1, \ldots, n)$ are subnormal, and that the C^*-algebra generated by them equals $\mathscr{C}$. Now we shall prove the following:

(a) *Commutator ideal J for $\mathscr{C}$ equals $\mathscr{K}(H^2(D))$,*

(b) $\sigma_\pi(T_{z_1}, \ldots, T_{z_n}) \subseteq \partial D$ $(n > 1)$.

Proof of (a). First note that $\mathscr{C}$ is irreducible. Indeed, if Q is an orthogonal projection $(Q \neq 0, 1)$ such that $QT_\varphi = T_\varphi Q$ for $\varphi \in C(\bar{D})$, then, writing $Q1 = g \in H^2(D)$, we have for polynomials φ and ψ

$$(g\varphi, \psi) = (Q\varphi, \psi) = (\varphi, Q\psi) = (\varphi, g\psi),$$

i.e.

$$\int_D g\varphi\bar{\psi}\,dV = \int_D \bar{g}\varphi\bar{\psi}\,dV.$$

Thus

$$\int_D (g-\bar{g}) \sum_i \varphi_i \bar{\psi}_i\,dV = 0$$

where φ_i, ψ_i are polynomials. By the Stone–Weierstrass theorem

$$\int_D (g-\bar{g})u\,dV = 0 \quad \text{for every } u \in C(\bar{D}).$$

Consequently, $\operatorname{Im} g = 0$ and so g is equal to 0 or to 1. It follows that $Q = 0$ or $Q = J$ and we get a contradiction.

(1) Assume also that $\partial D \subseteq \partial\sigma(T_z)$

Now we prove the inclusion $\mathscr{C} \supset \mathscr{K}(H^2(D))$. Since φ is irreducible, it is enough to prove that $\mathscr{C} \cap \mathscr{K}(H^2(D)) \neq \{0\}$ (2). By Theorem 2.1 in [9] we have that for every $s \in C(\bar{D})$, $s \neq 0$ on ∂D, s smooth in D, the operator $(I-P)s\colon H^2(D) \to \mathscr{L}^2(D)$ is compact.

Analyzing the proof of this theorem, we have noted that the theorem holds also for s which vanishes in some points of ∂D. But for T_{z_i} $(i = 1, \ldots, n)$ we have

$$T_{z_i}^* T_{z_i} - T_{z_i} T_{z_i}^* = Pz_i(J-P)\bar{z}_i.$$

Since $(J-P)\bar{z}_i$ are compact, $T_{z_i}^* T_{z_i} - T_{z_i} T_{z_i}^*$ are compact also. Denote by

$$\pi\colon \mathscr{C} \to \mathscr{C}/\mathscr{K}(H^2(D))$$

the natural homomorphism. Then, as we have seen, $\pi(T_{z_i})$ is the normal element in $\mathscr{C}/\mathscr{K}(H^2(D))$ for $i = 1, \ldots, n$. Since $\mathscr{C}/\mathscr{K}(H^2(D))$ is generated by $\pi(T_{z_i})$ $(i = 1, \ldots, n)$, it follows that $\mathscr{C}/\mathscr{K}(H^2(D))$ is commutative algebra and so $J \subset \mathscr{K}(H^2(D))$. But as it is well known [5], $\mathscr{K}(H^2(D))$ cannot contain non-trivial closed ideals, and consequently $J = \mathscr{K}(H^2(D))$. The proof of (a) is complete.

Proof of (b). The inclusion in (b) follows from Theorem 2.3 in [9] which states that T_s is compact for $s \in C(\bar{D})$, smooth in D, and such that $s|_{\partial D} \equiv 0$. It is easy to check that this theorem is also true for arbitrary $s \in C(\bar{D})$ not necessarily smooth in D [9]. Now recall that a joint spectrum of elements $x_1, \ldots, x_n$ of a commutative Banach algebra A is equal to

$$\{\eta(x_1), \ldots, \eta(x_n),\ \eta \in \operatorname{Sp} A\} \overset{\text{df}}{=} \sigma(x_1, \ldots, x_n)$$

where $\operatorname{Sp} A$ denotes a spectrum of A. Denote by $\tau\colon \mathscr{C}/\mathscr{K}(H^2) \to C(\sigma_\pi)$ *-isomorphism and assume that $\tau[T_{z_i}] = \xi_i$, $\xi_i \in C(\sigma_\pi)$. Then we have

$$\begin{aligned}(\xi_1(p_1), \ldots, \xi_n(p), p \in \sigma_\pi) &= \sigma(\xi_1, \ldots, \xi_n) = \sigma(\tau[T_{z_1}], \ldots, \tau[T_{z_n}])\\ &= \sigma([T_{z_1}], \ldots, [T_{z_n}])\\ &= \{\eta(T_{z_1}+\mathscr{K}), \ldots, \eta(T_{z_n}+\mathscr{K}),\ \eta \in \operatorname{Sp}\mathscr{C}/\mathscr{K}\}\\ &= \sigma_\pi(T_{z_1}, \ldots, T_{z_n}).\end{aligned}$$

Consequently, if T_s is compact then $s|_{\sigma_\pi} \equiv 0$ and (b) follows easily.

Now if $n = 1$ then it is well known that $\partial\sigma(T) \subseteq \sigma_\pi(T)$ for arbitrary $T \in L(H)$, so in this case we have the equality in (b).

The proof of the theorem is now complete.

There arises a natural question whether the inclusion in (b) can be replaced by the equality. We shall prove that it is possible for certain $D \subset C^n$. Let D be a bounded domain in C^n. Denote by $P(D)$ the Banach

(2) See Theorem 5.39 in [5].

algebra of all functions in $\bar{D}$ which are uniformly approximated by polynomials on $\bar{D}$. Let $A(D)$ denote the Banach algebra of all functions, continuous on $\bar{D}$, which are holomorphic in D. Now we prove the following lemma.

LEMMA. *Let D be a bounded domain with smooth boundary. Assume that $D = \{z \in U,\ \varrho(z) < 0\}$, where U is a neighborhood of $\bar{D}$ and $\varrho\colon U \to \mathbf{R}$ is a strictly plurisubharmonic C^2 function. Assume also that $A(D) = P(D)$. Then we have the equality*

$$\sigma_\pi(T_{z_1}, \ldots, T_{z_n}) = \partial D.$$

Proof. By (b) in Theorem 1.1 the inclusion $\sigma_\pi(T_{z_1}, \ldots, T_{z_n}) \subseteq \partial D$ holds. Denote by $\mathscr{A}$ the smallest commutative closed subalgebra of $L(H^2(D))$ containing T_{z_i} $(i = 1, \ldots, n)$. Let $\Gamma(\mathscr{A})$ denote the Shilov boundry of $\mathscr{A}$. By the result of [8], every point $(\eta(T_{z_1}), \ldots, \eta(T_{z_n}))$, where $\eta \in \Gamma(\mathscr{A})$, belongs to $\sigma_\pi(T_{z_1}, \ldots, T_{z_n})$. But it is easy to check that $\mathscr{A}$ is equal to $A(D)$. Indeed, if $\sigma_r(\tilde{T})$ denotes the right joint spectrum of the set $\tilde{T} = (T_{z_1}, \ldots, T_{z_n})$ then $D \subset \sigma_r(\tilde{T})$. Moreover, for every polynomial $p(z_1, \ldots, z_n)$ we have

$$p(D) \subseteq p(\sigma_r(\tilde{T})) = \sigma_r(p(\tilde{T})) \subseteq \sigma(p(\tilde{T}))$$

so

$$\|p\|_\infty \leqslant r(p(\tilde{T})) \leqslant \|p(\tilde{T})\| \leqslant \|p\|_\infty.$$

Since $A(D) = P(D)$ and $\|p\|_\infty = \|p(\tilde{T})\|$, $A(D)$ is equal to $\mathscr{A}$. By Theorem 15.3 of [7], $\Gamma(A(D)) = \partial D$, and so $(\eta(T_{z_1}), \ldots, \eta(T_{z_n})) \in \partial D$. The proof is complete.

Combining Theorem 1.1 and the Lemma we get

THEOREM 2.1. *Let $D = \{z \in U,\ \varrho(z) < 0\}$, where $\varrho\colon U \to \mathbf{R}$ is a strictly plurisubharmonic function of class C^2. Assume that $A(D) = P(D)$. Then $\mathscr{C}/\mathscr{K}(H^2(D))$ is isometrically isomorphic with $C(\partial D)$.*

Following Douglas [5] we can reformulate Theorem 2.1. Namely we can say that there exists a *-homomorphism ϱ from $\mathscr{C}$ onto $C(\partial D)$ such that the sequence

$$(0) \to \mathscr{K}(H^2(D)) \xrightarrow{i} \mathscr{C} \xrightarrow{\varrho} C(\partial D) \to (0)$$

is exact, where i denotes the inclusion map, $\varrho(T_\varphi) = \varphi$.

We can extend this result to the matrix case.

Denote by M_k the C^*-algebra of all $k \times k$ matrices with complex entries. Let $\bigoplus_{i=1}^{k} \mathscr{L}^2(D)$ be the direct sum of k copies of $\mathscr{L}^2(D)$ and $\bigoplus_{i=1}^{k} H^2(D)$ the direct sum of k copies of $H^2(D)$. For bounded measurable M_k valued function φ we define the Toeplitz operator on $\bigoplus_{i=1}^{k} H^2(D)$ by

$$T_\varphi f = \hat{P}(\varphi \cdot f)$$

where $\hat{P}$ is an orthogonal projection of $\bigoplus_{i=1}^{k} \mathscr{L}^2(D)$ onto $\bigoplus_{i=1}^{k} H^2(D)$. Let $\mathscr{C}_{M_k}$ be the C^*-algebra generated by $\{T_\varphi,\ \varphi \in C_{M_k}(\bar{D})\}$ where $C_{M_k}(\bar{D})$ is the C^*-algebra of all continuous M_k valued functions in $\bar{D}$. Recall next that the tensor product of a C^*-algebra A with M_k is isomorphic to the C^*-algebra of $k \times k$ matrices with entries from A.

Now applying Proposition 2 of [6], we get the following

THEOREM 3.1. *Let D be as in Theorem 2.1. Then there exists a *-homomorphism τ from $\mathscr{C}_{M_k}$ onto $C_{M_k}(\partial D)$ such that the sequence*

$$(0) \to \mathscr{K}\Big(\bigoplus_{i=1}^{k} H^2(D)\Big) \xrightarrow{i} \mathscr{C}_{M_k} \xrightarrow{\tau} C_{M_k}(\partial D) \to (0)$$

is exact.

From Theorem 3.1 we derive some corollaries.

COROLLARY 1. *Let $s \in C_{M_k}(\bar{D})$. The Toeplitz operator T_s is Fredholm if and only if* $\det s|_{\partial D} \neq 0$.

COROLLARY 2. *If $\varphi \in A(D)$ then $\|T_\varphi\| = \|\varphi\|_\infty$* (because $\Gamma(A(D)) = \partial D$).

COROLLARY 3 (see [5]). *If φ_{ij} $(i, j = 1, \ldots, p)$ are functions in $C_{M_k}(\bar{D})$ such that $\sum_{i=1}^{p} \prod_{j=1}^{p} T_{\varphi_{ij}}$ is compact, then*

$$\sum_{i=1}^{p} \prod_{j=1}^{p} \varphi_{ij}|_{\partial D} = 0.$$

Proof. It is enough to note that

$$\operatorname{Ker} \tau = \mathscr{K}\Big(\bigoplus_{i=1}^{k} H^2(D)\Big).$$

We conclude this section by the remark concerning the problem L. A. Coburn has rised in [2]. He asked whether for every function $\varphi \in L^\infty(\mu)$ (μ the surface measure on the unit sphere S^{2n-1})

$$\|T_\varphi\| = \|\varphi\|_\infty.$$

By the above work of Coburn it is true for $\varphi = \chi_\sigma$, when $\operatorname{int}\sigma \neq \emptyset$. One can try to prove it by the same method as in the proof of Theorem 1.1, but we did not succeed.

2. In this section we will prove some simple properties of Toeplitz operators on the n-dimensional torus Γ^n.

Let D be the open unit disc. Deddens proved in [3] that for every $\varphi \in H^\infty(D)$, $\psi \in H^\infty(D)$ such that $\psi(D) \nsubseteq \sigma(T_\varphi)$ there exists no operator $X \neq 0$

intertwining T_φ and T_ψ, i.e.

$$XT_\varphi = T_\psi X.$$

This theorem is also true for $n > 1$.

THEOREM 1.2. *Let $\varphi \in H^\infty(D^n)$, $\psi \in H^\infty(D^n)$ be such that $\psi(D^n) \nsubseteq \sigma(T_\varphi)$. Then there exists no operator $X \neq 0$ intertwining T_φ and T_ψ.*

Proof. The proof is almost the same as that in [3]. Assume that there exists an operator X satisfying the equality

$$(*) \qquad XT_\varphi = T_\psi X.$$

For $\lambda \in D^n$, $z \in D^n$, denote by

$$h_\lambda(z) = \frac{1}{(1-\lambda_1 z_1)\ldots(1-\lambda_n z_n)}$$

the Cauchy kernel. Now note that for every open set $A \subset D^n$ the set $\{h_\lambda\}_{\lambda\in A}$ spans $H^2(D^n)$. Write

$$N = \psi(D^n) \cap \big(C \setminus \sigma(T_\varphi)\big).$$

N is an open set. According to $(*)$ we have

$$T_\varphi^* X^* = X^* T_\psi^*.$$

Thus, by the known property of the Cauchy kernel we get

$$T_\varphi^* X^* h_\lambda = \overline{\psi(\bar\lambda)} X^* h_\lambda \quad \text{for} \quad \bar\lambda \in \psi^{-1}(N),$$

and the proof follows easily.

Next note that, for every $\varphi \in H^\infty(D^n)$, $\sigma(T_\varphi) = \overline{\varphi(D^n)}$ (the closure), which follows immediately from the equality $T_\varphi^* h_\lambda = \bar\varphi(\bar\lambda) h_\lambda$. Thus we can reformulate Theorem 1.2 in the following way.

THEOREM 1.2′. *If $\varphi \in H^\infty(D^n)$, $\psi \in H^\infty(D^n)$ are such that $\psi(D^n) \nsubseteq \overline{\varphi(D^n)}$ then there exists no operator $X \neq 0$ intertwining T_φ and T_ψ.*

Deddens asked in [4] whether there exists a compact operator in the commutant of the Toeplitz operator T_φ, $\varphi \in H^\infty(D)$. We have obtained a partial answer to this question.

Namely we have the following

THEOREM 2.2. *If $\varphi \in H^\infty(D^n)$ ($\varphi \neq$ const), then there exists no compact operator K, with $\sigma(K) \neq \{0\}$ in the commutant of T_φ.*

Proof. Assume that for a certain compact operator K with $\sigma(K) \neq \{0\}$ we have

$$(**) \qquad T_\varphi K = KT_\varphi.$$

Then for $0 \neq \mu \in \sigma(K)$ there exists a finite number of linearly independent vectors $f_1, \ldots, f_p$, $f_i \in H^2(D^n)$, which form a basis in the linear space H_μ, corresponding to the point μ.

From $(**)$ we have

$$K(\varphi \cdot f_i) = \mu(\varphi \cdot f_i), \quad i = 1, \ldots, p.$$

But

$$(+) \qquad \varphi f_i = \sum_{j=1}^{p} a_{ij} f_j, \quad a_{ij} \in C,\ i = 1, \ldots, p.$$

Since $f_i \neq 0$ m-almost everywhere on Γ^n (m Lebesgue measure on Γ^n), the system of equations $(+)$ has a non-zero solution m-almost everywhere. Thus

$$\det \begin{vmatrix} a_{11}-\varphi & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22}-\varphi & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots & a_{pp}-\varphi \end{vmatrix} = 0$$

m-almost everywhere. It follows that $\varphi \equiv$ const; we get a contradiction.

Note added in proof. Observe that the equality $A(D) = P(D)$, assumed in Lemma, holds for polynomially convex D. Indeed, by the theorem of Henkin (Mat. Sbor. 78 (120): 4 (1969), p. 631) every $f \in A(D)$ can be uniformly approximated on D by functions holomorphic in a neighborhood of D. Hence, by the Oka–Weil theorem, the equality holds.

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