[8] M. M. Day, *Normed linear spaces*, Berlin/Göttingen/Heidelberg 1958.
[9] J. Dugundji, *Topology*, Boston, Mass. 1966.
[10] M. Edelstein, A. C. Thompson, *Contractions, isometries and some properties of inner-product spaces*, Indag. Math. 29 (1967), S. 326–331.
[11] T. Figiel, *On nonlinear isometric embeddings of normed linear spaces*, Bull. Acad. Polon. Sci. 16 (1968), S. 185–188.
[12] F. Forelli, *The isometries of H^p*, Canad. J. Math. 18 (1964), S. 721–728.
[13] W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math. 26 (1966), S. 133–136.
[14] — *Linearization of isometric embeddings of Banach spaces*, Bull. Acad. Polon. Sci. 16 (1968), S. 189–193.
[15] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. 54 (1951), S. 325–338.
[16] N. Lal, S. Merill, *Isometries of H^p spaces of the torus*, Proc. Amer. Math. Soc. 31 (1972), S. 465–471.
[17] J. Lamperti, *On the isometries of some function spaces*, Pac. J. Math. 8 (1958), S. 459–466.
[18] M. Landsberg, *Lineare topologische Räume, die nicht lokalkonvex sind*, Math. Zeitschr. 65 (1956), S. 104–112.
[19] K. de Leeuw, W. Rudin, J. Werner, *The isometries of some function spaces*, Proc. Amer. Math. Soc. 11 (1960), S. 694–698.
[20] A. Lindenbaum, *Contributions à l'étude de l'espace métrique I*, Fund. Math. 8 (1926), S. 209–222.
[21] G. Lumer, *On the isometries of reflexive Orlicz spaces*, Ann. Inst. Fourier (Grenoble) 13 (1963), S. 99–109.
[22] — *Isometries of Orlicz spaces*, Bull. Amer. Math. Soc. 68 (1962), S. 28–30.
[23] P. Mankiewicz, *On Extension of Isometries in Normed Linear Spaces*, Bull. Acad. Polon. Sci. 20 (1972), S. 367–371.
[24] S. Mazur, S. M. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. (Paris) 194 (1932), S. 946–948.
[25] D. P. Milman, *Изометрия и экстремальные точки*, Докл. АН СССР 59 (1948), S. 1241–1244.
[26] A. L. Paterson, *Isometries between B^*-algebras*, Proc. Amer. Math. Soc. 22 (1969), S. 570–572.
[27] A. I. Plotkin, *Об изометрических операторах в пространствах суммируемых аналитических и гармонических функций*, Докл. АН СССР 185 (1969), S. 995–997.
[28] — *Об изометрических операторах на подпространствах в L^p*, Докл. АН СССР 193 (1970), S. 537–539.
[29] J. Rätz, *On isometries of generalized inner product spaces*, SIAM J. Appl. Math. 18 (1970), S. 6–9.
[30] S. Rolewicz, *Metric linear spaces*, Monogr. Mat., Tom 56, Warszawa 1972.
[31] — *A generalization of the Mazur-Ulam theorem*, Studia Math. 31 (1968), S. 501–505.
[32] A. K. Roy, *Extreme points and linear isometries of the Banach space of Lipschitz functions*, Canad. J. Math. 20 (1968), S. 1150–1164.
[33] B. Russo, *Isometries of L^p spaces associated with finite von Neumann algebras*, Bull. Amer. Math. Soc. 74 (1968), S. 228–232.
[34] — *Isometries of the trace class*, Proc. Amer. Math. Soc. 23 (1969), S. 213.
[35] S. Simons, *The sequence spaces $l(p_\nu)$ and $m(p_\nu)$*, Proc. London Math. Soc. (3) 15 (1965), S. 422–436.
[36] A. Vogt, *Maps which preserve equality of distance*, Studia Math. 65 (1973), S. 43–48.

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Extensions by mollifiers in Besov spaces*

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Abstract. An operator E of extension from lower dimensional subspaces for functions in Besov spaces is constructed using Friedrichs mollifiers. E has the useful property that for u defined on a hyperplane in $\boldsymbol{R}^n$ the support Eu is contained in the union of cones with vertices in the support of u and axes perpendicular to the hyperplane. Also if support of u is compact then so is the support of Eu.

1. Introduction. In this section we shall set up the notations, recall certain facts concerning Besov spaces and state the problem to be dealt with in the paper. Most of the facts about Besov spaces quoted below can be found in [1].

For a (complex, real or vector valued) function u defined in $\boldsymbol{R}^n$ we denote by $\Delta_h^k u$ the kth forward difference with increment $h \in \boldsymbol{R}^n$. If $\boldsymbol{R}^n$ is represented as $\boldsymbol{R}^n = \boldsymbol{R}^m \times \boldsymbol{R}^l$ with $x = (x', x'')$, $x' \in \boldsymbol{R}^m$, $x'' \in \boldsymbol{R}^l$ the corresponding partial differences are denoted by $\Delta_{h',x'}^k$, $\Delta_{h'',x''}^k$, $h' \in \boldsymbol{R}^m$, $h'' \in \boldsymbol{R}^l$.

The symbol $\| \ \|_p$, $1 \leqslant p \leqslant \infty$, is used to denote the L^p norm on $\boldsymbol{R}^n$ and, with notations as above, $\|u(\cdot, x'')\|_p$, $\|u(x', \cdot)\|_p$ denote the norms of $u(x', x'')$ as a function of x' with x'' fixed or respectively as a function of x'' with x' fixed.

The *Besov norm* $\| \ \|_{a,p,\theta}$, $a > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$ is defined by

$$\|u\|_{a,p,\theta} = \Big[\|u\|_p^p + \Big(\int_{\boldsymbol{R}^n} |h|^{-n-\theta a} \|\Delta_h^k u\|_p^\theta \, dh\Big)^{p/\theta}\Big]^{1/p} \tag{1.1}$$

where k is an integer $k > a$ and for $\theta = \infty$ the integral in parantheses is replaced by $\sup\{|h|^{-a}\|\Delta_h u\|_p;\ h \neq 0\}$.

The different choices of $k > a$ give rise to equivalent norms, this is why k is suppressed in the notation.

A norm equivalent to (1.1) is given by the formula

$$\Big[\|u\|_p^p + \Big(\int_{\boldsymbol{R}^m} |h'|^{-m-\theta a} \|\Delta_{h',x'}^{k'} u\|_p^\theta \, dh' + \int_{\boldsymbol{R}^l} |h''|^{-l-\theta a} \|\Delta_{h'',x''}^{k''} u\|_p^\theta\Big)^{p/\theta}\Big]^{1/p} \tag{1.2}$$

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where, as above $\boldsymbol{R}^n = \boldsymbol{R}^m \times \boldsymbol{R}^l$, $x = (x', x'')$, $k', k'' > \alpha$. Similar equivalent norms arise from the decomposition of $\boldsymbol{R}^n$ into cartesian products of more than two factors.

Yet another norm equivalent to (1.1) is given by the formula

$$\text{(1.3)} \qquad \Big[\|u\|_p^p + \Big(\int \sum_{|i|=m} |h|^{-n-\theta(\alpha-m)} \|\Delta_h^k D^i u\|_p^\theta \, dh\Big)^{p/\theta}\Big]^{1/p}$$

where m is an integer $0 \leqslant m < \alpha$ and $k > \alpha - m$.(1)

To simplify the notation we will not introduce distinct symbols for norms (1.1), (1.2) and (1.3); it will be clear from the context which one of them is used at each instant.

The space of (equivalence classes relative to the class $\mathfrak{A}_0$ of sets of Lebesgue measure 0 of) functions with finite norm $\| \ \|_{\alpha,p,\theta}$ is with this norm a Banach space referred to as *Besov space* and usually denoted by $B^\alpha_{p,\theta}(\boldsymbol{R}^n)$. The version (1.3) of the norm implies that

(1.4) *$B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ consists of all functions $u \in L^p(\boldsymbol{R}^n)$ whose distribution derivatives $D^i u \in B^{\alpha-|i|}_{p,\theta}(\boldsymbol{R}^n)$ for all i, $|i| < \alpha$.*

It is known ([1], 8.9.1) that $B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ can be characterized as the space of Bessel potentials, $B^\alpha_{p,\theta}(\boldsymbol{R}^n) = G_\alpha * B^0_{p,\theta}(\boldsymbol{R}^n)$(2) where $B^0_{p,\theta}(\boldsymbol{R}^n)$ is certain Banach space of distributions on $\boldsymbol{R}^n$ and $G_\alpha *$ denotes the operator of convolution with the n-dimensional Bessel kernel of order α,

$$\text{(1.5)} \qquad G_\alpha(x) = (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}(1+|\xi|^2)^{-\alpha/2}\, d\xi .$$

This fact and various properties of G_α (see [2], [4], [6]) have several consequences, some of them we list below.

(1.6) *The class $C_0^\infty(\boldsymbol{R}^n)$ with norm $\| \ \|_{\alpha,p,\theta}$ has the perfect functional completion relative to the exceptional class $\mathfrak{B}^\alpha_{p,\theta,n}$ described as follows. $A \subseteq \boldsymbol{R}^n$ belongs to $\mathfrak{B}^\alpha_{p,\theta,n}$ if and only if there is an ε, $0 < \varepsilon < \min(\alpha, 1)$, and a function $v \in B^\varepsilon_{p,\theta}(\boldsymbol{R}^n)$, $v \geqslant 0$ such that $A \subset \{x \in \boldsymbol{R}^n;\ \int G_{\alpha-\varepsilon}(x-y)v(y)\,dy = +\infty\}$.*

It is easy to check that the above definition of $\mathfrak{B}^\alpha_{p,\theta,n}$ is independent of ε.

In order not to complicate the notation we shall from now on use $B^\alpha_{p,\theta}$ to denote the perfect functional completion described by (1.6) instead of imperfect complection rel$\mathfrak{A}_0$ introduced before. We remark that (1.4) can be restated with distribution derivatives replaced by pointwise derivatives (see [2]).

(1) $i = (i_1, \ldots, i_n)$, $|i| = i_1 + \ldots + i_n$.

(2) In [1] the operator $G_\alpha *$ is denoted by I_α.

(1.7) *For fixed p, θ and $\alpha > 0$ the spaces $B^\alpha_{p,\theta}$ form an interpolation family (obtained by the analytic interpolation method, see the introduction [5]).*

Denote by u^L the Lebesgue correction of a function $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, i.e.

$$\lim_{r\to 0} \int_{|z|\leqslant 1} |u(x+rz) - u^L(x)|\, dz = 0 .$$

If $u^L(x)$ exists then x is called a *Lebesgue point* of u; a theorem of Lebesgue asserts that almost every point is a Lebesgue point of u, also $u^L = u$ a.e.

(1.8) *If u is equal almost everywhere to a function in $B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ then $u^L \in B^\alpha_{p,\theta}$, in particular the set of points where u^L is undefined is in $\mathfrak{B}^\alpha_{p,\theta,n}$ (see [3]).*

If $\mathfrak{A}$ is a class of subsets of a set X and $F \subset X$ then we denote by $\mathfrak{A}|_F$ the class $\{A \cap E,\ A \in \mathfrak{A}\}$. The next proposition describes the restriction and extension properties of spaces $B^\alpha_{p,\theta}$. We consider $\boldsymbol{R}^{n-1}$ as being canonically identified with a subspace of $\boldsymbol{R}^n$, $\boldsymbol{R}^{n-1} = \{(x', x_n) \in \boldsymbol{R}^n;\ x_n = 0\}$.

PROPOSITION 1.1. *If $\alpha > 1/p$, $1 \leqslant \theta \leqslant \infty$ then $\mathfrak{B}^\alpha_{p,\theta,n}|_{\boldsymbol{R}^{n-1}} \subset \mathfrak{B}^{\alpha-1/p}_{p,\theta,n-1}$ and $u \to u|_{\boldsymbol{R}^{n-1}}$ defines a bounded linear operator (of restriction) of $B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ into $B^{\alpha-1/p}_{p,\theta}(\boldsymbol{R}^{n-1})$. Also for every $\beta > 0$ there is a bounded operator (of extension) E: $B^\beta_{p,\theta}(\boldsymbol{R}^{n-1}) \to B^{\beta+1/p}_{p,\theta}(\boldsymbol{R}^n)$ with the property that $Ev|_{\boldsymbol{R}^{n-1}} = v$ for every $v \in B^\beta_{p,\theta}(\boldsymbol{R}^{n-1})$. In particular, the restriction operator in the first statement is onto.*

We note that by the first part of the proposition $\boldsymbol{R}^{n-1}$ is not an exceptional set for $B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ so that $u \to u|_{\boldsymbol{R}^{n-1}}$ is well defined on $B^\alpha_{p,\theta}(\boldsymbol{R}^{n-1})$.

If Ω is a domain in $\boldsymbol{R}^n$ one can define $B^\alpha_{p,\theta}(\Omega)$ as the space of all functions of the form $u|_\Omega$, $u \in B^\alpha_{p,\theta}(\boldsymbol{R}^n)$ with the restriction norm $\|v\|_{\alpha,p,\theta,\Omega} = \inf\{\|u\|_{\alpha,p,\theta};\ u|_\Omega = v\}$. The norms similar to (1.1), (1.2), (1.3) can be also directly defined on Ω and for domains satisfying mild regularity conditions can be shown to be equivalent to the restriction norm ([1], [3]). $B^\alpha_{p,\theta}(\Omega)$ is the perfect functional completion of $C_0^\infty(\boldsymbol{R}^n)|_\Omega$ rel$\mathfrak{B}^\alpha_{p,\theta}|_\Omega$. Proposition 1.1 can be also stated for $B^\alpha_{p,\theta}(\Omega)$.

Various explicit operators of extension are known, (see e.g. [1] in general case, [2], [4] in special cases); the objective of the paper is to prove that such an extension operator can be defined by the formula

(1.9)

$$(Eu)(x) = \varphi(x_n) u_e(x) = \varphi(x_n) \int_{\boldsymbol{R}^{n-1}} e(z') u(x' - x_n z')\, dz', \qquad x = (x', x_n) \in \boldsymbol{R}^n,$$

where e is a sufficiently smooth real valued function on $\boldsymbol{R}^{n-1}$ with support in the ball $\{|z'| \leqslant 1\}$ satisfying the condition $\int e(z')\,dz' = 1$ and $\varphi \in C_0^\infty(\boldsymbol{R}^1)$ is equal to 1 in a neighborhood of 0. In particular, if u has compact support in $\boldsymbol{R}^{n-1}$ then Eu defined by (1.9) has compact support contained in the set $\{x \in \boldsymbol{R}^n, \operatorname{dist}(x', \operatorname{supp} u) \leqslant x_n \leqslant M\}$ where $\operatorname{supp}\varphi \subset [-M, M]$.

The latter property of E is instrumental in describing spaces $B^a_{p,\theta}$ on manifolds with singularities ([6]). The other known extension formulas mentioned above in general do not preserve compactness of support.

We remark that a formula similar to (1.9), with a specific choice of e was used by Gagliardo [8] to describe the restrictions to $\boldsymbol{R}^{n-1}$ of functions in the Sobolev's spaces $W^1_p(\boldsymbol{R}^n)$, $p > 1$.

The operators referred to in this paper as *Friedrichs mollifiers* are also called *Sobolev's averages*.

In this research we were mainly interested in the case $\theta = p$ (actually $\theta = p = 2$), we decided to included the case of arbitrary θ in instances when such generalization did not involve additional complications in proofs or actually resulted in a clearer picture of the situation.

To simplify notations we shall use the letter C, possibly with subscripts to denote positive constants which may be different at different instances.

2. Transformations related to Friedrichs mollifiers. For $x \in \boldsymbol{R}^n$ we write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$.

In this section we are interested in various properties of the formula

$$(2.1)\quad u_e(x) = \int_{\boldsymbol{R}^{n-1}} e(z')u(x' - x_n z')\,dz' = |x_n|^{1-n}\int_{\boldsymbol{R}^{n-1}} e(x_n^{-1}z')u(x'-z')\,dz'$$

where e is a bounded measurable function with support in the unit ball $\{z' \in \boldsymbol{R}^{n-1};\ |z'| \leqslant 1\}$. Clearly, $u \mapsto u_e$ is defined for $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$; we shall study the properties of $u \mapsto u_e$ as a transformation between Besov spaces.

THEOREM 2.1. ([1].) *Suppose that $e \in C^2_0(\{|z'| \leqslant 1\})$. Then* (2.1) *defines a bounded linear transformation of $B^a_{p,\theta}(\boldsymbol{R}^{n-1})$ into $B^{a+(1/p)}_{p,\theta}(\boldsymbol{R}^{n-1} \times (-M, M))$ for any $M > 0$, $a > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$.*

Proof. Using Young's inequality we get

$$\|u_e(\cdot, x_n)\|_p \leqslant \|e\|_1 \|u\|_p,$$

implying that

$$(2.2)\quad \|u_e\|_p \leqslant (2M)^{1/p}\|e\|_1\|u\|_p.$$

We note next that with $D_i = \partial/\partial x_i$ we have

$$D_i u_e = (D_i u)_e, \qquad i = 1, \ldots, n-1, \qquad D_n u_e = \sum_{i=1}^{n-1}(D_i u)_{e_i}$$

where $e_i(z') = -e(z')z_i$; similar formulas are also valid for higher order derivatives. The remarks of Section 1 concerning the norms (1.3) and the interpolation property (1.7) imply that it is sufficient to prove the theorem for $0 < a < \varepsilon$ for any $\varepsilon > 0$ [3]. We use the version (1.2) of the norm

[3] The result for $0 < a < \varepsilon$ implies the corresponding result for $m < a < m+\varepsilon$, $m \geqslant 0$ an integer. The remaining values of a are taken care of by interpolation.

and consider first the case $p < \infty$. For $1 \leqslant \theta < \infty$ and some $k > a + (1/p)$ we shall obtain estimates of the form

$$(2.3)\quad \int_{\boldsymbol{R}^{n-1}} \|\Delta^k_{h',x'}u_e\|^\theta_p\,|h'|^{1-n-\theta(a+1/p)}\,dh' \leqslant C\|u\|^\theta_{a,p,\theta}$$

and

$$(2.4)\quad \int_{\boldsymbol{R}^1} \|\Delta^k_{h,x_n}u_e\|^\theta_p\,|h|^{-1-\theta(a+1/p)}\,dh \leqslant C\|u\|^\theta_{a,p,\theta}.$$

For $\theta = \infty$ we will obtain the estimates

$$(2.5)\quad \text{(i)}\qquad \sup |h'|^{-a-(1/p)}\|\Delta^k_{h'}u_e\|_p \leqslant C\|u\|_{a,p,\infty},$$

$$\text{(ii)}\qquad \sup |h|^{-a-(1/p)}\|\Delta^k_{h,x_n}u_e\|_p \leqslant C\|u\|_{a,p,\infty}.$$

We first prove (2.5)(i) and (2.3); taking $k = 3$ and using the last expression for u_e in (2.1), we can write

$$(2.6)\quad \Delta^3_{h',x'}u_e(x', x_n) = |x_n|^{1-n}\int_{\boldsymbol{R}^{n-1}} \Delta^2_{h',z'}e(x_n^{-1}z')\,\Delta_{h',x'}u(x'-z')\,dz'.$$

By the mean-value theorem and the properties of e

$$|x_n|^{1-n}\int |\Delta^2_{h',z'}e(x_n^{-1}z')|\,dz' \leqslant \min(C_1|x_n|^{-2}|h'|^2, C_2) \equiv \Phi(x_n, h')$$

where $C_1 = \|\Delta^2 e\|_1$, $C_2 = 4\|e\|_1$.

Applying Young's inequality to (2.6) we get

$$\|\Delta^3_{h',x'}u_e(\cdot, x_n)\|_p \leqslant \Phi(x_n, h')\|\Delta_{h'}u\|_p$$

and

$$(2.7)\quad \|\Delta^3_{h'}u_e\|^p_p \leqslant \int \Phi(x_n, h')^p\,dx_n\,\|\Delta_{h'}u\|^p_p = C_p|h'|\,\|\Delta_{h'}u\|^p_p$$

where $C_p = (2p-1)^{-1}C_1^{1/2}C_2^{p-1/2} + (C_1C_2)^{p/2}$.

Note that the same argument for $k = 2$ produces (2.7) with a constant containing $(p-1)^{-1}$ as a factor (making $p = 1$ seem to be an exceptional case).

The inequality (2.7) implies (2.5)(i).

For $1 \leqslant \theta < \infty$ we have

$$\int_{\boldsymbol{R}^{n-1}} \|\Delta^3_{h'}u_e\|^\theta_p |h'|^{1-n-\theta(a+(1/p))}\,dh' \leqslant C_p^{\theta/p}\int_{\boldsymbol{R}^{n-1}} \|\Delta_{h'}u\|^\theta_p\,|h'|^{1-n-\theta a}\,dh'$$

which is (2.3).

The proof of (2.4), (2.5)(ii) is more involved due to the peculiar way the variable x_n appears in (2.1); we take $k = 2$.

We first remark that

(2.8)

$$\int_{\mathbf{R}^1} |h|^{-1-\theta(\alpha+(1/p))} \|\Delta^2_{h,x_n} u_e\|_p^\theta \, dh = 2\int_0^\infty h^{-1-\theta(-\alpha+(1/p))} \Big(\int_{\mathbf{R}^n} |\Delta^2_{h,x_n} u_e(x)|^p \, dx\Big)^{\theta/p} dh$$

$$\leqslant 2^{1/p} 3^{\theta-1} \int_0^\infty h^{-1-\theta(\alpha+(1/p))} \Big[\Big(\int_{0\leqslant x_n\leqslant h} \ldots dx\Big)^{\theta/p} + \Big(\int_{h\leqslant x_n} \ldots dx\Big)^{\theta/p} + \Big(\int_{x_n\leqslant 0} \ldots dx\Big)^{\theta/p}\Big] dh.$$

The last expression in the bracket can be estimated as follows

(2.9)

$$\Big(\int_{x_n\leqslant 0} \ldots dx\Big)^{1/p} \leqslant \Big(\int_{-3h\leqslant x_n\leqslant 0} \ldots dx\Big)^{1/p} + \Big(\int_{x_n\leqslant -3h} \ldots dx\Big)^{1/p}$$

$$\leqslant 2^{1/p'} \Big(\int_{|x_n|\leqslant 3h} |\Delta_{h,x_n} u_e(x)|^p \, dx\Big)^{1/p} + \Big(\int_{x_n\leqslant -h} |\Delta^2_{-h,x_n} u_e(x)|^p dx\Big)^{1/p}.$$

Similarly we get for the first term

$$\Big(\int_{0\leqslant x_n\leqslant h} \ldots dx\Big)^{1/p} \leqslant 2^{1/p'} \Big(\int_{|x_n|\leqslant 2h} |\Delta_{h,x_n} u_e(x)|^p \, dx\Big)^{1/p}. \tag{2.10}$$

Note that the last term in (2.9) can be reduced to the same form as the middle term in (2.8) and hence can be treated in the same manner.

It follows that it suffices to estimate the integrals

$$\int_0^\infty h^{-1-\theta(\alpha+(1/p))} \Big[\int_{|x_n|\leqslant 3h} |\Delta_{h,x_n} u_e(x)|^p \, dx\Big]^{\theta/p} dh \tag{2.11}$$

and

$$\int_0^\infty h^{-1-\theta(\alpha+(1/p))} \Big[\int_{x_n\geqslant h} |\Delta^2_{h,x_n} u_e(x)|^p \, dx\Big]^{\theta/p} dh. \tag{2.12}$$

In (2.11) we use the formula

$$\Delta_{h,x_n} u_e(x) = \int_{\mathbf{R}^{n-1}} e(z') \big[u\big(x' - (x_n+h) z'\big) - u(x' - x_n z')\big] \, dz'.$$

Using the continuous version of Minkowski's inequality we get

$$\|\Delta_{h,x_n} u_e(\cdot, x_n)\|_p^p \leqslant \Big(\int_{\mathbf{R}^{n-1}} |e(z')| \, \|\Delta_{hz'} u\|_p \, dz'\Big)^p.$$

Integration with respect to x_n, $|x_n| \leqslant 3h$ gives

$$\|\Delta_{h,x_n} u_e\|_p \leqslant (6h)^{1/p} \int_{\mathbf{R}^{n-1}} |e(z')| \, \|\Delta_{hz'} u\|_p \, dz'. \tag{2.13}$$

We note that the right-hand side in (2.13) can be estimated by

$$C h^\alpha \sup_{z'} (|z'|^{-\alpha} \|\Delta_{z'} u\|_p) \quad \text{with} \quad C = 6^{1/p} \int |e(z')| \, |z'|^\alpha \, dz'$$

and (2.13) gives in this case (2.5)(ii). For $\theta < \infty$ we substitute (2.13) into (2.11), use Hölder's inequality, represent the integration with respect to z' in polar coordinates, and make the change of variables $h \mapsto hr = t$ to get

$$\int_0^\infty h^{-1-\theta(\alpha+1/p)} \|\Delta_{h,x_n} u_e\|_p^\theta \, dh \leqslant 6^{\theta/p} \int_0^\infty h^{-1-\theta\alpha} \Big(\int_{\mathbf{R}^{n-1}} |e(z')| \, \|\Delta_{hz'} u\|_p \, dz'\Big)^\theta dh$$

$$\leqslant 6^{\theta/p} \int_0^\infty h^{-1-\theta\alpha} \Big(\int_{\mathbf{R}^{n-1}} |e(z')| \, \|\Delta_{hz'} u\|_p^\theta \, dz'\Big) \Big(\int_{\mathbf{R}^{n-1}} |e(z')| \, dz'\Big)^{\theta/\theta'} dh$$

$$= C \int_\Sigma \int_0^1 \int_0^\infty h^{-1-\theta\alpha} |e(r\omega)| \, \|\Delta_{hr\omega} u\|_p^\theta \, r^{n-2} \, dr \, dh \, d\omega$$

$$= C \int_\Sigma \int_0^1 \int_0^\infty t^{-1-\theta\alpha} e(r\omega) \, \|\Delta_{t\omega} u\|_p^\theta \, r^{n-2+\theta\alpha} \, dt \, dr \, d\omega$$

$$\leqslant C \Big(\max_\Sigma \int_0^1 |e(t\omega)| \, r^{n+2+\theta\alpha} dr\Big) \int_\Sigma \int_0^\infty t^{-1-\theta\alpha} \|\Delta_{t\omega} u\|_p^\theta \, dt \, d\omega.$$

The last factor is equal to $\int_{\mathbf{R}^{n-1}} |h'|^{1-n-\theta\alpha} \|\Delta_{h'} u\|_p^\theta \, dh'$ and in the present case we get the desired estimate for (2.11) and therefore for (2.4). Note that in the above formulas Σ is the sphere $\{\omega \in \mathbf{R}^{n-1};\ |\omega| = 1\}$.

In (2.12) we use the formula (note that after the integration with respect to z' the coefficient of $u(x')$ is zero)

$$\Delta^2_{h,x_n} u_e(x) = \int_{\mathbf{R}^{n-1}} \Delta^2_{h,x_n} [x_n^{1-n} e(x_n^{-1} z')] [u(x' - z') - u(x')] \, dz'.$$

Using the mean-value theorem and the properties of e, we get the estimate

(2.14)

$$\Phi(h, x_n, z') \equiv |\Delta^2_{h,x_n} [x_n^{1-n} e(x_n^{-1} z')]| \leqslant C x_n^{-1-n} h^2 \chi_{x_n+2h}(z') \leqslant C x_n^{-1-n} h^2 \chi_{3x_n}(z'),$$

where $C = n(n-1)\|e\|_\infty + 2n\|\nabla e\|_\infty + \|\nabla^2 e\|_\infty$ and χ_r denotes the characteristic function of the ball $\{z';\ |z'| \leqslant r\}$.

As above we use the continuous form of Minkowski's inequality to get

$$(\|\Delta^2_{h,x_n} u_e(\cdot, x_n)\|_p)^p \leqslant \Big(\int_{\mathbf{R}^{n-1}} \Phi(h, x_n, z') \|\Delta_{-z'} u\|_p \, dz'\Big)^p \tag{2.15}$$

integrating with respect to x_n, $x_n \geqslant h$ we get

$$(2.16)\qquad \|\Delta^2_{h,x_n} u_e\|^p_{L^p(x_n\geqslant h)} \leqslant \int_h^\infty \Big(\int_{R^{n-1}} \Phi(h, x_n, z')\|\Delta_{-z'}u\|_p dz'\Big)^p dx$$

$$\leqslant \Big(\int_{R^{n-1}} \Big[\int_h^\infty \Phi(h, x_n, z')^p \|\Delta_{-z'}u\|^p_p dx_n\Big]^{1/p} dz'\Big)^p.$$

Using (2.14), we get

$$(2.17)\qquad \int_h^\infty \Phi(h, x_n, z')^p dx_n \leqslant C \int_{\max(h,(|z'|/3))}^\infty h^{2p} x_n^{-(n+1)p} dx_n$$

$$= C\big((n+1)p-1\big)^{-1}\min\big(h^{1-(n-1)p}, h^{2p}(\tfrac{1}{3}|z'|)^{1-(n+1)p}\big)$$

$$\equiv \Psi(h, z')^p.$$

Substituting in (2.16), we get

$$(2.18)\qquad \|\Delta^2_{h,x_n} u_e\|_{L^p(x_n\geqslant h)} \leqslant \int_{R^{n-1}} \Psi(h, z')\|\Delta_{-z'}u\|_p dz'.$$

For $\theta = \infty$ we get

$$\|\Delta^2_{h,x_n} u_e\|_{L^p(x_n\geqslant h)} \leqslant \Big(\int_{R^{n-1}} \Psi(h, z')|z'|^a dz'\Big)\sup_{h'} |h'|^{-a}\|\Delta_{h'}u\|_p$$

and by (2.17)

$$(2.19)\qquad \int \Psi(h, z')|z'|^\gamma dz'$$

$$= C\Big[\int_{|z'|\leqslant 3h} h^{n-1+1/p}|z'|^\gamma + 3^{n+(1/p')} \int_{|z'|\geqslant 3h} h^2|z'|^{-n-1+1/p+\gamma} dz'\Big] = C_1 h^{\gamma+1/p}$$

which yields the desired estimate with $\gamma = a$.

For $\theta < \infty$ we substitute (2.18) into (2.12) and use (2.19) (with $\gamma = 0$) to get

$$(2.20)\qquad \int_0^\infty h^{-1-\theta(a+(1/p))}\|\Delta^2_{h,x_n} u_e\|^\theta_p dh$$

$$\leqslant \int_0^\infty h^{-1-\theta(a+1/p)}\Big(\int_{R^{n-1}} \Psi(h, z')\|\Delta_{-z'}u\|_p dz'\Big)^\theta dh$$

$$\leqslant \int_0^\infty h^{-1-\theta(a+1/p)}\Big(\int_{R^{n-1}} \Psi(h, y')dy'\Big)^{\theta/\theta'} \int_{R^{n-1}} \Psi(h, z')\|\Delta_{-z'}u\|^\theta_p dz' dh$$

$$= C_1^{\theta/\theta'} \int_{R^{n-1}} \Big[\int_0^\infty h^{-1-\theta(a+1/p)+\theta/\theta' p}\Psi(h, z')dh\Big]\|\Delta_{-z'}u\|^\theta_p dz'.$$

From the definition of Ψ

$$(2.21)\qquad \int_0^\infty h^{-1-\theta(a+1/p)+\theta/\theta' p}\Psi(h, z')dh$$

$$= C\big((n+1)p-1\big)^{-1}\Big[\int_0^{|z'|/3} h^{1-\theta(a+1/p)+\theta/\theta' p}(\tfrac{1}{3}|z'|)^{(1/p)-n-1}dh +$$

$$+\int_{|z'|/3}^\infty h^{-1-\theta(a+1/p)+\theta/\theta' p+1/p+1-n}dh\Big] = C_2|z'|^{1-n-\theta a}$$

provided that

$$(2.22)\qquad a < \frac{1}{\theta}(2-1/p).$$

Substitution of (2.21) into (2.20) gives the desired estimate for (2.12) for a subject to (2.21). The proof is complete for $p < \infty$.

For $p = \infty$ we have

$$(2.23)\qquad \|\Delta_{h'}u_e\|_\infty \leqslant \int |e(z')|\,\|\Delta_{h'}u\|_\infty dz'$$

and

$$(2.24)\qquad \|\Delta_{h,x_n}u_e\|_\infty \leqslant \int |e(z')|\,\|\Delta_{hz'}u\|_\infty dz',$$

(2.23) yields immediately the desired estimate for (2.3), (2.5)(i) with $k = 1$. From (2.24) we obtain (2.5)(ii) with $k = 1$ and $C = \int |e(z')| \times \times |z'|^a dz'$; also for $\theta < \infty$

$$\int_{R^1} \|\Delta_{h,x_n}u_e\|^\theta_\infty |h|^{-1-\theta a}dh = 2\int_0^\infty \|\Delta_{h,x_n}u_e\|^\theta_\infty h^{-1-\theta a}dh$$

$$\leqslant \Big(\int_{R^{n-1}} |e(z')|dz'\Big)^{\theta/\theta'} \int_0^\infty \int_0^1 \int_{\Sigma'} |e(r\omega)|\,\|\Delta_{hr\omega}u\|^\theta_\infty h^{-1-\theta a} r^{n-2} d\omega\, dr\, dh$$

$$\leqslant C\int_0^\infty \int_\Sigma t^{-1-\theta a}\|\Delta_{t\omega}u\|^\theta_\infty d\omega\, dt,$$

$$C = \|e\|_1^{\theta/\theta'} \max_{\omega\in\Sigma} \int_0^1 |e(r\omega)|r^{n-2+\theta a},$$

which is (2.4).

Remarks. (1) For $\theta = p < \infty$ an alternative estimate for (2.12) can be obtained with $a < 1$ instead of (2.22).

(2) The above proof could be simplified if we knew that $B^a_{p,\theta}$, $p < \infty$, is an interpolation space between $B^a_{p,1}$ and $B^1_{p,\infty}$. A result of this nature is contained in [9], but unfortunately it does not include the cases $\theta = 1$ and $\theta = \infty$.

(3) Theorem 2.1 implies that the operator (1.9) $u \mapsto \varphi(x_n) u_e(x)$ is bounded from $B^{a}_{p,\theta}(\boldsymbol{R}^{n-1})$ into $B^{a+(1/p)}_{p,\theta}(\boldsymbol{R}^n)$, $a > 0$.

3. Extension operators defined by mollifiers. We consider now the transformation $u \mapsto u_e$ defined by (2.1) with e subject to condition:

$$\int e(x')\,dx' = 1, \qquad e(x') = 0 \text{ for } |x'| > 1. \tag{3.1}$$

If e is bounded, measurable and satisfies (3.1) then $\lim\limits_{x_n \to 0} u_e(x', x_n) = u^L(x')$ at every Lebesgue point of $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^{m-1})$.

With the above hypotheses on e we also have

PROPOSITION 3.1. (i) *If $u \in L^p_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$, $1 < p \leqslant \infty$ then $u^L_e(x)$ exists at almost every point of $\boldsymbol{R}^{n-1}$ and $u^L(x', 0) = u(x')$ a.e.*

(ii) *If $u \in B^a_{p,\theta}$ for some $a > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$, then $u^L_e(x', 0)$ exists* exc. $\mathfrak{B}^a_{p,\theta,n-1}$ *and $u^L_e(x', 0) = u(x')$* exc. $\mathfrak{B}^a_{p,\theta,n-1}$.

Proof. We have for $x = (x', 0)$

(3.2)
$$r^{-n} \int\limits_{|y| \leqslant r} |u_e(x+y) - u(x)|\,dy \leqslant r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|x_n| \leqslant r} |u_e(x'+y', y_n) - u(x')|\,dy_n\,dy'$$
$$\leqslant r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} |u_e(x'+y', y_n) - u(x'+y')|\,dy' +$$
$$+ 2r^{1-n} \int\limits_{|y'| \leqslant r} |u(x'+y') - u(x')|\,dy'.$$

The second term converges to zero as $r \to 0$ for every x' s.t. $u^L(x') = u(x')$, i.e. almost everywhere.

For the first term we have

$$I_r(x') \equiv r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} |u_e(x'+y', y_n) - u(x'+y')|\,dy_n\,dy'$$
$$= r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} \Big|\int e(z')[u(x'+y'-y_n z') - u(x'+y')]\,dz'\Big|\,dy_n\,dy'$$
$$\leqslant \|e\|_\infty r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} \int\limits_{|z'| \leqslant 1} |u(x'+y'-y_n z') - u(x'+y')|\,dz'\,dy_n\,dy'.$$

Consider now the function

$$v_\varrho(y') = \sup_{0 < r \leqslant \varrho} \int\limits_{|z'| \leqslant 1} |u(y'+rz') - u(y')|\,dz'; \tag{3.3}$$

$v_\varrho(y') \searrow 0$ as $\varrho \searrow 0$ at every Lebesgue point y' of u.

Since by hypothesis $u \in L^p_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$, $p > 1$, by K. T. Smith's generalization of Hardy-Littlewood's theorem [8]

$$\sup_{0 < r \leqslant 1} \int\limits_{|z'| \leqslant 1} |u(y'+rz')|\,dz' \in L^p_{\mathrm{loc}}(\boldsymbol{R}^{n-1})$$

and

$$v_\varrho \in L^p_{\mathrm{loc}}(\boldsymbol{R}^{n-1}) \subset L^1_{\mathrm{loc}}(\boldsymbol{R}^{n-1}).$$

Hence we can write for $0 < r \leqslant \varrho$

$$I_r(x') \leqslant \int\limits_{|z'| \leqslant 1} v_\varrho(x'+rz')\,dz'$$

and

$$\limsup_{r \to 0} I_r(x') \leqslant \limsup_{r \to 0} \int\limits_{|z'| \leqslant 1} v_\varrho(x'+rz')\,dz' = v_\varrho(x')$$

for almost every x'. Letting $\varrho \to 0$ we get (i).

The proof of (ii) depends on the mean-value theorem for the Bessel kernel ([2], p. 418) and is similar to the argument used in [3], § 0, Th. I.

$$\int\limits_{|y| \leqslant 1} G_a(x+ry)\,dy \leqslant C G_a(x), \qquad x \neq 0,\ r \leqslant 1, \tag{3.4}$$

with a constant C independent of r.

Pick $0 < \varepsilon < \min(a, 1)$, by (1.6) there is $v \in B^\varepsilon_{p,\theta}(\boldsymbol{R}^{n-1})$ such that

$$u(x') = \int G_{a-\varepsilon}(x'-y') v(y')\,dy' \text{ exc. } \mathfrak{B}^a_{p,\theta,n-1}. \tag{3.5}$$

Also $|v| \in B^\varepsilon_{p,\theta}$ and u is defined pointwise by (3.5) outside of the set

$$A = \{x' \in \boldsymbol{R}^{n-1};\ \int G_{a-\varepsilon}(x'-y')|v(y')|\,dy' = \infty\} \in \mathfrak{B}^a_{p,\theta}.$$

Using (3.4) we get

$$r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} |u_e(x'+y', y_n) - u(x')|\,dy$$
$$= r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} \Big[\int e(z') \int [G_{a-\varepsilon}(x'+y'-y_n z'-t') - G_{a-\varepsilon}(x'-t')] v(t')\,dt'\Big]\,dy$$
$$\leqslant \|e\|_\infty \int \Big[r^{-n} \int\limits_{|y'| \leqslant r} \int\limits_{|y_n| \leqslant r} \int\limits_{|z'| \leqslant 1} |G_{a-\varepsilon}(x'+y'-y_n z'-t') -$$
$$- G_{a-\varepsilon}(x'-t')|\,dy\,dz'\Big]\,|v(t')|\,dt'$$
$$= \|e\|_\infty \int F(r, x', t')\,|v(t')|\,dt'.$$

Since $G_{a-\varepsilon}(z')$ is smooth for $z' \neq 0$, it follows that $F(r, x', t') \xrightarrow[r \to 0]{} 0$ for $t' \neq x'$. Also using (3.4) we get

$$F(r, x', t')\,|v(t')| \leqslant (C+1) G_{a-\varepsilon}(x'-t')\,|v(t')| \in L^1(\boldsymbol{R}^{n-1})$$

for $x' \notin A$. By the Lebesgue dominated convergence theorem we get the result.

Remarks. (1) The first part of the proposition suggests that $u_e^L(x', 0) = u^L(x')$ for every Lebesgue point of u. This we were unable to prove; if it were true the part (ii) would follow by Th. I, § 0, [3].

(2) The hypothesis $u \in L^p_{\mathrm{loc}}$, $p > 1$ (or more generally, $u \in (L^1 \log^+ L^1)_{\mathrm{loc}}$) seems somewhat artificial; it would be interesting to see if (i) remains valid with the hypothesis $u \in L^1_{\mathrm{loc}}$.

As an immediate consequence of Proposition 3.1 we get

THEOREM 3.2. *If $u \in B^a_{p,\theta}(\boldsymbol{R}^{n-1})$, $a > 0$, $1 \leqslant p \leqslant \infty$ $1 \leqslant \theta \leqslant \infty$ and $e \in C_0^2(\{|x'| \leqslant 1\})$ satisfies (3.1) then $u_e|_{\boldsymbol{R}^{n-1}} = u$ exc. $\mathfrak{B}^a_{p,\theta}$. In particular, $u \to u_e$ is an extension.*

Proof. By Theorem 2.1, $u_e \in B^{a+1/p}_{p,\theta}(\boldsymbol{R}^n)$ and by (1.8) $u_e = u_e^L$ exc. $\mathfrak{B}^{a+1/p}_{p,\theta,n}$ also $u = u^L$ exc $\mathfrak{B}^a_{p,\theta,n-1}$ and $u_e^L|_{\boldsymbol{R}^{n-1}} = u^L$ by Proposition 3.1.

The definition of u_e implies that

$$\operatorname{supp} u_e \subset \{(x', x_n) \in \boldsymbol{R}^n;\ d(x', \operatorname{supp} u) \leqslant |x_n|\}.$$

Using singular multipliers [3] this permits us to construct for every $\varepsilon > 0$ an extension operator E: $B^a_{p,\theta}(\boldsymbol{R}^{n-1}) \to B^{a+(1/p)}(\boldsymbol{R}^n)$ with the property that

$$\operatorname{supp} Eu \subset \big\{(x', x_n) \subset \boldsymbol{R}^{n-1};\ x' \in \operatorname{supp} u,\ |x_n| \leqslant \varepsilon d\big(x', \partial(\operatorname{supp} u)\big)\big\}$$

(cf. [4]). The existence of such operators is very useful in defining the spaces $B^a_{p,\theta}$ on manifolds with singularities.

4. *p*-restrictions. (4) In this section we restrict our attention to the case $\theta = p$, we use the notation $B^a_{p,p} = B^a_p$. We also assume unless otherwise indicated that $1 < p < \infty$. The first part of Proposition 1.1 is not valid for $a = 1/p$; $(n-1)$-dimensional subspaces of $\boldsymbol{R}^n$ are exceptional sets for $B^{1/p}_p$. It is easy to check that for $0 < a < 1/p$ the conditions

$$u_+ = u|_{\boldsymbol{R}^n_+} \in B^a_p(\boldsymbol{R}^n_+), \qquad u_- = u|_{\boldsymbol{R}^n_-} \in B^a_p(\boldsymbol{R}^n_-)$$

imply that $u \in B^a_p(\boldsymbol{R}^n)$. Here we use the notation

$$\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n;\ x_n > 0\}, \qquad \boldsymbol{R}^n_- = \{x \in \boldsymbol{R}^n;\ x_n < 0\}.$$

The above statement is not valid if $a = 1/p$ which suggests that in spite of absence of pointwise restrictions $u_+|_{\boldsymbol{R}^{n-1}}$, $u_-|_{\boldsymbol{R}^{n-1}}$ the functions u_+ and u_- must satisfy certain conditions in order that $u \in B^{1/p}_p(\boldsymbol{R}^n)$. The concept is made precise by the following definition due to N. Aronszajn who also suggested some of the results below.

(4) In [6] these are referred to as abstract restrictions ($p = 2$).

A function $v \in L^p(\boldsymbol{R}^n_+)$, $p \geqslant 1$, has zero p-restriction to $\boldsymbol{R}^{n-1}$, $u|^p_{\boldsymbol{R}^{n-1}} = 0$, if and only if

(4.1)
$$\int_{\boldsymbol{R}^n_+} x_n^{-1} |v(x)|^p\,dx < \infty \quad \textit{or equivalently} \quad \int_0^1 \int_{\boldsymbol{R}^{n-1}} x_n^{-1} |v(x)|^p\,dx'\,dx_n < \infty.$$

Two functions u, $v \in L^p(\boldsymbol{R}^n_+)$ have the same p-restrictions to $\boldsymbol{R}^{n-1}$ if $u - v|^p_{\boldsymbol{R}^{n-1}} = 0$.

The relevance of this notion to the question raised at the beginning of the section is clear from the following proposition.

PROPOSITION 4.1. *Suppose that $u \in L^p(\boldsymbol{R}^n)$ and $u_\pm = u|_{\boldsymbol{R}^n_\pm} \in B^{1/p}_p(\boldsymbol{R}^n_\pm)$; let $v(x', x_n) = u(x', -x_n)$, $x_n > 0$, $x' \in \boldsymbol{R}^{n-1}$. Then $u \in B^{1/p}_p(\boldsymbol{R}^n)$ if and only if u_+ and v have the same p-restriction to $\boldsymbol{R}^{n-1}$.*

Proof. If $u \in B^{1/p}_p(\boldsymbol{R}^n)$ then

(4.2)
$$\int_{\boldsymbol{R}^1} \int_{\boldsymbol{R}^1} \int_{\boldsymbol{R}^{n-1}} |x_n - y_n|^{-2} |u(x', x_n) - u(x', -y_n)|^p\,dx\,dy_n$$
$$= \int_0^\infty \int_0^\infty \int_{\boldsymbol{R}^{n-1}} \ldots + \int_{-\infty}^0 \int_{-\infty}^0 \int_{\boldsymbol{R}^{n-1}} \ldots + 2\int_0^\infty \int_{-\infty}^0 \int_{\boldsymbol{R}^{n-1}} \ldots \leqslant \|u\|_{1/p,p} < \infty.$$

Thus we can write

$$\int_{\boldsymbol{R}^n_+} x_n^{-1} |u(x) - v(x)|^p\,dx = \int_0^\infty \int_{\boldsymbol{R}^{n-1}} x_n^{-1} |u(x', x_n) - u(x', -x_n)|^p\,dx'\,dx_n$$
$$= \int_{\boldsymbol{R}^{n-1}} \int_0^\infty \int_{-\infty}^0 (x_n - y_n)^{-2} |u(x', x_n) - u(x', -x_n)|^p\,dy_n\,dx_n\,dx'$$
$$\leqslant 2^{p-1} \int_{\boldsymbol{R}^{n-1}} \Big[\int_0^\infty \int_{-\infty}^0 (x_n - y_n)^{-2} \big(|u(x', x_n) - u(x', y_n)|^p +$$
$$+ |u(x', y_n) - u(x', -x_n)|^p\big)\,dy_n\,dx_n\Big]\,dx'.$$

The first integral is bounded by the last integral in (4.2) and the second can be estimated by

$$\int_{-\infty}^0 \int_{-\infty}^0 \int_{\boldsymbol{R}^{n-1}} |x_n - y_n|^{-2} |u(x', y_n) - u(x', x_n)|^p\,dx_n\,dy_n\,dx'$$

which is the second integral in (4.2).

Sufficiency. If $u_\pm \in B_p^{1/p}(\mathbf{R}_+^n)$ then

(4.3)
$$\int_0^\infty \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} |x'-y'|^{-n} |u(x', \pm x_n) - u(x', \pm y_n)|^p dx'\, dy'\, dx_n \leqslant \|u_\pm\|^p_{B_p^{1/p}(\mathbf{R}_\pm^n)}$$

and

(4.4)
$$\int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n-1}} |x_n - y_n|^{-2} |u(x', \pm x_n) - u(x', \pm y_n)|^p dx'\, dx_n\, dy_n \leqslant \|u_\pm\|^p_{B_p^{1/p}(\mathbf{R}_\pm^n)}.$$

Adding the two integrals in (4.3) we obtain the estimate

$$\int_{\mathbf{R}^{n-1}} |h'|^{-n} \|\Delta_{h',x'} u\|_p^p dx' \leqslant \|u_+\|_{B_p^{1/p}(\mathbf{R}_+^n)} + \|u\|_{B_p^{1/p}(\mathbf{R}_-^n)}.$$

Thus it remains to estimate the integral $\int_{\mathbf{R}^1} |h|^{-2} \|\Delta_{h,x_n} u\|_p^p dh$ which can be written as in (4.2). The first two integrals on the right-hand side of (4.2) are estimated by means of (4.4); for the third one we can write

$$\int_{\mathbf{R}^{n-1}} \int_0^\infty \int_{-\infty}^0 (x_n - y_n)^{-2} |u(x', x_n) - u(x', y_n)|^p dy_n\, dx$$
$$\leqslant 2^{p-1} \int_{\mathbf{R}^{n-1}} \Big[\int_0^\infty \int_{-\infty}^0 (x_n - y_n)^{-2} |u(x', x_n) - u(x', -x_n)|^p dy_n\, dx_n +$$
$$+ \int_{-\infty}^0 \int_{-\infty}^0 (x_n + y_n)^{-2} |u(x', x_n) - u(x', y_n)|^p dy_n\, dx_n \Big] dx'$$
$$\leqslant 2^{p-1} \Big[\|u\|^p_{B_p^{1/p}(\mathbf{R}_-^n)} + \int_{\mathbf{R}_+^n} x_n^{-1} |u(x', x_n) - (x', -x_n)|^p dx \Big].$$

Remarks. (1) The notion of p-restrictions, or some equivalent concept is essential in establishing compatibility conditions for Bessel potentials on manifolds with singularities (see [6]). Proposition 4.1 is a special case of such compatibility conditions.

(2) If $p = 1$ the compatibility condition becomes more complicated due to the fact that differences of order at least 2 appear in the definition of the norm in B_1^1.

(3) The definition of zero p-restrictions suggests that one could identify restrictions of functions in $B_p^{1/p}(\mathbf{R}_+^n)$ to $\mathbf{R}^{n-1}$ with elements of the quotient space $B_p^{1/p}(\mathbf{R}_+^n)/\mathring{B}_p^{1/p}(\mathbf{R}_+^n)$ where $\mathring{B}_p^{1/p}(\mathbf{R}_+^n) = \{u \in B_p^{1/p}(\mathbf{R}_+^n);\ u|^p_{\mathbf{R}^{n-1}} = 0\}$. This formal identification does not seem very useful because $\mathring{B}_p^{1/p}(\mathbf{R}_+^n)$ is not closed and actually is dense in $B_p^{1/p}(\mathbf{R}_+^n)$.

We next describe some relations between pointwise properties of functions and p-restrictions.

PROPOSITION 4.2. *If $u \in L^p(\mathbf{R}^n)$, $p \geqslant 1$, and $u|^p_{\mathbf{R}^{n-1}} = 0$ then $(|u|^p)^L(x', 0) = 0$ for a.e. $x' \in \mathbf{R}^{n-1}$. In particular, $u^L(x', 0) = 0$ for a.e. $x' \in \mathbf{R}^{n-1}$.*

Proof. By Fubini's theorem, $x_n^{-1}|u(x', x_n)|^p \in L^1(\mathbf{R}_+)$ for a.e. $x' \in \mathbf{R}^{n-1}$ and therefore

$$v_r(x') = \int_0^r x_n^{-1} |u(x', x_n)|^p dx' \underset{r\to 0}{\rightarrow} 0, \quad \text{for a.e. } x',\ v_r \in L^1(\mathbf{R}^{n-1}).$$

We have for $0 < \varrho \leqslant r$

$$\varrho^{-n} \int_{|y| \leqslant \varrho} |u(x'+y', y_n)|^p dy \leqslant \varrho^{-n} \int_{|y| \leqslant \varrho} \int_0^\varrho |u(x'+y', y_n)|^p dy_n\, dy'$$
$$\leqslant \varrho^{1-n} \int_{|y'| \leqslant \varrho} v_r(x'+y')\, dy'.$$

Letting $\varrho \to 0$ and using the fact that $v_r \in L^1$, we get

$$\limsup_{\varrho\to 0} \varrho^{-n} \int_{|y| \leqslant \varrho} |u(x'+y', y_n)|^p dy \leqslant v_r(x') \text{ a.e.}$$

and since $v_r(x') \underset{r\to 0}{\rightarrow} 0$ a.e., this completes the proof.

The converse to Proposition 4.2 does not seem to be true but we do not have a counterexample. ([5])

The next proposition gives some insight into properties of the extension operator $u \to u_e$ in the exceptional case.

PROPOSITION 4.3. *Let e_1, e_2 be bounded measurable functions satisfying (3.1) and $u \in L^2(\mathbf{R}^{n-1})$. Define $u_i = u_{e_i}$, $i = 1, 2$ by (2.1). Then u_1 and u_2 have the same 2-restrictions to $\mathbf{R}^{n-1}$.*

Proof. We have to show that the integral

$$I = \int_0^\infty \int_{\mathbf{R}^{n-1}} |u_1(x', x_n) - u_2(x', x_n)|\, x_n^{-1} dx_n$$

is finite. Using Fourier transform with respect to the variable x', the definition of u_i and Parseval's equality, we get

$$I = \int_0^\infty \int_{\mathbf{R}^{n-1}} |\hat{e}_1(x_n \xi') - \hat{e}_2(x_n \xi')|^2 x_n^{-1} |u(\xi')|^2 d\xi'\, dx_n$$

([5]) Added in proof. See [6] for a counterexample for $p = 2$.

and to arrive at the desired conclusion it suffices to show that the function

$$\varphi(\xi') = \int_0^\infty t^{-1}|e_1(t\xi') - e_2(t\xi')|^2 dt$$

is bounded.

Using polar coordinates $\xi' = |\xi'|\omega$, $|\omega| = 1$, $\varphi(\xi')$ can be written in the form

$$(4.5)\quad \varphi(\xi') = \int_0^\infty |e_1(t\omega) - e_2(t\omega)| t^{-1} dt$$
$$\leqslant \int_0^1 |e_1(t\omega) - e_2(t\omega)|^2 t^{-1} dt + \int_1^\infty |e_1(t\omega) - e_2(t\omega)|^2 t^{-1} dt.$$

Since $\hat{e} = \hat{e}_1 - \hat{e}_2 \in C^\infty(\boldsymbol{R}^{n-1})$ and $\hat{e}(0) = 0$, it follows from the second order Taylor's formula that the first integral is bounded. From boundedness of e and compactness of its support it follows that $D^i \hat{e} \in L^2(\boldsymbol{R}^{n-1})$ for all multiindices i; this implies, by the known theorems about restrictions, that $\hat{e}$ belongs to L^2 on all lines with a fixed bound for the norms, hence the last integral in (4.5) is bounded uniformly in ξ and the proof is complete.

It would be interesting to see if the content of Proposition 4.3 remains valid for $p \neq 2$, perhaps with some additional regularity conditions on e_1, e_2.

The following result is in certain sense complements Proposition 4.3 and Theorem 3.2 in an exceptional case.

PROPOSITION 4.4. *If e satisfies* (3.1), *is bounded and measurable then E defined by* (1.9) *is a bounded transformation of $L^2(\boldsymbol{R}^{n-1})$ into $P^{1/2}(\boldsymbol{R}^n) = B_2^{1/2}(\boldsymbol{R}^n)$.*

Proof. Let $v(x) = (Eu)(x) = \varphi(x_n)\int e(y')u(x' - x_n y')dy'$, where $\varphi \in C_0^\infty(-1, 1)$, $\varphi = 1$ in a neighborhood of $x_n = 0$. We have to estimate the integrals

$$(4.6)\quad I_1 = \int_{\boldsymbol{R}^1} \|v(\cdot, x_n)\|_{1/2}^2 dx_n, \qquad I_2 = \int_{\boldsymbol{R}^1}\int_{\boldsymbol{R}^1} \|\Delta_{t,x_n} v(\cdot, x_n)\|^2 |t|^{-2} dt\, dx_n$$

in terms of $\|u\|^2$.

The first integral in (4.6) can be written in terms of Fourier transform of v with respect to x':

$$I_1 = (2\pi)^{n-1} \int_{-1}^1 \int_{\boldsymbol{R}^{n-1}} |\varphi(x_n)\hat{e}(x_n\xi')|^2 (1 + |\xi'|^2)^{1/2} |\hat{u}(\xi')|^2 d\xi' dx_n.$$

To obtain the desired estimate we have to verify that the function

$$\Phi_1(\xi') = (1 + |\xi'|^2)^{1/2} \int_{-1}^1 |\varphi(x_n)\hat{e}(x_n\xi')|^2 dx_n \leqslant C(1 + |\xi'|^2)^{1/2} \int_{-1}^1 |\hat{e}(x_n\xi')|^2 dx_n$$

is bounded. For $\xi' = |\xi'|\omega'$, and $|\xi'| \geqslant 1$

$$\int_{-1}^1 |e(x_n\xi')|^2 dx_n \leqslant \frac{1}{|\xi'|} \int_{-\infty}^\infty |e(t\omega')|^2 dt \leqslant \frac{C}{|\xi'|} \quad \text{and} \quad \Phi_1(\xi') \leqslant C\frac{(1 + |\xi'|^2)^{1/2}}{|\xi'|}$$

with C independent of ω' (see the proof of Proposition 4.3). For $|\xi'| \leqslant 1$ we have $|\hat{e}(x_n\xi')| \leqslant 1 + C_1|\xi'|$ where

$$C_1 = \max_{|\xi'| \leqslant 1} |\nabla \hat{e}(\xi')| \quad \text{and} \quad \Phi_1(\xi') \leqslant 2^{1/2}(1 + C_1).$$

The second integral in (4.6) can also be represented in terms of partial Fourier transform:

$$I_2 = (2\pi)^{n-1} \int_{\boldsymbol{R}^{n-1}}\int_{\boldsymbol{R}^1}\int_{\boldsymbol{R}^1} |\Delta_{t,x_n}[\varphi(x_n)\hat{e}(x_n\xi')|^2 |\hat{u}(\xi')|^2 t^{-2} dt dx_n d\xi'$$

and as before we have to check that the function

$$\Phi_2(\xi') = \int_{\boldsymbol{R}^1}\int_{\boldsymbol{R}^1} |\Delta_t[\varphi(x_n)\hat{e}(x_n\xi')]|^2 t^{-2} dt\, dx_n$$

is bounded. The latter is easily recognized as the square of the difference term in the $P^{1/2}(\boldsymbol{R}^1)$ norm of the function $x_n \to \varphi(x_n)\hat{e}(x_n\xi')$. It follows, by the smoothness properties of φ, that this is estimated by a constant multiple of the corresponding term involving $\hat{e}$ only. For $\xi' \neq 0$, $\xi' = |\xi'|\theta'$

$$\int_{\boldsymbol{R}^1}\int_{\boldsymbol{R}^1} t^{-2}|\Delta_t \hat{e}(x_n\xi')|^2 dx_n dt = \int_{\boldsymbol{R}^1}\int_{\boldsymbol{R}^1} t^{-2}|\Delta_{t,x_n}\hat{e}(x_n\theta')|^2 dx_n dt$$

which is part of the $P^{1/2}$ norm of the restriction of $\hat{e}$ to the line through the origin in direction of θ'. As already remarked in the proof of Proposition 4.3, the derivatives of all orders of $\hat{e}$ are in L^2, and by known restriction theorems the last integral is bounded independently of θ'. The proof is complete.

Remarks. The estimates used in Section 2 do not seem to yield the statement of Proposition 4.4.

Propositions 4.3 and 4.4 allow us to elaborate on the notion of 2-restriction. A function $v \in P^{1/2}(\boldsymbol{R}^n)$ has *2-restriction* to $\boldsymbol{R}^{n-1}$ equal to $u \in L^2(\boldsymbol{R}^{n-1})$ if for some bounded measurable e satisfying (3.1)

$$(v - u_e)|_{\boldsymbol{R}^{n-1}}^2 = 0.$$

By Proposition 4.4 the above definition is independent of the choice of e; if the condition is satisfied for some e then it also holds for all e.(6)

(6) A more satisfactory definition is given in [6].

References

[1] S. M. Nikolskii, *Approximation of functions of several variables and imbedding theorems*, (in Russian), Moscow 1969.

[2] N. Aronszajn, K. T. Smith, *Theory of Bessel potentials. Part I*, Ann. Inst. Fourier, 11 (1961), pp. 385–475.

[3] R. Adams, N. Aronszajn, K. T. Smith, *Theory of Bessel potentials. Part II*, ibidem 17, 2 (1967), pp. 1–135.

[4] N. Aronszajn, F. Mulla, P. Szeptycki, *On spaces of potentials connected with L^p-classes*, ibidem 13, 2 (1963), pp. 211–306.

[5] R. D. Adams, N. Aronszajn, M. S. Hanna, *Theory of Bessel potentials. Part III*, ibidem 19, 2 (1969), pp. 279–338.

[6] N. Aronszajn, P. Szeptycki, *Theory of Bessel potentials. Part IV*, to appear in Ann. Inst. Fourier.

[7] K. T. Smith, *A generalization of an inequality of Hardy and Littlewood*, Can. J. Math. 8 (1956), pp. 157–170.

[8] E. Gagliardo, *Caratterrizzazioni della tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Padova 27 (1957), pp. 284–305.

[9] Mitchell H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space*, Journal of Math. and Mech. 14, no. 5, (1965), pp. 821–840.

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Toeplitz operators related to certain domains in C^n

by

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Abstract. Venugopalkrisna in [9] investigated the Toeplitz operators in strongly pseudoconvex domain $D \subset C^n$, $n > 1$. Among other he proved that Toeplitz operator with continuous symbol φ (smooth in D) is Fredholm if ∂D is smooth and if φ does not vanish on ∂D. On the other hand, Coburn identified C^*-algebra generated by Toeplitz operators on odd spheres, modulo compact operators [2]. We shall identify the C^*-algebra generated by Toeplitz operators in strongly pseudoconvex domain D modulo the compact operators. We shall also prove some simple properties of Toeplitz operators on the n-dimensional torus Γ^n.

1. Let $L(H)$ be the algebra of all linear bounded operators in a complex Hilbert space H and let $\mathscr{K}(H)$ be the ideal of all compact operators in H.

DEFINITION 1.1. For any bounded set D in C^n, denote by $\mathscr{L}^2(D)$ the *space of functions f: $D \to C$ which are square integrable with respect to the Lebesgue measure dV in C^n.*

DEFINITION 2.1. Denote by $H^2(D)$ the *space of all $f \in \mathscr{L}^2(D)$, which are holomorphic in D.*

We shall denote by P: $\mathscr{L}^2(D) \to H^2(D)$ the *orthogonal projection onto the subspace $H^2(D)$.*

The definition of Toeplitz operator associated with a function $\varphi \in L^\infty(dV)$ (bounded, measurable in D) reads as follows:

DEFINITION 3.1. Let $\varphi \in L^\infty(dV)$. The *Toeplitz operator* T_φ: $H^2(D) \to H^2(D)$ is defined by

$$T_\varphi f = P(\varphi \cdot f).$$

Let B be the closed unit ball in C^n and let μ be the usual surface measure on $\partial B = S^{2n-1}$. Then one can define the *Hardy space* $H^2(\mu)$ on ∂B as a closed subspace of all functions in $\mathscr{L}^2(\mu)$ which are holomorphic in the int B [2].

The definition of a Toeplitz operator on $H^2(\mu)$ is just the same as Definition 3.1. Let $\mathscr{C}$ be a C^*-algebra generated by Toeplitz operators T_φ ($\varphi \in C(\partial B)$) on $H^2(\mu)$. Then it was proved by Coburn in [2] that the C^*-algebra $\mathscr{C}/\mathscr{K}(H^2(\mu))$ is isometrically isomorphic with $C(\partial B)$. We shall prove