Extensions by mollifiers in Besov spaces

by

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Abstract. An operator of extension from lower dimensional subspaces for functions in Besov spaces is constructed using Friedrichs mollifiers. It has the useful property that for $u$ defined on a hyperplane in $\mathbb{R}^n$ the support $\Delta u$ is contained in the union of cones with vertices in the support of $u$ and axes perpendicular to the hyperplane. Also if the support of $u$ is compact then so is the support of $\Delta u$.

1. Introduction. In this section we shall set up the notations, recall certain facts concerning Besov spaces and state the problem to be dealt with in the paper. Most of the facts about Besov spaces quoted below can be found in [1].

For a (complex, real or vector valued) function $u$ defined in $\mathbb{R}^n$ we denote by $A_{\lambda}u$ the $\lambda$th forward difference with increment $\lambda \cdot R^n$. If $R^n$ is represented as $R^n = R^n_+ \times R$ with $x = (x', x_0)$, $x' = R^n_0$, $x_0 = R$, the corresponding partial differences are denoted by $A_{\lambda x_0} A_{\lambda x'}$. The symbol $|| u ||_{p, R^n_0}$ is used to denote the $L^p$ norm on $R^n_0$ and, with notations as above, $|| u(\cdot, x_0') ||_{p, R^n_0}$, $|| u(x', \cdot) ||_{p, R}$ denote the norms of $u(x', x_0')$ as a function of $x'$ with $x_0'$ fixed or respectively as a function of $x_0'$ with $x'$ fixed.

The Besov norm $|| u ||_{p, R^n_0}$ is a $0 < 1 < p < \infty$, $1 < \theta < \infty$ is defined by

\[ || u ||_{p, R^n_0} = \left( \int \left( \int |h|^{-n+\theta} || a \nabla^k u ||_{p, \mathbb{C}^n}^p \right)^{\frac{\theta p}{n-p}} \right)^{\frac{1}{\theta p}} \]

where $k$ is an integer $k > a$ and for $\theta = \infty$ the integral in parentheses is replaced by $\sup |h|^{-n-\theta} || a \nabla^k u ||_{p, \mathbb{C}^n}$; $\theta \neq 0$.

The different choices of $k > a$ give rise to equivalent norms, this is why $k$ is suppressed in the notation.

A norm equivalent to (1.1) is given by the formula

\[ || u ||_{p, R^n_0} = \left( \int |h|^{-n-\theta} || a \nabla^k u ||_{p, \mathbb{C}^n}^p \right)^{\frac{1}{\theta p}} \]

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where, as above $R^n = R^n \times R$, $a = (x', x'')$, $x', x'' > \alpha$. Similar equivalent norms arise from the decomposition of $R^n$ into cartesian products of two factors.

Yet another norm equivalent to (1.1) is given by the formula

$$
\|u\|_m^2 + \left( \sum_{|\alpha| \leq m} \left( \int_{R^n} |D^\alpha u(x)|^2 dx \right)^{\frac{2}{m}} \right)^{\alpha

where $m$ is an integer $0 < m < a$ and $\alpha > a - m$.  

To simplify the notation we will not introduce distinct symbols for norms (1.1), (1.2) and (1.3); it will be clear from the context which one of them is used at each instant.

The space of (equivalence classes relative to the class $\mathcal{U}$ of sets of Lebesgue measure 0 of) functions with finite norm $\|u\|_{\mathcal{U}, a,b}$ is with this norm a Banach space referred to as Besov space and usually denoted by $B^a_{p,q}(R^n)$. The version (1.3) of the norm implies that

$$B^a_{p,q}(R^n) \quad \text{consists of all functions} \quad u \in L^p(R^n) \quad \text{whose distribution derivatives} \quad D^\alpha u \in L^{p \alpha}_q(R^n) \quad \text{for all} \quad \alpha, \|u\|_{B^a_{p,q}(R^n)} < \infty.$$  

It is known (1.1), (9.6.1) that $B^a_{p,q}(R^n)$ can be characterized as the space of Bessel potentials, $B^a_{p,q}(R^n) = G^a_{p,q}(R^n)$ where $G^a_{p,q}(R^n)$ is certain Banach space of distributions on $R^n$ and $G^a_{p,q}$ denotes the operator of convolution with the $n$-dimensional Bessel kernel of order $a$,

$$G^a_{p,q}(x) = (2a)^{-\frac{n}{2}} \int_{R^n} e^{-|x-y|^2/2a} dy.$$  

This fact and various properties of $G^a_{p,q}$ (see [2], [4], [6]) have several consequences, some of them we list below.

(1.6) The class $C^a_{\infty}(R^n)$ with norm $\|u\|_{C^a_{\infty}}$ has the perfect functional completion relative to the exceptional class $B^a_{p,q,d}$ described as follows. $a \in R^n$ belongs to $B^a_{p,q,d}$ if and only if there is an $\epsilon > 0$ such that

$$a \in L^p(R^n) : G^a_{p,q}(x) = \frac{|x|^{-\frac{n}{2} + \epsilon}}{\int_{R^n} e^{-|y|^2/2a} dy}$$

It is easy to check that the above definition of $B^a_{p,q,d}$ is independent of $\epsilon$.

In order not to complicate the notation we shall from now on use $B^a_{p,q,d}$ to denote the perfect functional completion described by (1.6) instead of imperfect completion $B^a_{p,q}$ introduced before. We remark that (1.4) can be restated with distribution derivatives replaced by pointwise derivatives (see [2]).

(1.7) For fixed $p, q$ and $\alpha > a > 0$ the spaces $B^a_{p,q}$ form an interpolation family (obtained by the analytic interpolation method, see the introduction [3]).

Denote by $u^\mathcal{U}$ the Lebesgue correction of a function $u \in L^p_{\mathcal{U}}(R^n)$, i.e.

$$\lim_{r \to 0} \int_{|x| < r} |u(x) - u^\mathcal{U}(x)| dx = 0.$$  

If $u^\mathcal{U}(x)$ exists then $x$ is called a Lebesgue point of $u$; a theorem of Lebesgue asserts that almost every point $x$ is a Lebesgue point of $u$, also $u = u^\mathcal{U}$ a.e.

(1.8) If $u$ is equal almost everywhere to a function in $B^a_{p,q,d}(R^n)$ then $u - u^\mathcal{U}$ is in $B^a_{p,q,d}$, in particular the set of points where $u^\mathcal{U}$ is undefined is in $B^a_{p,q,d}$ (see [3]).

If $p = \{x \in X : F \in X \}$ then we denote by $F, \mathcal{U}$, the class $A \in R(E, X, \mathcal{U})$. The next proposition describes the restriction and extension properties of spaces $B^a_{p,q,d}$. We consider $R^n - \{\}$ as being canonically identified with a subspace of $R^n$, $R^n - \{\} = (x', x_n = 0) ; x_n = 0$.

Proposition 1.1. If $a \geq 1, p, q \leq 1 \leq \infty$ then $B^a_{p,q,d}(R^n - \{\}) \subset B^a_{p,q,d}(R^n - \{\})$ and $u \in B^a_{p,q,d}(R^n - \{\})$ defines a bounded linear operator (of restriction) of $B^a_{p,q,d}(R^n)$ into $B^a_{p,q,d}(R^n - \{\})$. Also for every $b > 0$ there is a bounded operator (of extension) $E : B^a_{p,q,d}(R^n - \{\}) \rightarrow B^a_{p,q,d}(R^n)$ with the property that $E u|_{R^n - \{\}} = u$ for every $u \in B^a_{p,q,d}(R^n - \{\})$. In particular, the restriction operator in the first statement is onto.

We note that by the first part of the proposition $R^n - \{\}$ is not an exceptional set for $B^a_{p,q,d}(R^n)$ so that $u \in B^a_{p,q,d}(R^n - \{\})$ is well defined on $B^a_{p,q,d}(R^n - \{\})$.

If $\Omega$ is a domain in $R^n$ one can define $B^a_{p,q,d}(\Omega)$ as the space of all functions of the form $u|_{\Omega}, u \in B^a_{p,q,d}(R^n)$ with the restriction norm $\|u\|_{B^a_{p,q,d}(\Omega)} = \inf \{\|u - u|_{\Omega}\|_{B^a_{p,q,d}(R^n)} : u \in B^a_{p,q,d}(R^n)\}$. The norms similar to (1.1), (1.2), (1.3) can be also directly defined on $\Omega$ and for domains satisfying mild regularity conditions can be shown to be equivalent to the restriction norm (1.1), (3). $B^a_{p,q,d}(\Omega)$ is the perfect functional completion of $C^a_{\infty}(\Omega)$ on relatively $B^a_{p,q,d}(\Omega)$.

Various explicit operators of extension are known, (see e.g. [1] in general case, [2], [4] in special cases); the objective of the paper is to prove that such an extension operator can be defined by the formula

$$<B^a_{p,q,d}(\Omega), \varphi> = \frac{1}{\varphi(a_n) \varphi(a)} \varphi(a_n) \int e^{(x', x')/\varphi(a_n)} \varphi(a) \frac{\varphi(a_n)}{\varphi(a)} \frac{\varphi(a_n)}{\varphi(a)} d^{x'},$$  

where $\varphi$ is a sufficiently smooth real valued function on $R^n$ with support in the ball $|x'| < 1$ satisfying the condition $\int e^{(x', x')/\varphi(a)} = 1$ and $\varphi(C^a_{\infty}(\Omega))$ is equal to 1 in a neighborhood of 0. In particular, if $u$ has compact support in $R^n - \{\}$ then $Bu$ defined by (1.9) has compact support contained in the set $\{x \in R^n, \text{dist}(x', \text{supp}(u)) < a_n < M\}$ where $\text{supp}(u) \subset [-M, M]$. 

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(1) $i = (i_1, \ldots, i_k)$, |$i| = i_1 + \ldots + i_k$.

(2) In [1] the operator $G_{\infty}$ is denoted by $I_a$. 

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(3) $\|u\|_{B^a_{p,q,d}}$ is independent of $\epsilon$. 

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(4) $\varphi(a_n) \varphi(a)$ 

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(5) $\varphi(a_n) \varphi(a)$ 

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(6) $\varphi(a_n) \varphi(a)$ 

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(7) $\varphi(a_n) \varphi(a)$ 

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(8) $\varphi(a_n) \varphi(a)$ 

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(9) $\varphi(a_n) \varphi(a)$ 

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The latter property of \( E \) is instrumental in describing spaces \( B^{\alpha}_{p,q}(\mathbb{R}^d) \) on manifolds with singularities ([6]). The other known extension formulas mentioned above in general do not possess compactness of support.

We remark that a formula similar to (1.9), with a specific choice of \( \epsilon \) was used by Gagliardo [8] to describe the restrictions to \( \mathbb{R}^{d-1} \) of functions in the Sobolev’s spaces \( W^{1,p}_0(\mathbb{R}^d) \), \( p > 1 \).

The operators referred to in this paper as Friedrichs mollifiers are also called Sobolev’s averages.

In this research we were mainly interested in the case \( \theta = p \) (actually \( \theta = p = 2 \)), we decided to include the case of arbitrary \( \theta \) in instances when such generalization did not involve additional complications in proofs or actually resulted in a clearer picture of the situation.

To simplify notations we shall use the letter \( C_\epsilon \) possibly with subscripts to denote positive constants which may be different at different instances.

2. Transformations related to Friedrichs mollifiers. For \( x \in \mathbb{R}^d \) we write \( z = (z', z_a) \), \( z' = (z_1, \ldots, z_{d-1}) \).

In this section we are interested in various properties of the formula

\[
(2.1) \quad u_{\epsilon}(z) = \int_{\mathbb{R}^{d-1}} \epsilon(x') u(z'-x_a) dz' = |z_a|^{-n} \int_{\mathbb{R}^{d-1}} \epsilon(x'z_a) u(z'-x_a) dz',
\]

where \( \epsilon \) is a bounded measurable function with support in the unit ball \( \{x' \in \mathbb{R}^{d-1} : |x'| \leq 1\} \). Clearly, \( u_{\epsilon} = \text{def} \) for \( u \in L_{\text{loc}}(\mathbb{R}^{d-1}) \); we shall study the properties of \( u_{\epsilon} \) as a transformation between Besov spaces.

Theorem 2.1. ([13]). Suppose that \( \epsilon \in L^1([|x'| \leq 1]) \). Then (2.1) defines a bounded linear transformation of \( B^{\alpha}_{p,q}(\mathbb{R}^{d-1}) \) into \( B^{\alpha}_{p,q}(\mathbb{R}^{d-1} \times (-M, M)) \) for any \( M > 0 \), \( 0 < \alpha < d \), \( 1 < p < \infty \), \( 1 < q < \infty \).

Proof. Using Young’s inequality we get

\[
|m_{x_1', x_a}|_p \leq |\epsilon|_{L^1} |z_a|_p,
\]

implying that

\[
|m_{x_1', x_a}|_p \leq (2M)^{1/p} |\epsilon|_{L^1} |z_a|_p.
\]

We note next that with \( D_i = \partial / \partial x_i \), we have

\[
D_i u_{\epsilon} = (D_i u)_{\epsilon}, \quad i = 1, \ldots, d-1, \quad D_a u_{\epsilon} = \sum_{i=1}^{d-1} (D_i u)_{\epsilon},
\]

where \( (D_i u)_{\epsilon} \) = \( -\epsilon(x'z_a) \); similar formulas are also valid for higher order derivatives. The remarks of Section 1 concerning the norms (1.3) and the interpolation property (1.7) imply that it is sufficient to prove the theorem for \( \theta = 1 \) for any \( \epsilon > 0 \). We use the version (1.2) of the norm

\[ (*) \text{ The result for } \theta = \epsilon < \epsilon \text{ implies the corresponding result for } m < \epsilon < m+1, \text{ } m > 0 \text{ an integer. The remaining values of } \epsilon \text{ are taken care of by interpolation.} \]

and consider first the case \( p < \infty \). For \( 1 < \theta < \infty \) and some \( k > a + 1/p \) we shall obtain estimates of the form

\[
(2.3) \quad \int_{\mathbb{R}^{d-1}} |A^{\alpha}_{\epsilon} u_{\epsilon}|_p^p |h'|^{-n-\theta+1} \, dh' \leq C |u|_{B^{\alpha}_{p,q}}^p,
\]

and

\[
(2.4) \quad \int_{\mathbb{R}^{d-1}} |A^{\alpha}_{\epsilon} u_{\epsilon}|_p^p |h'|^{-n-\theta+1} \, dh' \leq C |u|_{B^{\alpha}_{p,q}}^p.
\]

For \( \theta = \infty \) we will obtain the estimates

\[
(2.5) \quad (i) \quad \sup \{ |h'|^{-n-\theta+1} \} \|A^{\alpha}_{\epsilon} u_{\epsilon}\|_p \leq C \|u\|_{B^{\alpha}_{p,q}}^p,
\]

(ii) \( \sup \{ |h'|^{-n-\theta+1} \} \|A^{\alpha}_{\epsilon} u_{\epsilon}\|_p \leq C \|u\|_{B^{\alpha}_{p,q}}^p \).

We first prove (2.5)(i) and (2.3); taking \( k = 3 \) and using the last expression for \( u_{\epsilon} \) in (2.1), we can write

\[
D_\alpha u_{\epsilon}(z') = z_a^{-1} \int_{\mathbb{R}^{d-1}} A^{\alpha}_{\epsilon} \epsilon(z_a^{-1} z') A^{\alpha}_{\epsilon} u(z' - z_a) \, dz'.
\]

By the mean-value theorem and the properties of \( \epsilon \)

\[
|m_{x_1', x_a}|_p \leq \int |A^{\alpha}_{\epsilon} \epsilon(z_a^{-1} z')| \, dz' \leq \min \{C_\epsilon |x_a|^{-1} |h'|^\alpha, C_\epsilon\} = \Phi(z_a, h'),
\]

where \( C_\epsilon = |d\epsilon|, C_\epsilon = 4 |\epsilon|_1 \).

Applying Young’s inequality to (2.6) we get

\[
|\epsilon(z_a^{-1} z')| \, dz' \leq C_\epsilon |z_a|^{-1} |h'|^\alpha \|A^{\alpha}_{\epsilon} u\|_p,
\]

and

\[
|\epsilon(z_a^{-1} z')| \, dz' \leq C_\epsilon |z_a|^{-1} |h'|^\alpha \|A^{\alpha}_{\epsilon} u\|_p,
\]

where \( C_\epsilon = (2p-1)^{-1} C_{\epsilon}^{p-2} (C_{\epsilon} C_{\epsilon})^{p-2} \).

Note that the same argument for \( p = 2 \) produces (2.7) with a constant containing \( (p-1)^{-1} \) as a factor (making \( p = 1 \) seems to be an exceptional case).

The inequality (2.7) implies (2.5)(i).

For \( 1 < \theta < \infty \) we have

\[
\int_{\mathbb{R}^{d-1}} |A^{\alpha}_{\epsilon} u_{\epsilon}|_p^p |h'|^{-n-\theta+1} \, dh' \leq C_\epsilon^p \int_{\mathbb{R}^{d-1}} \|A^{\alpha}_{\epsilon} u\|_p^p |h'|^{-n-\theta} \, dh',
\]

which is (2.3).

The proof of (2.4), (2.5)(ii) is more involved due to the peculiar way the variable \( x_a \) appears in (2.1); we take \( k = 2 \).
We first remark that

$$\int_{\mathbb{R}^n} \mu_{\sigma}^{[-h, h]}(x) dx = 2 \int_{-\infty}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx = 2 \int_{-\infty}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

$$= 2 \int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx = 2 \int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

and (2.13) gives in this case (2.5) (ii). For \( \theta < \infty \) we substitute (2.13) into (2.11), use H"older's inequality, represent the integration with respect to \( x \) in polar coordinates, and to make the change of variables \( u \rightarrow h \) to get

$$\int_{\mathbb{R}^n} \mu_{\sigma}^{[-h, h]}(x) dx = 2 \int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

$$= 2 \int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

Similarly we get for the first term

$$\int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

Note that the last term in (2.9) can be reduced to the same form as the middle term in (2.8) and hence can be treated in the same manner.

It follows that it suffices to estimate the integrals

$$\int_{\mathbb{R}^n} \mu_{\sigma}^{[-h, h]}(x) dx$$

and

$$\int_{\mathbb{R}^n} \mu_{\sigma}^{[-h, h]}(x) dx$$

In (2.11) we use the formula

$$\mu_{\sigma}^{[-h, h]}(x) = \int_{\mathbb{R}^n} e(x') [u(x' - \sigma + h) - u(x' - \sigma - h)] dx'. $$

Using the continuous version of Minkowski's inequality we get

$$\|A_{\sigma} u_{\sigma}(\cdot, x)\|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \|e(x')\|_{L^p} dx' \right)^{1/p}.$$ 

Integration with respect to \( x_1, x_2 \), \( |x_1| \leq h \), gives

$$\|A_{\sigma} u_{\sigma}(\cdot, x)\|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \|e(x')\|_{L^p} dx' \right)^{1/p}.$$ 

We now that the right-hand side in (2.13) can be estimated by

$$O(h) \|e(x')\|_{L^p} \quad \text{with} \quad C = h^{1/p} \int_{\mathbb{R}^n} |e(x')| dx'$$

and (2.13) gives in this case (2.5) (ii). For \( \theta < \infty \) we substitute (2.13) into (2.11), use H"older's inequality, represent the integration with respect to \( x \) in polar coordinates, and to make the change of variables \( u \rightarrow h \) to get

$$\int_{\mathbb{R}^n} \mu_{\sigma}^{[-h, h]}(x) dx = 2 \int_{0}^{\infty} \mu_{T_{\sigma}^h(x)}^{[-h, h]}(x) dx$$

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Similarly we get for the first term

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In (2.11) we use the formula

$$\mu_{\sigma}^{[-h, h]}(x) = \int_{\mathbb{R}^n} e(x') [u(x' - \sigma + h) - u(x' - \sigma - h)] dx'. $$

Using the continuous version of Minkowski’s inequality we get

$$\|A_{\sigma} u_{\sigma}(\cdot, x)\|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \|e(x')\|_{L^p} dx' \right)^{1/p}.$$ 

Integration with respect to \( x_1, x_2 \), \( |x_1| \leq h \), gives

$$\|A_{\sigma} u_{\sigma}(\cdot, x)\|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \|e(x')\|_{L^p} dx' \right)^{1/p}.$$ 

Using the mean-value theorem and the properties of \( e \), we get the estimate

$$\Phi(h, x_1, x') = \|A_{\sigma} [e(x' - \sigma - h' x')] \|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \|e(x')\|_{L^p} dx' \right)^{1/p}.$$ 

where \( C = \|u(0)| + 2\|u(0)| + \|P^2 u(0)\|_{L^p} \) and \( \chi' \) denotes the characteristic function of the ball \( \{x' ; |x'| \leq r \} \).

As above we use the continuous form of Minkowski’s inequality to get

$$\|A_{\sigma} u_{\sigma}(\cdot, x)\|_{L^p} \lesssim \left( \int_{\mathbb{R}^n} \Phi(h, x_1, x') dx' \right)^{1/p}.$$ 

(2.15)
integrating with respect to \( x_n, x_n \gg h \) we get

\[
(2.16) \quad \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)} \leq \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n, x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
\quad \leq \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n, x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
= \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n, x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
= \int_0^\infty \Phi(h, x_n, x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx'.
\]

Using (2.14), we get

\[
(2.17) \quad \int_0^\infty \Phi(h, x_n, x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \leq \mathcal{O} \int_0^\infty h^{\frac{1}{p} - (s+1)p} (x_n')^{-1} \min \{ (1 + (s+1)p)^p (x_n')^{1-s} + (s+1)p \} \, dx \, dx' \\
= \mathcal{O} \int_0^\infty h^{\frac{1}{p} - (s+1)p} (x_n')^{-1} \min \{ (1 + (s+1)p)^p (x_n')^{1-s} + (s+1)p \} \, dx \, dx'.
\]

Substituting in (2.16), we get

\[
(2.18) \quad \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)} \leq \int_0^\infty \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx'.
\]

For \( \theta = \infty \) we get

\[
\|A_{\infty} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)} \leq \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\infty} u_n(x_n', x')|^p \, dx \, dx'.
\]

which yields the desired estimate with \( \gamma = \alpha \).

For \( \theta < \infty \) we substitute (2.18) into (2.12) and use (2.19) (with \( \gamma = \theta \)) to get

\[
(2.20) \quad \int_0^\infty h^{1-\theta(s+1)p} \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)}^p \, dh \leq \int_0^\infty h^{1-\theta(s+1)p} \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)}^p \, dh \\
\cdot \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
\leq \int_0^\infty h^{1-\theta(s+1)p} \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
\cdot \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
= C h^{1-\theta(s+1)p} \int_0^\infty \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx'.
\]

From the definition of \( \Psi' \),

\[
(2.21) \quad \int_0^\infty h^{1-\theta(s+1)p} \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)}^p \, dh \\
\leq \mathcal{O} \int_0^\infty h^{1-\theta(s+1)p} (\|e\|_{\mathcal{L}^p(\mathbb{R}^N)})^{p-1} \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)}^p \, dh \\
\leq \mathcal{O} \int_0^\infty h^{1-\theta(s+1)p} \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx' \\
\leq \mathcal{O} \int_0^\infty h^{1-\theta(s+1)p} \int_{\mathbb{R}^N} \Phi(h, x_n', x_n') |A_{\theta} u_n(x_n', x')|^p \, dx \, dx'.
\]

provided that

\[
(2.22) \quad \alpha < \frac{1}{\theta} (2-1/p).
\]

Substitution of (2.21) into (2.20) gives the desired estimate for (2.12) for a subject to (2.21). The proof is complete for \( p < \infty \).

For \( p = \infty \) we have

\[
(2.23) \quad \|A_{\theta} u_n \|_{\mathcal{L}^\infty(\mathbb{R}^N)} \leq \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx \\
\leq \mathcal{O} \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx \\
\leq \mathcal{O} \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx \\
\leq \mathcal{O} \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx.
\]

which yields immediately the desired estimate for (2.3), (2.5) (i) with \( k = 1 \). From (2.24) we obtain (3.5) (ii) with \( k = 1 \) and \( \gamma = 1 \) \( \times \times |\gamma|^p \, dx' \); also for \( \theta < \infty \)

\[
(2.24) \quad \|A_{\theta} u_n \|_{\mathcal{L}^p(\mathbb{R}^N)} \leq \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx \\
\leq \mathcal{O} \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx \\
\leq \mathcal{O} \int_0^\infty |e(x)| |A_{\theta} u_n| \, dx.
\]

Remarks. (1) For \( \theta = p < \infty \) an alternative estimate for (2.12) can be obtained with \( \alpha < 1 \) instead of (2.22).

(2) The above proof could be simplified if we knew that \( \mathcal{E}_{p, q} \mathcal{E}_{p, q} \), \( p < \infty \), is an interpolation space between \( \mathcal{E}_{p, q} \mathcal{E}_{p, q} \) and \( \mathcal{E}_{p, q} \mathcal{E}_{p, q} \). A result of this nature is contained in [9], but unfortunately it does not include the cases \( \theta = 1 \) and \( \theta = \infty \).
Theorem 3.1 implies that the operator \( (1.9) \) \( u \rightarrow \varphi(a_u)u_y(x) \) is bounded from \( B^*_{p,q}(\mathbb{R}^{n-1}) \) into \( B^*_{q,r}(\mathbb{R}^n) \), \( a > 0 \).

3. Extension operators defined by mollifiers. We consider now the transformation \( u \rightarrow u_y \) defined by (3.1) with \( \varepsilon \) subject to condition:

\[
(3.1) \quad \int e(\varepsilon(x))\,dx' = 1, \quad e(\varepsilon(x')) = 0 \text{ for } |x'| > 1.
\]

If \( e \) is bounded, measurable and satisfies (3.1) then \( \lim u_y(x, x_0) = u(e(x)) \) at every Lebesgue point of \( u \in L^p_{loc}(\mathbb{R}^n) \).

With the above hypotheses on \( e \) we also have

**Proposition 3.1.** (i) If \( u \in L^p_{loc}(\mathbb{R}^n) \), \( 1 < p < \infty \) then \( u_y(x) \) exists at almost every point of \( \mathbb{R}^{n-1} \) and \( u_y(x', 0) = u(x') \) a.e.

(ii) If \( u \in B^*_{p,q}(\mathbb{R}^{n-1}) \) for some \( a > 0 \), \( 1 < p < \infty \), \( 1 < q < \infty \), then \( u_y(x', 0) \) exists a.e. \( B^*_{p,q}(\mathbb{R}^{n-1}) \) and \( u_y(x', 0) = u(x') \) a.e. \( B^*_{p,q}(\mathbb{R}^{n-1}) \).

**Proof.** We have for \( x = (x', 0) \)

\[
(3.2) \quad r^{-n} \int_{|y'| < r} \left| \int_{|y'| < r} u(x + y - u(y')dy' \right| \leq r^{-n} \int_{|y'| < r} \int_{|y'| < r} \left| \int_{|y'| < r} u(x + y + y' - z_x') - u(x + y')dy_x' \right| dy_x' \, dy'.
\]

The second term converges to zero as \( r \to 0 \) for every \( x' \) s.t. \( u(x') = u(x) \), i.e. almost everywhere.

For the first term we have

\[
I_r(x') = r^{-n} \int_{|y'| < r} \left| \int_{|y'| < r} u(x + y + y' - z_x') - u(x + y')dy_x' \right| dy_x' \, dy'.
\]

Consider now the function

\[
(3.3) \quad v_y(y') = \sup_{0 < r < 1} \int_{|x'| < 1} \left| u(x + y + rz_x') - u(y') \right| dx_x' \, dz_x' \, dy_x'.
\]

Since \( v_y(y') \) is a.e. \( y' \) at every Lebesgue point \( y' \) of \( u \),

\( v_y(y') \) is smooth for \( y' \neq 0 \), it follows that \( F(r, x', t') \to 0 \) for \( t' \neq t' \).

Also using (3.4) we get

\[
E(r, x', t') \to 0 \quad \text{as} \quad r \to 0.
\]

Hence we can write for \( 0 < r < \varepsilon \)

\[
I_r(x') = \int_{|x'| < 1} v_y(x' + rz_x') dx_x' \, dy_x'.
\]

\[
\limsup_{r \to 0} I_r(x') = \int_{|x'| < 1} v_y(x' + rz_x') dx_x' \, dy_x'.
\]

for almost every \( x' \). Letting \( r \to 0 \) we get (i).

The proof of (ii) depends on the mean-value theorem for the Bessel kernel (see [2], p. 418) and is similar to the argument used in [3], § 6, Th. I.

\[
(3.4) \quad \int_{|x'| < 1} \vartheta_y(x + ry)dy \leq C \vartheta_y(x), \quad x \neq 0, \quad r \ll 1,
\]

with a constant \( C \) independent of \( r \).

Pick \( 0 < r < \min(a, 1) \), by (1.6) there is \( v \in B^*_{p,q}(\mathbb{R}^{n-1}) \) such that

\[
(3.5) \quad u(x') = \int_{|x'| < 1} \vartheta_y(x + ry)dy \quad \text{a.e.} \quad B^*_{p,q}(\mathbb{R}^{n-1}).
\]

Also \( v \in B^*_{p,q}(\mathbb{R}^{n-1}) \) and \( u \) is defined pointwise by (3.5) outside of the set

\[
A = \{ x' \in \mathbb{R}^{n-1} : \int_{|x'| < 1} \vartheta_y(x + ry)dy = \infty \} \subseteq B^*_{p,q}(\mathbb{R}^{n-1}).
\]

Using (3.4) we get

\[
r^{-n} \int_{|y'| < r} \left| \int_{|y'| < r} u(x + y + y' - z_x') - u(x + y')dy_x' \right| dy_x' \, dy'.
\]

Consider now the function

\[
(3.3) \quad v_y(y') = \sup_{0 < r < 1} \int_{|x'| < 1} \left| u(x + y + rz_x') - u(y') \right| dx_x' \, dz_x' \, dy_x'.
\]

Since \( v_y(y') \) is smooth for \( y' \neq 0 \), it follows that \( E(r, x', t') \to 0 \) for \( t' \neq t' \).

Also using (3.4) we get

\[
E(r, x', t') \to 0 \quad \text{as} \quad r \to 0.
\]

Hence we can write for \( 0 < r < \varepsilon \)

\[
I_r(x') \leq \int_{|x'| < 1} v_y(x' + rz_x') dx_x' \, dy_x'.
\]

\[
\limsup_{r \to 0} I_r(x') \leq \int_{|x'| < 1} v_y(x' + rz_x') dx_x' \, dy_x'.
\]

for almost every \( x' \). By the Lebesgue dominated convergence theorem we get the result.
Remarks. (1) The first part of the proposition suggests that \( u^N_u(x', 0) = u^N_u(x') \) for every Lebesgue point of \( u \). This we were unable to prove; if true the second part (ii) would follow from Th. 1, § 6, [3].

(2) The hypothesis \( u \in \mathcal{L}^{p}_{\text{loc}} \) (or more generally, \( u \in \mathcal{L}^{(p, \log^+ L^1)_{\text{loc}}} \)) seems somewhat artificial; it would be interesting to see if it remains valid with the hypothesis \( u \in \mathcal{L}^{1}_{\text{loc}} \).

As an immediate consequence of Proposition 3.1 we get

Theorem 3.2. If \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) \( a > 0 \), \( 1 \leq p < \infty \), \( 1 \leq \alpha \leq 1 \) and \( \varepsilon \in C_0^\infty (\{ |x'| < 1 \}) \) satisfies (3.1) then \( \bar{u} \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \). In particular, \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) is an extension.

Proof. By Theorem 2.1, \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) and by (1.8) \( u = u^p \) exc. \( \mathcal{H}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) also \( u = u^\varepsilon \) exc. \( \mathcal{H}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) and \( \mathcal{H}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) = u^\varepsilon \) by Proposition 3.1.

The definition of \( u \) implies that

\[
\text{supp } u \subset \{(x', a) \in \mathbb{R}^{n-1}; d(x', \text{supp } u) \leq |a|\}.
\]

Using singular multipliers [3] this permits us to construct for every \( \varepsilon > 0 \) an extension \( E : \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \rightarrow \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^{n-1}) \) with the property that

\[
\text{supp } E u \subset \{(x', a) \in \mathbb{R}^{n-1}; d(x', \text{supp } u) \leq \varepsilon |a|\}
\]

(cf. [4]). The existence of such operators is very useful in defining the spaces \( \mathcal{B}^{p}_{\text{loc}} \) on manifolds with singularities.

4. \( p \)-restrictions. In this section we restrict our attention to the case \( \theta = p \); we use the notation \( \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n) = \mathcal{B}^{p}_{\text{loc}} \). We also assume unless otherwise indicated that \( 1 < p < \infty \). The first part of Proposition 1.1 is not valid for \( a = 1/p \) \((-n-1)\)-dimensional subspaces of \( \mathbb{R}^n \) are exceptional sets for \( \mathcal{B}^{p}_{\text{loc}} \). It is easy to check that for \( 0 < a < 1 \) \( p \)

\[
u_{\varepsilon} = |u|_{\mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n)}^n;
\quad \nu_{\varepsilon} = |u|_{\mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n)}^n
\]

imply that \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n) \). Here we use the notation

\[
\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n; x_n > 0 \};
\quad \mathbb{R}^n_- = \{ x \in \mathbb{R}^n; x_n < 0 \}.
\]

The above statement is not valid if \( a = 1/p \) which suggests that in spite of the use of pointwise restrictions \( u_{\varepsilon} \in \mathcal{M}^{p}_{\text{loc}}(\mathcal{M}^{p}_{\text{loc}}(\mathbb{R}^n)) \). The concepts are made precise by the following definition due to N. Aronszajn who also suggested some of the results below.

\[A \text{ function } v \in \mathcal{L}^{p}(\mathbb{R}^n), p > 1, \text{ has zero } p \text{-restriction to } \mathbb{R}^{n-1}, \text{ if and only if}
\]

\[
\int_{\mathbb{R}^{n-1}} |v(x)|^p dx < \infty \quad \text{or equivalently}
\]

\[
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int \int_{\mathbb{R}^{n-1}} |v(x)|^p dx \, dy \, dz \, dx < \infty.
\]

Two functions \( u, v \in \mathcal{L}^{p}(\mathbb{R}^n) \) have the same \( p \)-restrictions to \( \mathbb{R}^{n-1} \) if

\[
u_{\varepsilon} = |u|_{\mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n)}^n = 0.
\]

The relevance of this notion to the question raised at the beginning of the section is clear from the following proposition.

Proposition 4.1. Suppose that \( u \in \mathcal{L}^{p}(\mathbb{R}^n) \) and \( u_{\varepsilon} \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n) \).

Let

\[
u_{\varepsilon} = u(x', a, a); \quad u_{\varepsilon} = |u(x', a)|^p dy \, dx.
\]

Then \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n) \) if and only if \( u_{\varepsilon} \) and \( v_{\varepsilon} \) have the same \( p \)-restriction to \( \mathbb{R}^{n-1} \).

Proof. If \( u \in \mathcal{B}^{p}_{\text{loc}}(\mathbb{R}^n) \) then

\[
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |v(x, a)|^p dx \, dy \, dz = \infty
\]

Thus we can write

\[
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |v(x, a)|^p dx \, dy \, dz = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |v(x, a)|^p dx \, dy \, dz
\]

The second integral can be estimated by

\[
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |v(x, a)|^p dx \, dy \, dz
\]

which is the second integral in (4.2).
Sufficiency. If \( u \in B_p^{(p,\infty)}(R_n^1) \) then

\[
(4.3) \quad \int \int_{R_n^1 \times R_n^1} |(x' - y')^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H} \leq \| u \|_{p,\infty}^p \| \mathcal{H} \|_{R_n^1}^p.
\]

and

\[
(4.4) \quad \int_0^1 \int_{R_n^1} \int_{R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H} \leq \| u \|_{p,\infty}^p \| \mathcal{H} \|_{R_n^1}^p.
\]

Adding the two integrals in (4.3) we obtain the estimate

\[
\int_0^1 \int_{R_n^1\times R_n^1} |(x' - y')^{-p} - u(x', \pm y_n)|^p dx' dy' \leq \| u \|_{p,\infty}^p \| \mathcal{H} \|_{R_n^1}^p.
\]

Thus it remains to estimate the integral \( \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \) which can be written as in (4.2). The first two integrals on the right-hand side of (4.2) are estimated by means of (4.4); for the third one we can write

\[
\int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \leq 2^{p-1} \int_0^1 \int_{R_n^1\times R_n^1} \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H}.
\]

\[
(4.5) \quad \frac{1}{2^{p-1}} \int_0^1 \int_{R_n^1\times R_n^1} \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H}.
\]

Remarks. (1) The notion of \( p \)-restrictions, or some equivalent concept is essential in establishing compatibility conditions for Hessian potentials on manifolds with singularities (see [8]). Proposition 4.1 is a special case of such compatibility conditions.

(2) If \( p = 1 \) the compatibility condition becomes more complicated due to the fact that differences of order at least 2 appear in the definition of the norm in \( B^1_1 \).

(3) The definition of zero \( p \)-restrictions suggests that one could identify restrictions of functions in \( B_p^{(p,\infty)}(R_n^1) \) to \( R_n^1 \) with elements of the quotient space \( B_p^{(p,\infty)}(R_n^1)/B_p^{(1,\infty)}(R_n^1) \) where \( B_p^{(1,\infty)}(R_n^1) = \{ u \in B_p^{(p,\infty)}(R_n^1) \mid u|_{R_n^1} \} \).

We next describe some relations between pointwise properties of functions and \( p \)-restrictions.

Theorem 4.2. If \( u \in L^p(R_n^1), p \geq 1 \), and \( u|_{R_n^1} \neq 0 \) then \( \| u \|_{p,\infty}^p \) \( \leq \frac{1}{2^{p-1}} \int_0^1 \int_{R_n^1\times R_n^1} \int_0^1 |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H} \). In particular, \( u|_{R_n^1} \neq 0 \) for a.e. \( x' \in R_n^1 \).

Proof. By Fubini's theorem, \( \int_0^1 \int_{R_n^1\times R_n^1} \int_0^1 |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H} \) for a.e. \( x' \in R_n^1 \) and therefore

\[
v_r(x') = \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H} \leq \| u \|_{p,\infty}^p \| \mathcal{H} \|_{R_n^1}^p.
\]

Let \( \epsilon \to 0 \) and using the fact that \( v_r \in L^1 \), we get

\[
\limsup \frac{\epsilon^{1-p} \int_0^1 |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy'}{\epsilon^{1-p} \int_0^1 |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy'} \leq \epsilon^{1-p} \int_0^1 |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy'.
\]

and since \( v_r(x') \to 0 \) a.e., this completes the proof.

The converse to Proposition 4.2 does not seem to be true but we do not have a counterexample. (*)

The next proposition gives some insight into properties of the extension operator \( u \mapsto u \) in the exceptional case.

Proposition 4.3. Let \( e_1, e_2 \) be bounded measurable functions satisfying (3.1) and \( u \in L^{p'}(R_n^1) \). Define \( u_1 = u e_1, u_2 = u e_2, u_3 = u e_1 e_2 \). Then \( u_1, u_2, u_3 \) have the same \( 2 \)-restrictions to \( R_n^1 \).

Proof. We have to show that the integral

\[
I = \frac{1}{2^{p-1}} \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H}
\]

is finite. Using Fourier transform with respect to the variable \( x' \), the definition of \( u_1 \) and Parseval's equality, we get

\[
I = \frac{1}{2^{p-1}} \int_0^1 \int_{R_n^1\times R_n^1} |(x_n - y_n)^{-p} - u(x', \pm y_n)|^p dx' dy' \, d\mathcal{H}.
\]

(*) Added in proof. See [6] for a counterexample for \( p = 2 \).
and to arrive at the desired conclusion it suffices to show that the function

$$
\varphi(\xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| e^{i\xi' \cdot x} - e^{i\xi' \cdot x} \right|^2 dx
$$

is bounded.

Using polar coordinates $\xi' = |\xi'| \omega$, $|\omega| = 1$, $\varphi(\xi')$ can be written in the form

$$(4.5) \quad \varphi(\xi') = \int_{0}^{1} |e^{i\xi' \cdot \xi}| \omega_1^2 d\xi$$

$$= \int_{0}^{1} \left| \xi_1 (\omega_1) - e^{i\xi' \cdot \xi} \right|^2 dt$$

Since $\hat{e} = \hat{e}_1 - \hat{e}_2 e^{i\omega_1 (\mathcal{R}^{n-1})}$ and $\hat{e}(0) = 0$, it follows from the second order Taylor's formula that the first integral is bounded. From boundedness of $\omega$ and compactness of its support it follows that $\mathcal{D}^{k} e \in C(\mathcal{R}^{n-1})$ for all multiindices $\hat{\omega}$; this implies, by the known theorems about restrictions, that $e$ belongs to $\mathcal{D}$ on all lines with a fixed bound for the norm, hence the last integral in (4.5) is bounded uniformly in $\xi'$ and the proof is complete.

It would be interesting to see if the content of Proposition 4.3 remains valid for $p \neq 2$, perhaps with some additional regularity conditions on $e_1, e_2$.

The following result is in certain sense complementary to Proposition 4.3 and Theorem 3.1 in an exceptional case.

**Proposition 4.4.** If $e$ satisfies (3.1), is bounded and measurable then $E$ defined by (3.9) is a bounded transformation of $L^{2} (\mathbb{R}^{n})$ into $L^{12} (\mathbb{R}^{n})$.

**Proof.** Let $v(x) = (2\pi)^{-1} \int_{\mathbb{R}^{n}} v(y) e^{i\xi' \cdot y} dy$, where $\xi' \in C_{0}^{-1} (-1, 1)$, $\varphi = 1$ in a neighborhood of $x_0 = 0$. We have to estimate the integrals

$$(4.6) \quad I_1 = \int_{\mathbb{R}^{n}} \left| v(\xi') \right|^{2} d\xi', \quad I_2 = \int_{\mathbb{R}^{n}} |A_{L_{2}} e^{i\xi' \cdot \xi} |^{2} d\xi d\xi'$$

in terms of $|u|^2$.

The first integral in (4.6) can be written in terms of Fourier transform of $v$ with respect to $\xi'$:

$$I_1 = (2\pi)^{-1} \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} v^{(n)}(x) e^{i\xi' \cdot \xi} dx \right|^{2} \xi' d\xi'$$

To obtain the desired estimate we have to verify that the function

$$\Phi_{1}(\xi') = (1 + |\xi'|^{10}) \int_{-\infty}^{\infty} |v(x) - v_{n}(x)|^{2} dx \leq C (1 + |\xi'|^{10}) \int_{-\infty}^{\infty} |\hat{v}(x)|^{2} dx$$

is bounded. For $|\xi'| = 1$, $|\xi'| \geq 1$

$$\int_{-\infty}^{\infty} |\hat{v}(x_{n})|^{2} dx_{n} \leq \frac{1}{|\xi'|} \int_{-\infty}^{\infty} |\hat{v}(x_{m})|^{2} dx_{m} \leq \frac{C}{|\xi'|}$$

with $C$ independent of $\omega'$ (see the proof of Proposition 4.3). For $|\xi'| \leq 1$ we have $|\hat{v}(x_{n})| \leq 1 + C_{2} |\xi'|$ where

$$C_{2} = \max_{|\xi'| \leq 1} |\hat{v}(\xi' \xi)|$$

and $\Phi_{1}(\xi') \leq 2^{12} (1 + C_{2})$.

The second integral in (4.6) can also be represented in terms of partial Fourier transform:

$$I_2 = (2\pi)^{-1} \int_{\mathbb{R}^{n}} \left| A_{L_{2}} e^{i\xi' \cdot \xi} \right|^{2} d\xi d\xi'$$

and as before we have to check that the function

$$\Phi_{2}(\xi') = \int_{\mathbb{R}^{n}} \left| A_{L_{2}} e^{i\xi' \cdot \xi} \right|^{2} d\xi d\xi'$$

is bounded. The latter is easily recognized as the square of the difference term in the $L^{12}(\mathbb{R}^{n})$ norm of the function $u_{n} \mapsto v(x_{n}) e^{i\xi' \cdot \xi}$. It follows, by the smoothness properties of $v$, that this is estimated by a constant multiple of the corresponding term involving $\varphi$ only. For $|\xi'| = 0$, $|\xi'| = 1$, $|\xi'| = \theta'$

$$\int_{-\infty}^{\infty} \left| A_{L_{2}} e^{i\xi' \cdot \xi} \right|^{2} d\xi d\xi'$$

which is part of the $L^{12}$ norm of the restriction of $e$ to the line through the origin in direction of $\theta'$. As already remarked in the proof of Proposition 4.3, the derivatives of all orders of $e$ are in $\mathcal{D}$, and by known restriction theorems the last integral is bounded independently of $\theta'$. The proof is complete.

**Remarks.** The estimates used in Section 2 do not seem to yield the statement of Proposition 4.4.

Propositions 4.3 and 4.4 allow us to elaborate on the notion of 2-restriction. A function $v \in L^{12}(\mathbb{R}^{n})$ has 2-restriction to $\mathcal{R}^{n-1}$ equal to $v \in L^{12}(\mathcal{R}^{n-1})$ if for some bounded measurable $\epsilon$ satisfying (3.1)

$$(v - u_{\epsilon})|_{\mathcal{R}^{n-1}} = 0$$

By Proposition 4.4 the above definition is independent of the choice of $\epsilon$; if the condition is satisfied for some $\epsilon$ then it also holds for all $\epsilon$. (*)

(*) A more satisfactory definition is given in [3].
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Toeplitz operators related to certain domains in $C^\alpha$

by

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Abstract. Using a method from [9] investigated the Toeplitz operators in strongly pseudoconvex domain $D \subset C^\alpha$, $\alpha > 1$. Among other he proved that Toeplitz operator with continuous symbol $\varphi$ smooth in $D$ is Fredholm if $\partial D$ is smooth and $\varphi$ does not vanish on $\partial D$. On the other hand, electronic $C^\alpha$-algebra generated by Toeplitz operators on odd spheres, module compact operators $[2]$. We shall identify the $C^\alpha$-algebra generated by Toeplitz operators in strongly pseudoconvex domain $D$ modulo the compact operators. We shall also prove some simple properties of Toeplitz operators on the $n$-dimensional torus $T^n$.

1. Let $L^2(H)$ be the algebra of all linear bounded operators on a complex Hilbert space $H$ and let $\mathcal{A}(H)$ be the ideal of all compact operators in $H$.

DEFINITION 1.1. For any bounded set $D$ in $C^n$, denote by $L^2(D)$ the space of functions $f: D \to C$ which are square integrable with respect to the Lebesgue measure $dV$ in $C^n$.

DEFINITION 2.1. Denote by $H^1(D)$ the space of all $f \in L^2(D)$, which are holomorphic in $D$.

We shall denote by $P: L^2(D) \to H^1(D)$ the orthogonal projection onto the subspace $H^1(D)$.

The definition of Toeplitz operator associated with a function $\varphi \in L^\infty(dV)$ (bounded, measurable in $D$) reads as follows:

DEFINITION 3.1. Let $\varphi \in L^\infty(dV)$. The Toeplitz operator $T\varphi: H^1(D) \to H^1(D)$ is defined by

$$T\varphi f = P(\varphi \cdot f).$$

Let $B$ be the closed unit ball in $C^n$ and let $\mu$ be the usual surface measure on $\partial B = S^{n-1}$. Then one can define the Hardy space $H^1(\mu)$ on $\partial B$ as a closed subspace of all functions in $L^2(\mu)$ which are holomorphic in the int $B$ [2].

The definition of a Toeplitz operator on $H^1(\mu)$ is just the same as Definition 3.1. Let $\mathcal{A}$ be a $C^\alpha$-algebra generated by Toeplitz operators $T\varphi$ ($\varphi \in C(\partial B)$) on $H^1(\mu)$. Then it was proved by Coburn in [2] that the $C^\alpha$-algebra $\mathcal{A} = \mathcal{A}(H^1(\mu))$ is isometrically isomorphic with $C(\partial B)$. We shall prove...