STUDIA MATHEMATICA, T. LIV. (1975)

An algebra of finitely additive measures

by

J. S. PYM (Sheffield) and H. L. VASUDEVA (Chandigarh)

Abstract. The algebra of all finitely additive measures on a discrete semigroup which is a totally ordered set with the multiplication max is studied. It is found that all proper maximal left (or right) ideals are two-sided and that the radical is the smallest closed ideal for which the quotient algebra is commutative. This quotient algebra is isomorphic with the measure algebra (in the usual sense) of a compact totally ordered totally disconnected semigroup.

In his paper [3], Olubummo studied the algebra of all finitely additive measures on the additive semigroup of positive integers. He was only partially successful in his aim of determining the maximal ideals of the algebra. The most complete description of the full measure algebra (in the usual sense) on a compact semigroup has been given by Hewitt and Zuckermann [2] and Ross [6] who considered totally ordered semigroups with the multiplication max. We are motivated by the success of these authors to investigate the algebra of finitely additive measures on a semigroup of this kind.

Although the underlying semigroup is commutative, the convolution of finitely additive measures may not be. We are therefore led to seek the form of maximal left and maximal right ideals. These we determine completely, and they turn out to be two-sided (Theorem 2.3). The quotient of the algebra by its radical is the (commutative) algebra of measures on the weakly almost periodic compactification of S—a compact, totally disconnected, totally ordered semigroup (Theorem 2.8). Its structure can be found from [2] or [6], but we indicate how our results can be used to rediscover some of the earlier work. Indeed, every algebra of measures on a semigroup is a quotient of an algebra of finitely additive measures on the semigroup with the discrete topology, and we illustrate in our final section how this fact can be exploited.

1. Preliminaries. Let S be a totally ordered set with a minimal element 0. An *interval* in S is a subset U of S with the property that $x, y \in U$ and $x \leqslant z \leqslant y$ imply $z \in U$. Among the intervals we find in particular the sets $]a, b[= \{x: a < x < b\}$, $[a, b[= \{x: a \leqslant x < b\}$, etc., and the sets $\{x\colon x \geqslant a\}$ which we shall denote by $[a, \infty[$, for $a, b \in S$. A *segment* is an interval containing 0 (so that U is a segment if and only if $x \in U$ and $y \leqslant x$ imply $y \in U$). We make S a semigroup by defining the product xy of x and y in S by

$$xy = \max\{x, y\}.$$

Clearly, 0 becomes an identity. We let $\mathscr{B}(S)$ denote the space of all (complex-valued) bounded functions on S. When it is given the supremum norm, its Banach dual $\mathscr{F}(S)$ can be identified with the space of all bounded finitely additive measures on the discrete set S. It is often convenient to use an integral notation for the value of the functional $\mu \in \mathscr{F}(S)$ at $f \in \mathscr{B}(S)$, and we shall write

$$\mu(f) = \int_S f(x)\,d\mu(x).$$

Let χ_E denote the characteristic function of $E \subseteq S$. The *restriction* μ_E of μ to E is defined by

$$\mu_E(f) = \mu(\chi_E f) = \int_S \chi_E(x) f(x)\,d\mu(x) = \int_E f(x)\,d\mu(x).$$

If $\mu = \mu_E$, we say μ is *carried* by E. If μ is carried by E, ν is carried by F and $x \leqslant y$ whenever $x \in E$ and $y \in F$, we shall express this fact loosely by saying carrier $\mu \leqslant$ carrier ν; of course, carriers are not unique, and the relationship will not hold for all carriers of μ and ν.

Now let both $\mu \in \mathscr{F}(S)$ and $f \in \mathscr{B}(S)$ be non-negative. Then if $x \leqslant y$,

$$\mu_{[0,x]}(f) \leqslant \mu_{[0,y]}(f).$$

For each segment U, the net $(\mu_{[0,x]}(f)\colon x \in U)$, with the ordering induced by that of S, is an increasing net of real numbers; it is bounded above by $\mu(f)$ and so it converges. We write

$$\mu_{U-}(f) = \lim_{x \in U} \mu_{[0,x]}(f). \tag{1}$$

(Notice that if U has a greatest element, $\mu_{U-}(f) = \mu_U(f)$.) It is clear that μ_{U-} extends to a linear functional on $\mathscr{B}(S)$ which is positive and hence continuous (i.e. in $\mathscr{F}(S)$). Moreover, if $x \in U$, $\mu_{U-} - \mu_{[0,x]}$ is positive too and therefore

$$\|\mu_{U-} - \mu_{[0,x]}\| = (\mu_{U-} - \mu_{[0,x]})(1) \to 0$$

since the constant function 1 is in $\mathscr{B}(S)$. Now $\mathscr{F}(S)$ is a vector lattice and so each of its elements is a simple linear combination of four positive elements. Thus we may use formula (1) to define μ_{U-} for any $\mu \in \mathscr{F}(S)$ and moreover the convergence of $\mu_{[0,x]}$ to μ_{U-} is in the norm of $\mathscr{F}(S)$.

Similar arguments can be used to justify the definition

$$\mu_{U+}(f) = \lim_{x \in S \setminus U} \mu_{[0,x[}(f) \tag{2}$$

for $\mu \in \mathscr{F}(S)$ and $f \in \mathscr{B}(S)$ (the limit is, of course, taken for decreasing x). Again, $\mu_{[0,x[} \to \mu_{U+}$ in the norm of $\mathscr{F}(S)$. If $S \setminus U$ has a minimal element, $\mu_{U+} = \mu_U$.

We shall need two other functionals derived from $\mu \in \mathscr{F}(S)$. They are

$$\mu_{\to U} = \mu_U - \mu_{U-} \quad \text{and} \quad \mu_{U\leftarrow} = \mu_{U+} - \mu_U. \tag{3}$$

An element of $\mathscr{F}(S)$ of either of these forms will be called a *limit functional*. Note that for each x, $\mu_{\to U}$ (resp. $\mu_{U\leftarrow}$) is carried by $]x, \infty[\cap U$ (resp. $[0, x[\cap (S \setminus U)$).

The *convolution product* $\mu * \nu$ of μ and ν in $\mathscr{F}(S)$ is defined by

$$\mu * \nu(f) = \int_S \cdot \int_S f(xy)\,d\nu(y) \cdot d\mu(x) \qquad (f \in \mathscr{B}(S)) \tag{4}$$

(see [7]). It is important to recognize that the order of integration cannot be interchanged so that convolution is not commutative (except in the trivial case when S is finite); this can be seen from results in [7], and below. Under convolution $\mathscr{F}(S)$ becomes a Banach algebra.

If U is a segment, then $\chi_U(x)\chi_U(y) = \chi_U(xy)$. It is thus easy to see from (4) that $(\mu*\nu)_U = \mu_U * \nu_U$ and so that $\mu \mapsto \mu_U$ is an algebra homomorphism. Continuity of convolution and of the limit operations in (1) and (2) shows that $\mu \mapsto \mu_{U-}$ and $\mu \mapsto \mu_{U+}$ are also homomorphisms. It is again immediate from (4) that $\mu*\nu(1) = \mu(1)\nu(1)$ and we deduce that each of the maps

$$\mu \mapsto \mu_U(1); \qquad \mu \mapsto \mu_{U-}(1); \qquad \mu \mapsto \mu_{U+}(1) \tag{5}$$

is a complex-valued homomorphism of $\mathscr{F}(S)$.

We can rewrite (4) in a way which is often useful. Since $xy = x$ whenever $y \leqslant x$, we have

$$\mu * \nu(f) = \int_S \Big(f(x)\nu_{[0,x]}(1) + \int_{y>x} f(y)\,d\nu(y)\Big)\,d\mu(x). \tag{6}$$

We shall derive three useful formulae from (6). *Suppose first that* carrier $\nu \leqslant$ carrier μ, say that ν is carried by the segment U and that every x in some carrier V of μ satisfies $x \geqslant y$ for all $y \in U$. If then $x \in V$, and $y > x$, we have $y \in S \setminus U$ so that $\int_{y>x} f\,d\nu = 0$. Moreover, $U \subseteq [0, x]$, whence $\nu_{[0,x]} = \nu$. Thus

$$\mu * \nu(f) = \int_V f(x)\nu_{[0,x]}(1)\,d\mu(x) = \nu(1)\mu(f),$$

i.e. $\mu * \nu = \nu(1)\mu$. Using similar arguments we can prove the rest of

(7) $$\mu * \nu = \nu * \mu = \nu(1)\mu.$$

Next suppose that μ is carried by the segment U and that $\nu = \nu_{\to U}$. Then for each $x \in U$, ν is carried by $]x, \infty[\cap U$. We therefore find

$$\mu * \nu(f) = \int_U \Big(f(x)\nu_{[0,x]}(1) + \int_{y>x} f(y)\,d\nu(y)\Big)\,d\mu(x)$$
$$= \int_U \big(0 + \nu(f)\big)\,d\mu(x) = \mu(1)\nu(f),$$

so that

(8) $$\mu * \nu = \mu(1)\nu.$$

Again if *μ is carried by $S \setminus U$ where U is a segment and $\nu = \nu_{U\leftarrow}$* we can use similar arguments to show that

(9) $$\mu * \nu = \nu(1)\mu.$$

We can see from these formulae that convolution is not commutative. For if in formula (8) we choose also $\mu = \mu_{\to U}$, we have $\mu * \nu = \mu(1)\nu$ while $\nu * \mu = \nu(1)\mu$ and it is easy (using the Hahn-Banach theorem) to infer the existence of functionals μ and ν for which these are distinct.

We need two final formulae. If $\mu = \mu_{U-}$, $\nu = \nu_{\to U}$, then by (7) $\mu_{[0,x]} * \nu = \nu * \mu_{[0,x]} = \mu_{[0,x]}(1)\nu$. We may take limits because of the norm-continuity of convolution to find

(10) $$\mu * \nu = \nu * \mu = \mu(1)\nu.$$

Similarly, if $\mu_{U+} = 0$ (i.e. $\mu = \mu - \mu_{U+}$) and $\nu = \nu_{U\leftarrow}$, then

(11) $$\mu * \nu = \nu * \mu = \nu(1)\mu.$$

2. Ideals. Let U be a segment, and write

$$J_U = \{\mu\colon \mu \text{ is carried by } U \text{ and } \mu(1) = 0\}.$$

2.1. Lemma. *For each segment U, J_U is a two-sided ideal.*

Proof. Let $\mu \in J_U$ and let ν be arbitrary. Then (using formula (7) and the fact that $\mu \to \mu_U$ is a homomorphism)

$$\nu * \mu = \nu_U * \mu + \nu_{S\setminus U} * \mu = \nu_U * \mu_U + \mu(1)\nu_{S\setminus U} = (\nu * \mu)_U$$

which is carried by U. Also from (5), $\nu * \mu(1) = \nu(1)\mu(1) = 0$. This proves that J_U is a left ideal, but it is now clear that it is also a right ideal.

Now let I be any maximal proper left (resp. right) ideal. We put

$$U(I) = \{x\colon \delta_x \notin I\}$$

where δ_x is, as usual, defined by $\delta_x(f) = f(x)$ $(f \in \mathscr{B}(S))$. Then $U(I)$ is a segment, for the identity δ_0 does not belong to I, and if $\delta_x \in I$ and $y \geqslant x$, then $\delta_y = \delta_y * \delta_x \in I$.

2.2. Lemma. *Let I be a maximal proper left (resp. right) ideal. Put $U = U(I)$.*

(i) *If $\mu_{U+} = 0$, then $\mu \in I$.*

(ii) *If $\mu = \mu_{U-}$, then $\mu \in I$ if and only if $\mu(1) = 0$.*

Proof. We give the proofs for left ideals only.

(i) Let $x \notin U$, so that $\delta_x \in I$. Then

$$\mu_{[x,\infty[} = \delta_x(1)\mu_{[x,\infty[} = \mu_{[x,\infty[} * \delta_x \in I$$

using (7). Since I is closed,

$$\mu = \mu - \mu_{U+} = \mu - \lim_{x \in S\setminus U} \mu_{[0,x[} = \lim_{x \in S\setminus U} \mu_{[x,\infty[} \in I.$$

(ii) First let $x \in U$ and suppose that $\nu \in I$ is carried by $[0, x]$. Then

$$\nu(1)\delta_x = \delta_x * \nu \in I$$

by (7). Since $\delta_x \notin I$, $\nu(1) = 0$.

Now consider $I + J_{[0,x]}$. Using Lemma 2.1 we see that it is a left ideal. Moreover, it is proper; for if $\delta_0 = \nu + \lambda$ where $\nu \in I$ and $\lambda \in J_{[0,x]}$ then ν must be carried by $[0, x]$ since both δ_0 and λ are, and then by what we have just proved, $\delta_0(1) = \nu(1) + \lambda(1) = 0 + 0 = 0$, which is false. The maximality of I now implies that $J_{[0,x]} \subseteq I$. Thus, if ν is carried by $[0, x]$ and $\nu(1) = 0$, then $\nu \in I$.

Now let $\mu = \mu_{U-}$. Write $k_x = \mu_{[0,x]}(1)$. As x increases in U, $k_x \to \mu_{U-}(1) = \mu(1) = k$ (say), by (1). The functional $\mu_{[0,x]} - k_x\delta_0$ is carried by $[0, x]$ and vanishes at 1; it is therefore in I. Since I is closed, its limit $\mu - k\delta_0 = \mu_{U-} - k\delta_0$ is also in I. Therefore, $\mu \in I$ if and only if $k = 0$, i.e. $\mu(1) = 0$. This completes the proof.

Lemma 2.2 deals with all except the limit functionals. We treat these in the proof of our first theorem.

2.3 Theorem. *Let U be a segment. Then*

$$\{\mu\colon \mu_{U-}(1) = 0\}, \quad \{\mu\colon \mu_U(1) = 0\}, \quad \{\mu\colon \mu_{U+}(1) = 0\}$$

are maximal two-sided ideals. Each maximal proper left ideal and each maximal proper right ideal is two-sided and has one of the above forms.

Proof. By (5) each of the given sets is a maximal two-sided ideal.

Let I be a maximal proper left ideal and let $U = U(I)$ as in Lemma 2.2. Naturally, we must distinguish three cases.

(i) Suppose for some $\lambda \in I$, $\lambda_{S\setminus U} \notin I$. Since by Lemma 2.2 (i), $\lambda - \lambda_{U+} \in I$, we see that

$$\lambda_{U\leftarrow} = \lambda_{U+} - \lambda_U = -(\lambda - \lambda_{U+}) + (\lambda - \lambda_U) = -(\lambda - \lambda_{U+}) + \lambda_{S\setminus U} \notin I.$$

Now let $\mu \in I$. Since $\mu - \mu_{U+} \in I$, again by 2.2(i), we see that $\mu_{U+} \in I$. Then, using formulas (7) and (9),

$$\mu_{U+}(1)\lambda_{U\leftarrow} = \mu_U(1)\lambda_{U\leftarrow} + \mu_{U\leftarrow}(1)\lambda_{U\leftarrow} = \lambda_{U\leftarrow} * (\mu_U + \mu_{U\leftarrow}) = \lambda_{U\leftarrow} * \mu_{U+} \in I.$$

But this is impossible unless $\mu_{U+}(1) = 0$. Thus $I \subseteq \{\mu: \mu_{U+}(1) = 0\}$, and since I is a maximal left ideal and the other set is a proper ideal, the inclusion must be an equality.

(ii) We now suppose that for all $\lambda \in I$, both $\lambda_{S\setminus U} \in I$ and $\lambda_{\to U} \in I$. Since $\mu = \mu_{U-} + \mu_{\to U} + \mu_{S\setminus U}$, we see that $\mu \in I$ implies $\mu_{U-} \in I$. But now Lemma 2.2(ii) applies to show that when $\mu \in I$, $\mu_{U-}(1) = 0$, and the proof is completed as before.

(iii) The final case is when $\lambda_{S\setminus U} \in I$ for all $\lambda \in I$ but $\nu_{\to U} \notin I$ for some $\nu \in I$. Since $\nu = \nu_{U-} + \nu_{\to U} + \nu_{S\setminus U}$, we see that $\nu_{U-} \notin I$ so that $\nu_{U-}(1) \neq 0$, using Lemma 2.2(ii).

Now let $\mu \in I$ and let $\sigma = \sigma_{\to U}$. As $\mu_{S\setminus U} \in I$, $\mu_U \in I$. By formulae (10) and (8)

$$\mu_{U-}(1)\sigma + \sigma(1)\mu_{\to U} = \sigma * (\mu_{U-} + \mu_{\to U}) = \sigma * \mu_U \in I.$$

Applying this formula in particular with $\mu = \nu$ shows that $\sigma \in I$ if and only if $\sigma(1) = 0$. But it also shows that $\sigma * \mu_U = (\sigma * \mu_U)_{\to U}$, and hence (using (5)) $\sigma(1)\mu_U(1) = \sigma * \mu_U(1) = 0$. Since σ is arbitrary, we may take $\sigma \notin I$, i.e. $\sigma(1) \neq 0$. This shows that $\mu \in I$ implies $\mu_U(1) = 0$. Again, the maximality of I ensures the conclusion.

The situation is not quite symmetrical between left and right ideals, but the proofs are sufficiently similar for us to simply sketch the methods. Let I be a maximal right ideal. First suppose that for all $\lambda \in I$, $\lambda_{S\setminus U} \in I$, but that for some $\nu \in I$, $\nu_{\to U} \notin I$; then an argument similar to that of (i) above shows that $I = \{\mu: \mu_U(1) = 0\}$. Secondly, suppose that for all $\lambda \in I$, both $\lambda_{S\setminus U} \in I$ and $\lambda_{\to U} \in I$; then an argument identical with (ii) above shows that $I = \{\mu: \mu_{U-}(1) = 0\}$. Finally, suppose that for some $\lambda \in I$, $\lambda_{S\setminus U} \notin I$; an argument similar to that of (iii) above shows that $I = \{\mu: \mu_{U+}(1) = 0\}$.

We see from Theorem 2.3 that the radical $\mathscr{R}$ of $\mathscr{F}(S)$, which is defined to be the intersection of the maximal proper left (or right) ideals, is in fact the intersection of kernels of complex homomorphisms. The quotient $\mathscr{F}(S)/\mathscr{R}$ is therefore commutative. Now $\mathscr{F}(S)/\mathscr{R}$ can be viewed as the restriction of $\mathscr{F}(S)$ to the polar $\mathscr{R}^\circ$ of $\mathscr{R}$ in $\mathscr{B}(S)$, and so if $f \in \mathscr{R}^\circ$, $\mu * \nu(f) = \nu * \mu(f)$ for all μ, ν in $\mathscr{F}(S)$.

We write

$$\mathscr{W} = \{f \in \mathscr{B}(S): \text{ for all } \mu, \nu \in \mathscr{F}(S),\ \mu * \nu(f) = \nu * \mu(f)\}.$$

According to Theorem 4.2 of [5] (using also 2.2(iii) of [4]) $f \in \mathscr{W}$ if and only if f is weakly almost periodic on S; this means that $f \in \mathscr{W}$ if and only if there do not exist sequences (x_n), (y_m) in S such that the two iterated limits $\lim_n \lim_m f(x_n y_m)$ and $\lim_m \lim_n f(x_n y_m)$ both exist and are distinct. However, there is a more convenient characterization in this case.

2.4. LEMMA. *$\mathscr{W} = \{f$: for each segment U, $\lim_{x\in U} f(x)$ (x increasing) and $\lim_{x\in S\setminus U} f(x)$ (x decreasing) exist$\}$.*

Proof. First, suppose $\lim_{x\in U} f(x)$ does not exist. Then the limit for either the real or the imaginary part of f does not exist, and so we may assume f to be real. Suppose

$$\liminf_{x\in U} f(x) = b < a = \limsup_{x\in U} f(x).$$

We construct sequences (x_n), (y_n) by induction as follows. If $x_1 < y_1 < \ldots < x_n < y_n$ are already chosen, take $x_{n+1} \in U$ with $x_{n+1} > y_n$ and $f(x_{n+1}) > a - 1/(n+1)$, and $y_{n+1} \in U$ with $y_{n+1} > x_{n+1}$ and $f(y_{n+1}) < b + 1/(n+1)$. It is easy to see then that one of the iterated limits is b and the other is a. Thus $f \notin \mathscr{W}$.

To prove the opposite inclusion, we take a sequence (x_n). Since each sequence contains a monotone subsequence, we may take (x_n) to be monotone, and we choose to consider the case when (x_n) is increasing. Write $U = \bigcup_n [0, x_n]$. Then if there is M such that $y_m \in U$ for $m > M$, we find

$$\lim_n \lim_m f(x_n y_m) = \lim_m \lim_n f(x_n y_m) = \lim_{x\in U} f(x)$$

when the third limit exists. If (y_m) is increasing but $y_m \notin U$ for some m, we interchange the roles of (x_n) and (y_m). If (y_m) is decreasing, we put $V = \{x: x < y_m \text{ for all } m\}$ and find (when $y_m \notin U$ for some m) that each iterated limit has the value $\lim_{x\in S\setminus V} f(x)$.

It is obvious from this lemma that $\mathscr{W}$ is a C^*-subalgebra of $\mathscr{B}(S)$, and so $\mathscr{W}$ is naturally isomorphic with the space $\mathscr{C}(W)$ of all continuous functions on some compact space W. Moreover, W can be identified with the set of non-trivial complex-valued homomorphisms of $\mathscr{W}$.

2.5. LEMMA. *If $h \in W$, there is a segment U such that either $h(f) = \lim_{x\in U} f(x)$ or $h(f) = \lim_{x\in S\setminus U} f(x)$ ($f \in \mathscr{W}$).*

Proof. Begin by observing that if $E \subseteq S$ is a set for which $\chi_E \in \mathscr{W}$, then $\chi_E^2 = \chi_E$, so that $h(\chi_E)$ is either 0 or 1. Also, if $E \subseteq F$, $\chi_E \chi_F = \chi_E$, whence $h(\chi_E) \leqslant h(\chi_F)$. Obviously, if U is a segment, $\chi_U \in \mathscr{W}$. We write $U = \bigcup\{V\colon V$ is a segment and $h(\chi_V) = 0\}$. If $x \in U$, then $h(\chi_{[0,x[}) = 0$.

Suppose $h(\chi_U) = 1$. Since for $x \in U$, we have $U = [0, x[\, \cup([x, \infty[\cap U)$, and h is linear, we see that $h(\chi_{[x,\infty[\cap U}) = 1$. Therefore, for $f \in \mathscr{W}$, $h(f) = h(f\chi_{[x,\infty[\cap U})$. The continuity of h now enables us to prove that $h(f) = \lim_{x \in U} f(x)$. In a similar way if $h(U) = 0$ we find $h(f) = \lim_{x \in S\setminus U} f(x)$.

This lemma enables us to describe the space W. From each segment $U \neq S$ we form two elements of W, U_l corresponding to $\lim_{x\in U} f(x)$ and U_r corresponding to $\lim_{x \in S\setminus U} f(x)$; when $U = S$, only U_l exists. These two are always distinct elements of W. However, it is possible that $U_r = V_l$ for some U, V; this occurs precisely when, for all $f \in \mathscr{W}$, $\lim_{x\in S\setminus U} f(x) = \lim_{x \in V} f(x)$, and so when, for some x, $U = [0, x[$ and $V = [0, x]$. The ordering of S can be transferred naturally to W: we put $U_l \leqslant U_r$ for every U, and if $U \subseteq V$, $U \neq V$ we put $U_r \leqslant V_l$. Then W has a minimal element $\{0\}_l$ and a maximal element S_l; we shall write simply 0 for $\{0\}_l$ and 1 for S_l.

We write $\tilde{f}$ for the function in $\mathscr{C}(W)$ corresponding to $f \in \mathscr{W}$ under the natural isomorphism. If U is a segment, $\chi_U \in \mathscr{W}$, and $\tilde{\chi}_U$ is 1 on $[0, U_l]$ and 0 on $[U_r, 1]$. We conclude that both these intervals are open in W.

We can deduce that every open interval in W is open. Indeed, for any U, we already have that $[0, U_r[\, = [0, U_l]$ and $]U_l, 1] = [U_r, 1]$ are open. If we write $V_x = [0, x]$ then

$$[0, U_l[\, = \bigcup [0, (V_x)_l],$$

where the union is taken over those $x \in U$ different from the maximal element of U, if it has one. Similarly,

$$]U_r, 1] = \bigcup\{[(V_x)_r, 1]\colon x \in S\setminus U,\ x \neq \min(S\setminus U)\}.$$

Thus, these intervals are open. It follows quite easily that W has the interval topology.

We now show that W is totally disconnected. Let $U \subseteq V$ and suppose that U and V correspond to different points of W, so that $U_r \neq V_l$. Then (see above) there is $x \in V\setminus U$ such that $V \neq [0, x]$. Therefore if $T = [0, x]$, $U_l \leqslant U_r \leqslant T_l$ and $T_r \leqslant V_l \leqslant V_r$. Because $[0, T_l]$ and $[T_r, 1]$ are open, the result follows.

The general theory of weakly almost periodic compactifications (e.g. [5]) asserts that W is a compact separately continuous semigroup containing a dense homomorphic image of S. In this case the homomorphism is given by $x \mapsto [0, x]_l$; and separate continuity then shows that multiplication in W is given by the operation max. It is easy to see then that multiplication in W is in fact (jointly) continuous. We sum up in the following statement.

2.6. LEMMA. *The space W is a compact totally disconnected ordered semigroup with multiplication* max *and the order topology.*

Write $\mathscr{M}(W) = \mathscr{C}(W)^*$; it is, of course, the space of (bounded) regular measures on W. The natural map $\mu \to \tilde{\mu}$ of $\mathscr{F}(S)$ onto $\mathscr{M}(W)$ given by

$$\tilde{\mu}(\tilde{f}) = \mu(f) \qquad (f \in \mathscr{W})$$

is an algebra homomorphism [5].

2.7. LEMMA. *The algebra $\mathscr{M}(W)$ is semisimple. The complex homomorphisms are given by the mappings*

$$\tilde{\mu} \mapsto \tilde{\mu}([0, U_l[), \qquad \tilde{\mu} \mapsto \tilde{\mu}([0, U_l]), \qquad \tilde{\mu} \mapsto \tilde{\mu}([0, U_r]),$$

for U a segmen of S.

Proof. The linear span of the functions $\{\tilde{\chi}_U\colon U$ is a segment$\}$ is obviously an algebra containing 1 (viz. $\tilde{\chi}_S$) and separating points (see the proof that W is totally disconnected above). Hence if $\mu \in \mathscr{F}(S)$ and $\tilde{\mu}(\tilde{\chi}_U) = 0$ for all $\tilde{\chi}_U$, $\tilde{\mu} = 0$. But $\tilde{\mu}(\tilde{\chi}_U) = \mu(\chi_U) = \mu_U(1)$, and since $\mu \mapsto \mu_U(1)$ is a homomorphism on F_S, $\tilde{\mu} \mapsto \tilde{\mu}(\tilde{\chi}_U)$ is a homomorphism on $\mathscr{M}(W)$. Thus $\mathscr{M}(W)$ is semisimple.

Every complex homomorphism on $\mathscr{M}(W)$ gives rise to a homomorphism on $\mathscr{F}(S)$ via the homomorphism $\mathscr{F}(S) \to \mathscr{M}(W)$. We have already found one family of these: for each segment U,

$$\tilde{\mu}([0, U_l]) = \tilde{\mu}(\tilde{\chi}_U) = \mu_U(1).$$

If we let $V_x = [0, x]$ for $x \in S$, then as $[0, (V_x)_l]$ is compact in W and $[0, (V_x)_l] \nearrow [0, U_l[$, we find

$$\tilde{\mu}([0, U_l[) = \sup_{x\in U} \tilde{\mu}([0, (V_x)_l]) = \sup_{x\in U} \mu_{[0,x]}(1) = \mu_{U-}(1),$$

so that $\tilde{\mu} \mapsto \tilde{\mu}([0, U_l[)$ is also a homomorphism (see (5)). In a similar way we find that

$$\tilde{\mu}([0, U_r]) = \mu_{U+}(1).$$

Since each homomorphism on $\mathscr{F}(S)$ is of one of these kinds, we see that we have found all the complex homomorphisms of $\mathscr{M}(W)$.

(The result of Lemma 2.7 is also an immediate consequence of Theorem 3.3 of [2].)

2.8. THEOREM. *The quotient of $\mathscr{F}(S)$ by its radical R is isomorphic with $\mathscr{M}(W)$.*

Proof. We saw above (before 2.4) that $R^\circ \subseteq \mathscr{W}$. Since $\mathscr{F}(S)/\mathscr{W}^\circ \cong \mathscr{M}(W)$ is semisimple, we have $R \subseteq \mathscr{W}^\circ$, or as $\mathscr{W}$ is a closed subspace $\mathscr{W} \subseteq R^\circ$. Thus $\mathscr{W} = R^\circ$, and the result follows.

For the further harmonic analysis of $\mathscr{M}(W)$ we refer readers to Hewitt and Zuckermann [2] and Ross [6].

3. Other algebras on S. (i) Let $\mathscr{U}$ be a space of functions on S with $\mathscr{W} \subseteq \mathscr{U} \subseteq \mathscr{B}(S)$ for which $\mathscr{U}^*$ is a convolution algebra. There are then natural quotient homomorphisms $\mathscr{F}(S) \to \mathscr{U}^* \to \mathscr{W}^* \cong \mathscr{M}(W)$. Since $\mathscr{U}^*$ must have an identity, each maximal proper left ideal corresponds to a maximal proper left ideal in $\mathscr{F}(S)$, and hence is two-sided and also corresponds to a maximal ideal in $\mathscr{M}(W)$. The theory for $\mathscr{U}^*$ can therefore be deduced from that for $\mathscr{F}(S)$ and $\mathscr{M}(W)$.

One such algebra is given by the following proposition.

3.1. THEOREM. *Let S be compact in the order topology and let $\mathscr{U}$ be the space of bounded measurable functions. Then $\mathscr{U}^*$ is a convolution algebra and $\mathscr{W} \subseteq \mathscr{U} \subseteq \mathscr{B}(S)$.*

Proof. Since each $f \in \mathscr{W}$ has one-sided limits at each point, $\mathscr{W} \subseteq \mathscr{U}$.

Now let $\nu \in \mathscr{F}(S)$ and let $f \in \mathscr{U}$. To show that $\mathscr{U}^*$ is a convolution algebra we need only prove that $x \to \int_S f(xy)\,d\nu(y)$ is in $\mathscr{U}$ (i.e. is measurable; see [5] esp. (1.4)). By standard arguments, we need consider only the case in which both ν and f are positive. Now (cf. formula (6))

$$\int_S f(xy)\,d\nu(y) = f(x)\nu_{[0,x]}(1) + \int_{y>x} f(y)\,d\nu(y).$$

Since ν is positive, $x \to \nu_{[0,x]}(1)$ is increasing and

$$x \to \int_{y>x} f(y)\,d\nu(y)$$

is decreasing, so that both these functions are monotone. The conclusion follows.

(ii) Let $\mathscr{V}$ be a space of functions on S with $\mathscr{V} \subseteq \mathscr{W}$ for which $\mathscr{V}^*$ is a convolution algebra. Then $\mathscr{V}^*$ is a quotient of $\mathscr{W}^* \cong \mathscr{M}(W)$ and so is abelian. It is not difficult to see that $\mathscr{V}^*$ is semi-simple and that its complex homomorphisms can be derived from Lemma 2.7.

In particular, if S has a compact topology for which its multiplication is separately continuous, we may take $\mathscr{V} = \mathscr{C}(S)$, the space of continuous functions on S. The inclusion $\mathscr{C}(S) \subseteq \mathscr{W} \cong \mathscr{C}(W)$ provides a surjective homomorphism $W \to S$, and it can be seen that the homomorphisms of (2.7) are of the form $\mu \mapsto \mu([0,x[)$ or $\mu \mapsto \mu([0,x])$ for $x \in S$, and $\mu \in \mathscr{C}(S)^*$. Thus we recover the results of [2].

(iii) Suppose that S has no identity. Then we can adjoin one in the usual way: form $S \cup \{0\}$ and define multiplication by $0x = x0 = x$ for all x. Then the order also extends $-0 \leqslant x$ for all x — and multiplication is still max. We may therefore deduce results in this situation from those we already have.

However, it may be of interest to point out that new proper maximal right ideals (naturally not modular in general) arise in this situation. We write

$$\mathscr{F}_0(S) = \{\mu:\ \mu = \mu_{\varnothing\leftarrow}\}.$$

(Where $\mu_{\varnothing\leftarrow} = \mu - \mu_{\varnothing+} = \mu - \lim_{x\in S} \mu_{]x,\infty[}$, the limit being taken for x decreasing. If S has an identity 0, this definition simply yields the multiples of δ_0.) Let h be any continuous linear functional on $\mathscr{F}_0(S)$, and write

$$I_h = \{\mu:\ h(\mu_{\varnothing\leftarrow}) = 0\}.$$

Let $\mu \in I_h$ and let $\nu \in \mathscr{F}(S)$. Then using formulae (9) and (11)

$$\begin{aligned}\mu * \nu &= \mu * \big(\nu_{\varnothing\leftarrow} + (\nu - \nu_{\varnothing\leftarrow})\big)\\ &= \nu_{\varnothing\leftarrow}(1)\,\mu + \mu(1)(\nu - \nu_{\varnothing\leftarrow}),\end{aligned}$$

whence

$$h\big((\mu*\nu)\big)_{\varnothing\leftarrow} = \nu_{\varnothing\leftarrow}(1)\,h(\mu) = 0,$$

and $\mu*\nu \in I_h$. Because I_h has codimension 1, it must be a maximal proper right ideal.

The asymmetry between left and right ideals is clearly apparent here, for there are no corresponding maximal left ideals; the arguments of Theorem 2.3 still apply to cover all cases. The reason is that the measures $\mu_{\to\varnothing}$ do not exist. A parallel construction to the one we have just made using the measures $\mu_{\to S}$ will provide minimal left ideals for which there is no right analogue.

Acknowledgement. H. L. Vasudeva would like to thank The British Council and University Grants Commission of India for enabling him to visit The University of Sheffield while this work was done.

References

[1] E. Hewitt and K. A. Ross, *Abstract harmonic analysis* I, Springer-Verlag, 1963.

[2] E. Hewitt and H. S. Zuckerman, *Structure theory for a class of convolution algebras*, Pacific J. Math. 7 (1957), pp. 913–941.

[3] A. Olubummo, *Finitely additive measures on the non-negative integers*, Math. Scandinavica 24 (1969), pp. 186–194.

[4] J. S. Pym, *The convolution of linear functionals*, Proc. London Math. Soc. 14 (1964), pp. 431–444.

[5] J. S. Pym, *The convolution of functionals on spaces of bounded functions*, ibid., 15 (1965), pp. 84–104.
[6] K. A. Ross, *The structure of certain measure algebras*, Pacific J. Math. 11 (1961), pp. 723–737.
[7] N. J. Young, *Semigroup algebras having regular multiplication*, Studia Math. 47 (1973), pp. 191–196.

UNIVERSITY OF SHEFFIELD, ENGLAND
PANJAB UNIVERSITY, CHANDIGARH, INDIA

Received January 21, 1974 (783)

STUDIA MATHEMATICA, T. LIV. (1975)

Isometrien in metrischen Vektorräumen

von

REINHARD WOBST (Dresden)

Zusammenfassung. Wir untersuchen, ob eine surjektive Isometrie zwischen metrischen Vektorräumen linear sein muß. Insbesondere wird ein Satz von Charzyński (1953) verallgemeinert und wesentlich kürzer bewiesen. Ein Satz von Rolewicz (1968) wird etwas verallgemeinert. Es wird gezeigt, daß jede Isometrie einer lokalkompakten metrischen Gruppe mit endlich vielen Komponenten in sich surjektiv sein muß. Die Gestalten aller surjektiven Isometrien der Räume $B(S)$ und $l(p_n)$ in sich werden bestimmt, und es wird eine spezielle Aussage über isometrische Einbettungen in l_p ($p\in(0,1)$) bewiesen.

§ 1. Einführung. Es seien (E, d); (F, h) reelle metrische Vektorräume (mit translationsinvarianter Metrik). Eine Abbildung $T\colon E\to F$ mit

$$d(x, y) = h(Tx, Ty) \qquad (x, y\in E)$$

heißt *Isometrie* von E in F.

Da für jede Isometrie auch die Abbildung $x\mapsto Tx-To$ eine Isometrie ist, dürfen wir $To = o$ annehmen (mit o bezeichnen wir das Nullelement eines Vektorraumes).

Eine bekannte Fragestellung ist, wann eine Isometrie T mit $To = o$ linear sein muß.

S. Mazur und S. M. Ulam [24] bewiesen, daß für den Fall normierter Räume jede surjektive Isometrie mit $To = o$ linear ist (vgl. S. Banach [4], S. 166). Mit analogen Fragen beschäftigten sich N. Aronszajn [2], J. A. Baker [3], Z. Charzyński [6], M. M. Day [8], S. Rolewicz [31] und A. Vogt [36]. Insbesondere bewies Charzyński folgendes Theorem:

Sind E und F metrische Vektorräume gleicher endlicher Dimension, und ist T eine Isometrie von E auf F mit $To = o$, so ist T linear.

Rolewicz ([30], S. 242) (vgl. Rolewicz [31]) zeigte:

Es seien X und Y zwei reelle lokalbeschränkte Räume mit den F-Normen $\|x\|_X$ bzw. $\|x\|_Y$. Außerdem seien für alle $x\in X$ und $y\in Y$ die Funktionen $\|tx\|_X$ und $\|ty\|_Y$ konkav für positive t.

Dann ist jede Isometrie U, die X auf Y mit $Uo = o$ abbildet, ein linearer Operator.