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Normal extensions of commutative subnormal operators

by

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Abstract. The present paper is concerned with the problem of the existence of commutative normal extensions of a commutative pair of subnormal operators. It is proved that this problem has a positive solution if one of these subnormal operators has the following properties: Its spectrum X has connected complement and it has a normal extension whose spectrum is contained in ∂X.

In what follows, H is a complex Hilbert space with inner product (x, y); $x, y \in H$, and norm $\|x\| = \sqrt{(x, x)}$; $x \in H$. $L(H)$ denotes the algebra of all linear bounded operators (shortly, operators) on H. For $T \in L(H)$, T^* is the adjoint of T. I_H, or shortly, I, denotes the identity operator. $T|_K$ is the restriction of the operator T to the subspace K. We denote by $\sigma(T)$ the spectrum of $T \in L(H)$.

The operator $A \in L(H)$ is called *subnormal* [2] if there is a space $K \supset H$ and a normal operator B on K such that $A = B|_H$. B is called a *normal extension* of A. Normal extension is called *minimal* if $K = K'$ for every space K' which reduces B and $H \subset K' \subset K$. B is a minimal normal extension of A if and only if $K = \bigvee_i B^{*i} H$. It follows that two minimal normal extensions are unitarily equivalent.

The present paper is concerned with the following problem. We are given two commuting subnormal operators. *Do there exist commutative normal extensions of these operators?* A positive answer is known for isometries, as shown by Ito [3]. To begin with we give several propositions.

PROPOSITION 1. *Suppose that the subnormal operator A in $L(H)$ is cyclic, i.e. for some $x \in H$ we have $H = \bigvee_i A^i x$* (x is called a *cyclic vector* for A), *and assume that the operator S commutes with A. Then S is subnormal and if B in $L(K)$ is the minimal normal extension of A, then there is a normal operator N which commutes with B such that $N|_H = S$.*

The above proposition follows from Theorem 1 of [4]. In this theorem it is shown that operators A and B are unitarily equivalent to multiplication operators $\tilde{A}$ and $\tilde{B}$ for some φ on $H^2(\mu)$ and $L^2(\mu)$, respectively, for suitable plane measure, $H^2(\mu)$ being the $L^2(\mu)$ closure of polynomials. It is also proved that S is unitarily equivalent to multiplication operator

$\tilde{S}$ for some $\psi \in H^\infty(\mu)$. The multiplication operator for ψ on $L^2(\mu)$ is a normal extension of $\tilde{S}$, and commutes with $\tilde{B}$.

Note that the following proposition follows from the corollary of Theorem 7 of [1], Lemma 3 of [3], and the Putnam–Fuglede theorem.

PROPOSITION 2. *Let $A \in L(H)$ be a subnormal operator and assume that A commutes with the normal operator $B \in L(H)$. If $N \in L(K)$ is the minimal normal extension of A, then there is the unique normal extension $L \in L(K)$ of B, which commutes with N.*

We now see that our problem of the existence of commutative normal extensions of commutative subnormal operators may be reduced to the following: let the operators A and S be subnormal and let S commute with A. *Does there exist a subnormal extension $\tilde{S}$ of S which commutes with the minimal normal extension of A?* If the above question has a positive solution we get a positive solution of our initial problem.

The theorem below gives a solution in the case of isometries commuting with subnormal operators.

THEOREM 1. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$, $A_\gamma \in L(H)$, be a commutative subnormal semigroup and $\{B_\gamma\}_{\gamma\in\Gamma}$, $B_\gamma \in L(K)$, be its minimal normal extension. If $\{V_\delta\}_{\delta\in\Delta}$, $V_\delta \in L(H)$, is a semigroup of isometries and V_δ commutes with A_γ for every δ and γ, then there is a unique semigroup $\{\tilde{V}_\delta\}_{\delta\in\Delta}$ of isometries which commutes with all B_γ, and is such that $V_\delta \subset \tilde{V}_\delta$ for every δ. If $\{V_\delta\}$ is a commutative semigroup, then $\{\tilde{V}_\delta\}$ is also commutative.*

Proof. By Theorem 1 of [3], the semigroup $\{A_\gamma\}$ is positive definite. We consider V_δ for arbitrary δ. For every finite number of $x_i \in H$ and γ_i we have

$$\sum_{ij}(A_{\gamma_i}V_\delta x_j, A_{\gamma_j}V_\delta x_i) = \sum_{ij}(V_\delta^* V_\delta A_{\gamma_i}x_j, A_{\gamma_j}x_i) = \sum_{ij}(A_{\gamma_i}x_j, A_{\gamma_j}x_i).$$

Now by Lemma 3 of [3], it follows that there is an operator $\tilde{V}_\delta$ such that $\tilde{V}_\sigma$ is an extension of V_δ and $\tilde{V}_\delta$ commutes with all B_γ. We shall show that $\tilde{V}_\delta$ is an isometry. It follows from the minimality of $\{B_\gamma\}$ that $K = \bigvee_{\gamma\in\Gamma} B_\gamma^* H$. For every element $\sum_i B_{\gamma_i}^* x_i$, where x_i are in H, we have

$$\tilde{V}_\delta\Big(\sum_i B_{\gamma_i}^* x_i\Big) = \sum_i B_{\gamma_i}^* \tilde{V}_\delta x_i = \sum_i B_{\gamma_i}^* V_\delta x_i.$$

Consequently, $\tilde{V}_\delta$ is unique and the following equation holds

$$\begin{aligned}\Big\|\tilde{V}_\delta\Big(\sum_i B_{\gamma_i}^* x_i\Big)\Big\|^2 &= \sum_{ij}(B_{\gamma_j}^* V_\delta x_j, B_{\gamma_i}^* V_\delta x_i) = \sum_{ij}(B_{\gamma_i} V_\delta x_j, B_{\gamma_j} V_\delta x_i)\\ &= \sum_{ij}(A_{\gamma_i} V_\delta x_j, A_{\gamma_j} V_\delta x_i) = \sum_{ij}(A_{\gamma_i} x_j, A_{\gamma_j} x_i)\\ &= \sum_{ij}(B_{\gamma_i}x_j, B_{\gamma_j}x_i) = \Big\|\sum_i B_{\gamma_i}^* x_i\Big\|^2,\end{aligned}$$

hence $\tilde{V}_\delta$ is a norm preserving on the dense subspace of K and is, consequently, an isometry. Since this extension is unique, for δ_1 and δ_2 we have that $\tilde{V}_{\delta_1+\delta_2} = \tilde{V}_{\delta_1}\tilde{V}_{\delta_2}$, hence $\{\tilde{V}_\delta\}$ is a semigroup of isometries. Now by Lemma 3 of [3], it follows that if V_{δ_1} commutes with V_{δ_2}, then $\tilde{V}_{\delta_1}$ commutes with $\tilde{V}_{\delta_2}$, which completes the proof.

THEOREM 2. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$, $A_\gamma \in L(H)$ be a subnormal commutative semigroup and let V be an isometry commutative with all A_γ. Then there is a unitary extension U of V and a normal extension $\{N_\gamma\}$ of $\{A_\gamma\}$ such that U commutes with all N_γ.*

Proof. We know by Theorem 1 that if $\{B_\gamma\}$ is the minimal normal extension of $\{A_\gamma\}$, then there exists the isometry $\tilde{V} \supset V$ such that $\tilde{V}$ commutes with all B_γ. For every B_γ there is a normal extension N_γ, commutative with the minimal unitary extension U of $\tilde{V}$. Since this extension N_γ is unique for every γ, it is a normal commutative semigroup.

We require some information on function algebras. If X is a compact subset of the complex plane having connected complement, then, by a well-known theorem of Walsh, the algebra $\mathscr{P}(X)$, the closure of the algebra of polynomials in $C(X)$, is a Dirichlet algebra on the boundary ∂X. Sarason [6] proved that if X has connected complement, then every connected component of $\operatorname{int}X$ is a Gleason part of the algebra $\mathscr{P}(X)$, and every non-trivial Gleason part has this form. If X has connected complement, we shall denote by $\{G_j\}$ the connected components of $\operatorname{int}X$ and by μ_j the representing measure on ∂X of any evaluation functional φ at the point $z \in G_j$. Note that μ_j is carried by ∂G_j (see [6]) and all measures for points in G_j are mutually absolutely continuous.

Now we will prove

THEOREM 3. *Let the operator $T \in L(H)$ be subnormal. Suppose that $X = \sigma(T)$ has connected complement and that there is a normal extension N of T such that $\sigma(N) \subset \partial X$. Then there exist subspaces H_j of H such that $H = H_0 \oplus \bigoplus_{j=1}^{\infty} H_j$, where every H_j reduces T and $T = T_0 \oplus \bigoplus_{j=1}^{\infty} T_j$.*

In addition, the following conditions hold:

1. *T_0 is normal, $\sigma(T_0) \subset \partial X$, and the spectral measure of T_0 and μ_j are mutually singular for every $j > 0$,*

2. *for every $j > 0$ there is a normal extension N_j of T_j such that $\sigma(N_j) \in \partial G_j$ and the spectral measure of N_j is absolutely continuous with respect to μ_j,*

3. *for every $j > 0$, $\sigma(T_j) = \bar{G}_j$.*

Proof. Let $N \in L(K)$ be the minimal normal extension of T such that $\sigma(N) \subset \partial X$, and let E be its spectral measure. Define the following subspaces of K:

$K_0 = \{x \in K: (Ex, x)$ is singular with regard to μ_j for every $j > 0\}$,

$K_j = \{x \in K: (Ex, x)$ is absolutely continuous with respect to $\mu_j\}$

for $j > 0$. It is known (see [6]) that the spaces $K_0, K_1, \ldots$, etc. are mutually orthogonal subspaces reducing N. Let $N_j = N|_{K_j}$. N_j is normal for every j. Let P_j be the projection of K on K_j and P the projection of K on H. Let $H_j = P_j H$. By Lemma 1 of [6], we have that $H = H_0 \oplus \bigoplus_{j=1}^{\infty} H_j$, where H_0 reduces N, and H_j for $j > 0$ is an invariant subspace for N. We shall show that H_j reduces T_j for every $j \geqslant 0$. Since spaces H_j are orthogonal and their orthogonal sum is equal to H, we have to show that H_j is invariant for T. Let $x \in H_j$. Then, by definition of H_j, $x \in K_j$. Consequently, $Nx = N_j x$. Since $x \in H$, we have $Tx = Nx$. It follows that $Tx = N_j x \in K_j$. By the definition of projection P_j, we have that $P_j Tx = Tx$. Since $Tx \in H$, $Tx = P_j Tx \in H_j$. We have shown that if $x \in H_j$, then $Tx \in H_j$, and consequently H_j reduces T for every $j \geqslant 0$.

Now we have that operator T has the form $T = T_0 \oplus \bigoplus_{j=1}^{\infty} T_j$, where $T_0 = T|_{H_0}$ and $T_j = T|_{H_j}$ for $j > 0$. Since T_0 is a restriction of T, and particularly of N, and H_0 reduces N, T_0 is normal. Now we consider the spectral measure of T_0. Spectral measure of N_0 is the restriction of the measure E to the space K_0. Since H_0 is a subspace of K_0 and reduces N, T_0 is a restriction of N to the space H_0. We have

$$(T_0 Px, y) = (N_0 Px, y) = \int \lambda d\big(E(\lambda) P_0 Px, y\big) \quad \text{for every } x, y \in K_0.$$

Since the measure (EPx, y) is singular with respect to μ_j for every $j > 0$ and $x, y \in K_0$, it follows that the spectral measure of T_0 is singular with respect to μ_j for every $j > 0$ and its carrier is contained in ∂X. Since the carrier of the spectral measure is equal to the spectrum of this operator, we have $\sigma(T_0) \subset \partial X$. We have proved that $T = T_0 \oplus \bigoplus_{j=1}^{\infty} T_j$, and hence condition 1.

Now we shall prove conditions 2 and 3. Since the spectral measure of N_j is absolutely continuous with respect to μ_j, its closed carrier, which is equal to the spectrum of N_j, is contained in ∂G_j. Since N_j is normal extension of T_j, condition 2 is proved.

By Theorem 1 of [6], we may conclude that for $j > 0$ the set $\bar{G}_j$ is a spectral set for T_j. It follows that the spectrum of T_j is contained in $\bar{G}_j$. Suppose now that $\sigma(T_{j_0}) \neq \bar{G}_{j_0}$ for some $j_0 > 0$. Let $Y_j = G_j \setminus \sigma(T_j)$. It follows then that there exists $\lambda_0 \in Y_{j_0}$. Since $\sigma(T_0) \subset \partial X$, $\lambda_0 \notin \sigma(T_0)$. Since the sets G_j are disjoint and open, $\lambda_0 \notin \bar{G}_j$ for $j \neq j_0$, and consequently, $\lambda_0 \notin \sigma(T_j)$ for $j \neq j_0$.

Finally, $\lambda_0 \notin \sigma(T_j)$ for $j \geqslant 0$. Let $x \in H$. Then x has the form $x = \sum_{j=0}^{\infty} x_j$, where $x_j \in H_j$. We consider the following vector

$$y = (\lambda_0 I_H - T)x = \bigoplus_{j=0}^{\infty} (\lambda_0 I_{H_j} - T_j)x = \sum_{j=0}^{\infty} (\lambda_0 I_{H_j} - T_j)x_j,$$

and write

$$(\lambda_0 I_{H_j} - T_j)x_j = y_j.$$

Since $\lambda_0 \notin \sigma(T_j)$, the bounded operators $(\lambda_0 I_{H_j} - T_j)^{-1}$ exist and

$$(\lambda_0 I_{H_j} - T_j)^{-1} y_j = x_j.$$

We shall show that the norms of operators $(\lambda_0 I_{H_j} - T_j)^{-1}$ are equi-bounded. The function $f(z) = (\lambda_0 - z)^{-1}$ has its only pole at λ_0. $\bar{G}_j$ is a spectral set for T_j and $\lambda_0 \notin \bar{G}_j$ for $j \neq j_0$. Consequently,

$$\|(\lambda_0 I_{H_j} - T_j)^{-1}\| \leqslant \|f\| = \sup_{z \in \bar{G}_j} |(\lambda_0 - z)^{-1}| \quad \text{for} \quad j \neq j_0.$$

Evidently,

$$\sup_{z \in \bar{G}_j} |(\lambda_0 - z)^{-1}| \leqslant \big(\operatorname{dist}(\lambda_0, \bar{G}_j)\big)^{-1}.$$

Now for every $j \neq j_0$ we have the inequality

$$\|(\lambda_0 I_{H_j} - T_j)^{-1}\| \leqslant \big(\operatorname{dist}(\lambda_0, \bar{G}_j)\big)^{-1}.$$

Since X is compact, we derive that there exists a constant $M_1 < \infty$ such that, for every $j \neq j_0$, $\big(\operatorname{dist}(\lambda_0, \bar{G}_j)\big)^{-1} \leqslant M_1$. Then for $M = \max\big(M_1, \|(\lambda_0 I_{H_{j_0}} - T_{j_0})^{-1}\|\big)$, we have $\|(\lambda_0 I_{H_j} - T_j)^{-1}\| \leqslant M$ for every $j \geqslant 0$, which implies that the operator $\bigoplus_{j=0}^{\infty} (\lambda_0 I_{H_j} - T_j)^{-1}$ exists and is bounded. Hence

$$\bigoplus_{j=0}^{\infty} (I_{H_j} - T_j)^{-1} y = \sum_{j=0}^{\infty} (I_{H_j} - T_j)^{-1} y_j = \sum_{j=0}^{\infty} x_j = x,$$

$$\text{i.e.} \quad \bigoplus_{j=0}^{\infty} (\lambda_0 I_{H_j} - T_j)^{-1} = (\lambda_0 I_H - T)^{-1}.$$

It follows that $\lambda_0 \notin \sigma(T)$, which is in contradiction with the assumption $\sigma(T) = X$. We have proved that for $j > 0$, $\sigma(T_j) = \bar{G}_j$, which completes the proof.

The above theorem implies

THEOREM 4. *Let T be an operator on H. Suppose that $X = \sigma(T)$ has connected complement and assume that there is a normal extension N on K of T such that $\sigma(N) \subset \partial X$. Then*

$$T = T_0 \oplus \bigoplus_{j=1}^{\infty} \varphi(S_j),$$

where

1. T_0 *is normal and* $\sigma(T_0) \subset \partial X$,

2. S_j *are unilateral shifts and* φ_j *are suitable conformal maps of the open unit disc onto* G_j $(j > 0)$.

Before proving the theorem few remarks are in order. The operator T on H is called *X-pure* (see [6]) if X is a spectral set for T, and there is no invariant subspace H' of H such that $H' \neq \{0\}$ and $T|_{H'}$ is normal with its spectrum contained in ∂X. If X is the spectral set for T and has connected complement, then T is the orthogonal sum of a normal operator with spectrum carried by ∂X and an X-pure operator. It follows that without any loss of generality we may suppose that operators T_j in Theorem 3 are X-pure. Let G be one of the non-trivial Gleason parts G_j, for a fixed arbitrary j, and μ the related measure μ_j. Let $H^\infty(\mu)$ denote the weak star closure in $L^\infty(\mu)$ of the algebra $P(\partial X)$. Since $H^\infty(\mu) = H^2(\mu)\cap L^\infty(\mu)$, for every $h \in H^\infty(\mu)$, there is a sequence of polynomials p_n convergent to h in $L^2(\mu)$-norm. This sequence (Prop. 6 of [6]) is almost uniformly convergent on G to an analytic function of G. Let h_G be the corresponding limit function.

Proof of Theorem 4. By Theorem 3, we have the decomposition $T = T_0 \oplus \bigoplus_{j=1}^{\infty} T_j$ with X-pure T_j for $j > 0$. Let N be a normal extension of T_j such that $\sigma(N_j) \subset \partial G_j$ (see Th. 3). Since the spectral measure of operator N_j is absolutely continuous with respect to μ_j, we may define for $h \in H^\infty(\mu_j)$ an operator $h(N_j) = \int h\,dE_j$, where E_j denotes the spectral measure of N_j. The related mapping $\Phi\colon H^\infty(\mu_j) \to L(K_j)$ is an algebra isomorphism of $H^\infty(\mu_j)$ and $H^\infty(N_j) = \{h(N_j)\colon h \in H^\infty(\mu_j)\}$. Since for X-pure operators the spectral measure E_j is mutually absolutely continuous with μ_j (Th. 3 of [6]), the map Φ is a homeomorphism relative to the weak star topology on $H^\infty(\mu_j)$ and weak operator topology on $H^\infty(N_j)$. The operators N_j and $h(N_j)$ have the same invariant subspaces since each is a weak limit of polynomials in the other (Prop. 12 of [6]). In particular, H_j is an invariant subspace for $h(N_j)$. Define $h(T_j) = h(N_j)|_{H_j}$. Thus we have the map

$$\chi\colon H^\infty(\mu_j) \to H^\infty(T_j) \stackrel{\text{df}}{=} \{h(T_j)\colon h \in H^\infty(\mu_j)\} \subset L(H).$$

This map is evidently an algebra isomorphism and a homeomorphism in the weak star topology for $H^\infty(\mu_j)$ and weak operator topology for $H^\infty(T_j)$. If $\varphi = h_G$, we may equivalently denote $\varphi(T_j) = h(T_j)$. Let h_j be an inner function in $H^\infty(\mu_j)$ such that $\psi_j = h_{jG_j}$ is a conformal map of G_j onto the open unit disc. By Proposition 7 of [6], h_j exists for every G_j. Since $|h_j| = 1$ a.e. μ_j on ∂G_j, the operator $h(N_j)$ is unitary. It follows that $S_j = \psi_j(T_j)$ is an isometry because it is a restriction of $H(N_j)$ to an invariant subspace. For every polynomial, we have $p(S_j) = (p\circ\psi_j)(T_j)$. Consequently, by weak continuity, in limit we have $\varphi_j(S_j) = T_j$. It is sufficient to prove that the operators S_j are D-pure (D = unit disc) because a D-pure isometry is a shift. Suppose that there is a subspace H_j' of H_j such that H_j is invariant for S_j and $S_j|_{H_j'}$ is unitary. Since the operators S_j and T_j have the same invariant subspaces, H_j' is invariant for T. Consequently, $T_j|_{H_j'}$ is normal and its spectrum is contained in ∂G_j, which contradicts the property of T_j being X-pure. This completes the proof.

Now we may prove the following.

THEOREM 5. *Let* $A \in L(H)$ *be a subnormal operator, and let* $N \in L(K)$ *be its minimal normal extension. Suppose that* $T \in L(H)$ *is subnormal and commutes with* A. *Assume that* T *has a normal extension* B *such that* $\sigma(B) \subset \partial\sigma(T)$. *If* $X = \sigma(T)$ *has connected complement then there is the subnormal extension* R *of* T *such that* R *commutes with* N.

Proof. By Theorems 3 and 4, we know that

$$H = H_0 \oplus \bigoplus_{j=1}^{\infty} H_j \quad \text{and} \quad T = T_0 \oplus \bigoplus_{j=1}^{\infty} T_j,$$

where $T_j \in L(H_j)$ and

1. T_0 is normal and $\sigma(T_0) \subset \partial X$,

2. T_j has the form $T_j = \varphi_j(S_j)$ for $j > 0$,

3. S_j is a unilateral shift and φ_j is a conformal map of the unit disc onto G_j.

By Theorem 2.1 of [5], the operator $A = A_0 \oplus \bigoplus_{j=1}^{\infty} A_j$, where $A_j \in L(H_j)$ and for every $j \geqslant 0$ $T_jA_j = A_jT_j$. Evidently, every A_j as a restriction of a subnormal operator is subnormal.

Let $K_j = \bigvee_{i\geqslant 0} N^{*i}H_j$. Evidently, the subspaces K_j reduce N and $K = K_0 \oplus \bigoplus_{j=1}^{\infty} K_j$. Write $N_j = N|_{K_j}$. N_j is the minimal normal extension of A_j. It is sufficient to prove that every operator T_j has a subnormal extension R_j which commutes with N_j. We consider operators A_0 and T_0. Since T_0 is normal, by Proposition 2, we have that T_0 has a normal extension which commutes with N_0.

Now we consider operators A_j and T_j, for $j > 0$. Note that the equation $T_j = \varphi_j(S_j)$ is equivalent to $S_j = \psi_j(T_j)$.

Let $\psi_j = h_{jG_j}$ and let p_n^j be the sequence of polynomials which converges to h_j in the weak star topology in $H^\infty(\mu_j)$. For every p_n^j we have $p_n^j(T_j)A_j = A_jp_n^j(T_j)$, which in the limit shows that $S_jA_j = A_jS_j$. Since S_j is an isometry, it follows from Theorem 1 that there exists an isometry

$V_j \in L(K_j)$ such that V_j is the extension of S_j and $V_j N_j = N_j V_j$. The same argument for φ_j and V_j in place of ψ_j and T_j shows that $R_j = \varphi_j(V_j)$ commutes with N_j. Since V_j and $\varphi_j(V_j)$ have the same invariant subspaces, R_j is an extension of T_j. Evidently, R_j is subnormal. Hence for every $j \geqslant 0$ we get a subnormal extension of T_j which commutes with N_j, and our proof is complete.

Proposition 2 and Theorem 5 yield the following

THEOREM 6. *Let $A \in L(H)$ be a subnormal operator and suppose that the operator $T \in L(H)$ commutes with A. Assume that:*

1. *$X = \sigma(T)$ has connected complement.*
2. *There is a normal extension B of T such that $\sigma(B) \subset \partial X$.*

Then there is a normal extension $R \in L(K)$ and a normal extension $N \in L(K)$ of T such that N commutes with R.

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Multipliers on Banach algebras

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Abstract. This paper is concerned with the study and application of (left, right, double) multipliers on Banach algebras. We consider mainly Banach algebras with bounded (left, right) approximate identities and Banach algebras which are dense $*$-subalgebras of dual B^*-algebras. More specifically, in this second group of Banach algebras we are primarily interested in multipliers on modular annihilator A^*-algebras.

Let A be a Banach algebra with a bounded right approximate identity. Let $M_r(A)$ be the algebra of all bounded linear right multipliers on A. It follows that $M_r(A)$ can be embedded into the second conjugate space A^{**} of A, when A^{**} is considered as a Banach algebra with an Arens product. By using this embedding of $M_r(A)$ into A^{**}, we obtain various properties of A, A^{**}, and $M_r(A)$. Similarly, if A has a bounded left approximate identity we can embed the algebra $M_l(A)$ of continuous linear left multipliers on A into A^{**}. We also consider $M_l(A)$ and $M_r(A)$ with respect to their weak operator topologies and study the groups of isometric and onto (left, right, double) multipliers under these topologies.

The last section of the paper is devoted to the study of multipliers on a modular annihilator A^*-algebra A. Here we show how (left, right, double) multipliers on A are related to (left, right, double) multipliers on the completion $\mathfrak{A}$ of A.

Introduction. Let A be a Banach algebra and let $M_l(A)$ (resp. $M_r(A)$) be the algebra of continuous linear left (resp. right) multipliers on A. Let $M(A)$ be the algebra of double multipliers (S, T) on A such that $S \in M_l(A)$ and $T \in M_r(A)$. It was shown by L. Maté [14] that if A has a bounded right approximate identity then $M_r(A)$ can be embedded anti-isomorphically in the second conjugate space A^{**} of A, when A^{**} is considered as a Banach algebra with Arens product $F * G$, $F, G \in A^{**}$. This embedding is given by the map $T \to T^{**}(E)$, where E is the right identity of $(A^{**}, *)$. In § 5 we gather together various results on the algebras of multipliers as well as A and A^{**} coming out of Maté's representation. For example, we show that the canonical image $\pi(A)$ is a right ideal of $(A^{**}, *)$ if and only if every $F \in A^{**}$ is of the form $F = T^{**}(E) + G$, where $T \in M_r(A)$ and $G \in A^{**}$ with the property that $\pi(A) * G = (0)$.

In § 6 we consider the algebras $M_l(A)$ and $M_r(A)$ with respect to their weak operator topologies. Let $\mathscr{S}(M_l(A))$ (resp. $\mathscr{S}(M_r(A))$) be the closed unit ball of $M_l(A)$ (resp. $M_r(A)$). We show that if A has a right