Admissible translates of stable measures

by

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Abstract. We investigate the structure of the set of admissible translates of a stable measure, and we obtain bounds on the size of this set. We then apply this to show that certain stable stochastic processes have no non-trivial admissible translates.

1. Introduction. It is the purpose of this paper to examine the set \( A_\sigma \) of the admissible translates of a stable measure \( \mu \) on a real separable Hilbert space. For the special case of a Gaussian measure the set \( A_\sigma \) can be described, completely, through the characteristic functional of \( \mu \) (see [5], Theorem 4.1). For a general infinitely divisible measure \( v \) Gikhman and Skorokhod ([5], Theorem 6.1) have obtained sufficient conditions for an element of the Hilbert space to be an admissible translate of \( v \). However, the conditions of Theorem 6.1 ([5]) are difficult to verify. Theorem 6.2 ([5]) simplifies the conditions in the case of a stable measure, but unfortunately Theorem 6.2 is false. In contrast to [5] our main goal is to obtain information on the structure of the set \( A_\sigma \) (see Pitcher [13]) and to obtain measure theoretic and algebraic bounds on the size of \( A_\sigma \). For example, we show that (i) \( A_\sigma \) is a cone in \( H \), and (ii) \( A_\sigma \) is a Borel set of \( \mu \)-measure zero.

The organization of the paper is as follows. Section 2 contains the preliminaries and Section 3 contains some general theorems on the structure of \( A_\sigma \). In Section 4 we specialize to stable measures and in Section 5 we restrict our attention to stable measures on a real separable Hilbert space. Section 6 contains some results, which are useful for the applications given in Section 7. We conclude in Section 8 with some questions, and some remarks on these questions.

2. Preliminaries. \( X \) (and \( Y \)) always denote a real, Hausdorff, topological vector space (RHTVS). \( \mathcal{B}(X) \) will denote the \( \sigma \)-algebra generated by the open sets of \( X \), and the sets in \( \mathcal{B}(X) \) will be referred to as \textit{Borel sets}. \( \mu \) will always represent a probability measure on \( \mathcal{B}(X) \) and \( \mathcal{B}(X) \) will denote the \( \mu \)-completion of \( \mathcal{B}(X) \). \( \mathcal{D}(X) \) will denote the set of probability measures on \( \mathcal{B}(X) \), and \( \mathcal{S}(X) \) will denote the weak star topology.

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If $\Phi$ is a measurable mapping of $(X, \mathcal{B}(X))$ into $(Y, \mathcal{B}(Y))$ and $\mu \ll \mathcal{B}(X)$, then $\mu^* \ll \mathcal{B}(Y)$ will be defined by

$$
\rho^*(E) = \mu(\Phi^{-1}(E))
$$

for all $E \in \mathcal{B}(Y)$. If $\Phi(x) = x + x$ (for some $x \in X$), then we write $\mu_\perp$ for $\rho^*$. If $\Phi(x) = -x$ (for some $x \in \mathcal{B}(Y \setminus \{0\})$, then we write $\rho^+$ for $\rho^*$.

**Definition 1.** An element $a \in X$ is said to be an admissible translate (resp., singular translate) of $\mu$ if $\mu_\perp$ is absolutely continuous (resp., singular) with respect to $\mu$ (denoted by $\mu \ll \mu_\perp$ and $\mu_\perp \perp \mu$, respectively).

Throughout $A_\perp$ (resp., $S_\perp$), denote the set of admissible (resp., singular) translates of $\mu$. In the case where $X$ has a non-trivial topological dual $X^*$, the characteristic functional of $\mu$ is the function on $X^*$ given by

$$
\hat{\mu}(a^*) = \int e^{i\langle x, a \rangle} \mu(dx),
$$

for all $a^* \in X^*$, where $\langle x, a \rangle = \langle x, a \rangle$ for $y \in X$ and $a^* \in X^*$. If $X$ is a Hilbert space, we identify $X^*$ with $X$ and $\langle x, a \rangle$ means the inner product of $x$ and $a$. For probability measures $\mu$ and $\nu$ on $X$, $\mu^* \ll \nu$ is defined by

$$
\mu^*(E) = \int \nu(E - x) \mu(dx)
$$

for all $E \in \mathcal{B}(X)$.

When we refer to a set $G$ as being a subgroup of $X$, we, of course, mean that $G$ is a subgroup under addition. Finally, $\alpha$ will denote Lebesgue measure on the real line.

3. In this section we present some general results on $A_\perp$.

Let $G_\perp$ (resp., $G_\perp^*$) be the intersection taken over subgroups $G_\perp$ of $X$ such that $G_\perp \in \mathcal{B}(X)$ and $\mu(G_\perp) > 0$ (resp., $\mu(G) = 1$).

**Definition 2.** A set $G \in X$ is said to be a cone if $x \in G$ and $\lambda > 0$ implies $x \in G$, and $x_1, x_2 \in G$ implies $x_1 + x_2 \in G$.

**Proposition 1.** Let $\mu$ be a regular, tight probability measure on $X$.

(i) If $a \not\in G$, then $\mu_\perp \ll \mu_\perp$.

(ii) If $a \not\in G$ and $\mu_\perp \ll \mu_\perp$, then $\mu_\perp \ll \mu_\perp$.

**Proof.** (i) Let $G$ be a subgroup of $X$ such that $G \in \mathcal{B}(X)$ and $\mu(G) = 1$. Then $x \in G$ implies $G \cap (G - x) = 0$. Hence $\mu_\perp(G) = \mu(G(\perp - x)) = 0$.

(ii) Let $G$ be a subgroup of $X$ such that $G \perp \in \mathcal{B}(X)$ and $\mu(G) > 0$, and let $a \in G$. We are to show $a \not\in G$. Since $G$ is regular and tight, we may choose $G_\perp \in G$ such that $G_\perp \in \mathcal{B}(X)$ and $\mu(G_\perp) > 0$ (see [19], Corollary 1.1). Let $H = \{\lambda x : \lambda \in \mathbb{R}, x \in \mathbb{R}\}$. $H$ is a Borel set in $\mathbb{R}$, since $X$ is an RMTVS. If $\mu(H) = 0$, then there exists an uncountable collection of positive real numbers $\{\lambda_n\}$ such that $H = \lambda_n a$ are pairwise disjoint. Hence $\{a_\perp - \lambda_n a\}$ are pairwise disjoint. However, $\mu_\perp \perp \mu_\perp$ implies $\mu(G_\perp - \lambda_n a) > 0$.

4. In this section we restrict ourselves to stable measures (defined below).

**Definition 3.** A probability measure $\mu$ on $X$ is said to be stable of index $\alpha$ if for any $\lambda > 0$, there exists $\lambda \in X$ such that $\mu(\cdot + \lambda) = (\mu^{\alpha})^{\cdot}$, where $\gamma = \lambda^{\alpha + r}$.

**Proposition 4.** A probability measure $\mu$ on $X$ is symmetric if $\mu(A) = \mu(-A)$ for all $A \in \mathcal{B}(X)$.

**Proposition 5.** $A_\perp$ is a cone.

**Proof.** Fix $0 < \lambda < 1$ and choose $r > 0$ such that $\lambda^{1 + r} = 1$. Suppose that $a \in A_\perp$ and $\mu(E) = 0$. Now there exists $x \in (\lambda, r)$ such that $\mu = \mu^{\alpha}$. Hence

$$
0 = \mu(E) = \mu(E + x) = \int \mu^\alpha(E + x) \mu^\alpha(dx)
$$

$$
= \int \mu(\lambda^{1 + r}(E + x)) \mu(dx).
$$

since $A_\perp$ is a cone. This is impossible. Hence $\mu(E) > 0$ and therefore $H = X$. This in turn implies $a \in A_\perp$.

For the remainder of this section we assume that $X$ is also a complete, separable metric space.

**Proposition 2.** $A_\perp \in \mathcal{B}(X)$.

**Proof.** Since $X$ is a complete separable metric space, there exists a compact metric space $K$, and a continuous injection $f : X \to K$ (see [9], Theorem 2.1, pp. 68–69). Now $f(X)$ is a Borel set of $K$ (see, e.g., Theorem 3.9 [12]). The map $f : X \to f(X)$ defined by $f(x) = (\mu_\perp^*)$ is continuous. By Theorems 2.10 and 3.1, [1], we know that the map $A : \mathcal{B}(X) \to f(X)$ defined by $A(x) = \text{absolutely continuous part of } x$ with respect to $\mu_\perp^*$ is $\mathcal{B}(X)$-measurable. Hence $\{x \in X : A(x) = f(x)\} \in \mathcal{B}(X)$.

We are done since

$$
A_\perp = \{x \in X : A(x) = f(x)\}.
$$

**Proposition 3.** Suppose that $A_\perp$ is a cone in $X$.

Then either $\mu(A_\perp) = 0$ or $A_\perp$ is finite dimensional.

**Proof.** By Proposition 2, $A_\perp$ is a Borel set in $X$.

Let $\nu = \rho \mu$. Then $A_\perp$, which is also a Borel set, contains $A_\perp - A_\perp$.

If $\mu(A_\perp) > 0$, then $\rho(A_\perp) > 0$. If $\gamma = \nu$ restricted to $A_\perp$ by Tjalln [4] (see also Tjalln [18]), $A_\perp$ is finite dimensional.

**Proposition 4.** Let $X$ and $Y$ be Borel and let $\nu$ be a probability measure on $X$.

Assume that $A : X \to Y$ is measurable and linear. Then $\nu(A^{-1}(Y)) = \nu(X)$ and therefore $\nu(A) \leq A^{-1}(Y)$.

Note that if $A$ is an injection then $A^{-1}(A) = A_\perp$.

4. In this section we restrict ourselves to stable measures (defined below).

**Definition 3.** A probability measure $\mu$ on $\mathcal{B}(X)$ is said to be stable of index $\alpha$ for any $\lambda > 0$, there exists $\lambda \in \mathcal{B}(X)$ such that $\mu(\cdot + \lambda) = (\mu^{\alpha})^{\cdot}$, where $\gamma = \lambda^{\alpha + r}$.
Therefore \( \mu(k^{-1}(-E + x) + k^{-1}ra) = 0 \) for \( \mu \)-almost all \( x \). Since \( a \in A_\mu \), we have \( 0 = \mu(k^{-1}(-E + x) + k^{-1}ra - a) = \mu(k^{-1}(E + s - \lambda a) + k^{-1}ra) \) for \( \mu \)-almost all \( x \). This yields

\[
\mu_a(E) = \mu_k(E + s - \lambda a) = \int \mu(k^{-1}(-E + x) + k^{-1}ra - a) \mu(dx) = 0.
\]

We have just shown \( a \in A_\mu \) implies \( \lambda \in A_\mu \) for \( 0 < \lambda < 1 \). Since \( 0 \in A_\mu \) and \( G = \{ 1; \mu \in \mu \} \) is a semigroup in \( B_1 \), \( G \) contains \( [0, \infty) \).

**Corollary 5.1.** If \( \mu \) is a symmetric stable measure on \( B(X) \), then \( A_\mu \) is a linear subspace of \( X \).

**Proof.** Since \( \mu \) is symmetric, \( A_\mu = -A_\mu \).

**Corollary 5.2.** If \( \mu \) is a stable measure on \( B(X) \), which is regular and tight, then \( A_\mu = -A_\mu = \mathbb{G} \).

**Proof.** Apply Proposition 1 (ii).

**Remark.** Let \( M_\mu \) (resp. \( M_\mu' \)) be the intersection of all linear subspaces \( M \) of \( X \) such that \( \mathcal{M}(\mathcal{M}) \neq 0 \) (resp. \( \mu(M) = 1 \)). It has been shown by Dudley and Kanter \cite{3} that \( \mu(M) \neq 0 \) implies \( \mu(M) = 1 \). Hence \( M_\mu = M_\mu' \). Hence in order to prove that for \( \mu \) symmetric we have, for every \( \alpha \times X \), either \( \mu_\alpha \perp \mu \) or \( \mu \sim \mu_\alpha \). It is sufficient to show that \( A_\mu = M_\mu \). (To see the sufficiency apply Proposition 1.) For a Gaussian measure on a real separable Hilbert space \( A_\mu = M_\mu \) (see \cite{19}, Theorem 5) and hence for such measures we have the above-mentioned dichotomy.

**Corollary 5.3.** If \( \mu \) is a stable measure, then either \( \mu(A_\mu - A_\mu) = 0 \) or \( A_\mu \) is finite dimensional.

**Proof.** Apply Proposition 3.

**Remark.** If \( A_\mu - A_\mu \) is finite dimensional and \( \mu(A_\mu - A_\mu) > 0 \), then \( A_\mu - A_\mu = \text{support of } \mu \) (by \cite{3}). Hence either \( \mu(A_\mu - A_\mu) = 0 \) or \( A_\mu - A_\mu \) = support of \( \mu \).

**5.** From this point on \( X \) will be separable Hilbert space.

Let \( \mu \) be a symmetric stable measure of index \( a \) on \( B(X) \). In \cite{9} Kuelbs has shown that there exists a symmetric, finite, positive Borel measure \( \Gamma \) on the unit sphere such that

\[
\tilde{\mu}(y) = \exp \left\{ - \int \frac{1}{\beta} \langle y, \theta \rangle^a \Gamma(d\theta) \right\} \quad (\text{see also } \cite{8}).
\]

We will use the notation \( \mu = [a, \Gamma] \).

At this point we give a counterexample to Theorem 6.2 \cite{3}. Choose a finite positive Borel measure \( \Gamma \) on the unit sphere of an infinite-dimensional (separable) Hilbert space, such that the support of \( \Gamma \) is all of \( S \).

Then, for \( a \in H \),

\[
a = \lim_{E \rightarrow E_a} \frac{1}{|E_a|} \int_{E_a} \left( \int \Gamma(d\theta) \right) \text{ where } E_a = \left\{ \theta \in S : \left\| \frac{a}{\|a\|} - \theta \right\| < \frac{1}{n} \right\}.
\]

Now, Theorem 6.3 \cite{3} implies that \( a \in A_\mu \Rightarrow \mu \cdot A_\mu = H \), contradicting Proposition 3. For another counterexample see \cite{30}.

Now let \( H_\alpha = \{ x \in X : \int \langle y, \theta \rangle^a \Gamma(d\theta) = 0 \} \). \( H_\alpha \) is a closed subspace of \( X \). Let \( H = H_\alpha \). Then it is easy to see that \( \mu(H) = 1 \). Now complete \( H \) with respect to the metric \( \| \cdot \|_{\alpha, r} \) given by

\[
\|y\|_{\alpha, r} = \left\{ \begin{array}{ll}
\|y\|^{\alpha, r}(S) & \text{if } a \geq 1, \\
\frac{1}{\alpha} \left( \frac{\|y\|^{\alpha, r}(S)}{\|y\|^{\alpha, r}(S)} \right)^{1/\alpha} & \text{if } 0 < a < 1.
\end{array} \right.
\]

Note that

\[
\|y\|_{\alpha, r} \leq \left\| \|y\|^{\alpha, r}(S) \right\| \quad \text{if } a \geq 1, \\
\|y\|^{\alpha, r}(S) \quad \text{if } 0 < a < 1,
\]

and hence \( \| \cdot \|_{\alpha, r} \) is continuous. Let \( B(a, \Gamma) \) denote this completed space.

We, therefore, have the continuous injection \( i : H \rightarrow B(a, \Gamma) \). Since \( i \) is one to one and has dense range, and since \( H \) is a Hilbert space, the adjoint map \( i^* : B^*(a, \Gamma) \rightarrow H^* \) is one to one and if \( a \geq 1 \) has dense range. Note, also, that \( i \) is a compact operator, and hence so is \( i^* \).

**Theorem 6.** If \( a \in X \setminus \{ (\alpha, r)^* B^*(a, \Gamma) \} \), then \( \mu_\alpha \perp \mu \).

**Proof.** It is enough to show that \( \mu_\alpha \perp \mu \) for \( a \in H \setminus \{ (\alpha, r)^* B^*(a, \Gamma) \} \), since \( \mu(H) = 1 \). We claim that there exists a sequence \( (b_n)_{n=1}^{\infty} \subseteq H \) such that \( \|\langle b_n, x\rangle\|_{\alpha, r} \rightarrow 0 \) and \( \langle a, b_n \rangle = 1 \) for all \( n \). Suppose not. Therefore for every sequence \( (b_n)_{n=1}^{\infty} \subseteq H \) such that \( \|\langle b_n, x\rangle\|_{\alpha, r} \rightarrow 0 \), we have \( \langle a, b_n \rangle \rightarrow 0 \). Hence if we define \( \delta \) in \( H \) by \( \delta(x) = \langle x, a \rangle \), then the above assumption can be rephrased as saying that \( \delta \) is continuous on \( i(H) \) in the metric \( \| \cdot \|_{\alpha, r} \). Therefore we can extend \( \delta \) to a continuous linear functional on \( B(a, \Gamma) \). But note that \( \langle x, \delta^* \rangle = \langle x, a \rangle = \langle a, x \rangle \) for all \( x \in H \). Hence \( \delta^* = a \) or \( a \in (\alpha, r)^* B^*(a, \Gamma) \), a contradiction.

Now choose \( (b_n) \subseteq H \) such that \( \|\langle b_n, x\rangle\|_{\alpha, r} \rightarrow 0 \) and \( \langle a, b_n \rangle = 1 \) for all \( n \). We have

\[
\int_{\alpha} \mu_{\alpha}(b) \mu(dx) = \mu(b_n) = \exp \left\{ - \|\delta^*\|^{\|b_n\|_{\alpha, r}} \right\} \rightarrow 1
\]

as \( n \rightarrow \infty \), where

\[
a = \left\{ \begin{array}{ll}
1 & \text{if } 0 < a < 1, \\
a & \text{if } 1 \leq a < 2.
\end{array} \right.
\]
Therefore \( \langle \cdot, b_n \rangle \to 0 \) in \( \mu \)-measure and hence some subsequence \((b_{n_k})\) of \((b_n)\) converges to zero for \( \mu \)-almost all \( x \). On the other hand,
\[
\int e^{i \langle \cdot, b_n \rangle} \mu_\alpha(dx) = e^{i \langle \cdot, b_n \rangle} \mu(b_n) = e^{i \mu(b_n)} - e^0 = 0.
\]
Therefore \( \langle \cdot, b_n \rangle \to 1 \) in \( \mu \)-measure, and thus a subsequence of \((b_{n_k})\) converges to one for \( \mu \)-almost all \( x \). Hence \( \mu \perp \mu_\alpha \).

**Corollary 6.1.** \( M' \leq I(B^*(a, I]) \).

**Proof.** Fix \( \alpha \in \mathcal{I} \backslash I(B^*(a, I]) \). For any sequence \((y_n) \subseteq H\), define
\[
M(y_n) = \{ x \in X : \lim \langle x, y_n \rangle = 0 \}.
\]

\( M(y_n) \) is clearly a measurable set which is also a linear subspace. By the proof of Theorem 6 we see that there exists a sequence \((b_n) \subseteq H\) such that \( \langle a, b_n \rangle = 1, \|b_n\|_H \to \infty \) and \( \mu(M(b_n)) = 1 \). However, \( a \notin M(b_n) \) and hence \( a \notin M' \).

In the next proposition we find a sufficient condition for the singularity of the symmetric stable measures. Let \( \mu \equiv \{ a_i, I_i \} \) \((i = 1, 2)\) be given.

**Proposition 7.** If \( (\| a_i \|_{H_i}) \subseteq H \), \( i = 1, 2 \) are not equivalent metrics on \( X \), then \( \mu \perp I \).

**Proof.** If \( (\| a_i \|_{H_i}) \subseteq H \), \( i = 1, 2 \) are not equivalent, there exists (for example) \( a_2 \in X \) such that \( \| a_2 \|_{H_1} \to \infty \) and \( \| a_2 \|_{H_2} = 1 \) for all \( a \). Hence \( \mu_1(a_2) = 1 \).

Thus there exists a subsequence \((a_{n_k})\) such that \( \mu(M(a_{n_k})) = 1 \) where \( M(a_{n_k}) \) is defined as in Corollary 6.1. By Dudley and Kanter's zero-one law [3], \( M(a_{n_k}) \) has \( \mu \)-measure zero or one. If \( \mu(M(a_{n_k})) = 1 \), then \( \mu(b_{n_k}) = 1 \), by the Bounded Convergence Theorem. However, \( \mu_1(b_{n_k}) = \exp(-\| b_{n_k} \|_{H_1}) = 1 \), a contradiction. Hence \( \mu_1(b_{n_k}) = 0 \) and thus \( \mu \perp \mu_1 \).

**6.** In this section we present results which will be useful in applications to stable processes.

**Proposition 8.** Let \( \{ l_\alpha \}_{\alpha} \subseteq \mathbb{R}^a \) be a sequence of independent random variables such that \( E(e^{itl_\alpha}) = e^{-|\alpha|^2} \), for some fixed \( 0 < a < 2 \). Let \( \mu \) be the measure on \( \mathbb{R}^a \) induced by the sequence \( \{ l_\alpha \} \). Then \( \mathbb{A}_2 = \{ x \in \mathbb{R}^a : \sum |a| < \infty \} \).

**Proof.** This follows from Shewp [15] (or LeCam [10]) and the fact that the stable density has finite Fisher information.

In the applications to stable processes we will only need that \( \sum |a| = \infty \) implies \( \mu \perp \mu \). This follows more easily from Dudley [2] (Theorem 2).

**Corollary 8.1.** Let \( \mu = \{ a, I \} \) be given where the support of \( I \) is the orthonormal set \( \{ e_{b} \} \) in \( X \) and \( I(e_b) = \lambda_b \). Then
\[
A_\mu = \left\{ x \in X : \sum_{b=1}^m \frac{|\langle x, e_b \rangle|^2}{\lambda_b^2} < \infty \right\}.
\]

**Proof.** Consider the map \( A : X \to \mathbb{R}^\mathbb{N} \) defined by
\[
A(x) = \left( \frac{|\langle x, e_b \rangle|^2}{\lambda_b^2} \right)_{b=1}^m,
\]
and note that the random variables \( \xi_b \) on \( \mathbb{R}^\mathbb{N} \) given by \( \xi_b(x) = x_b \) satisfy the hypotheses of Proposition 8.

**Remark.** Under the hypotheses of Corollary 8.1 it is easy to see that if \( a \geq 1 \).

\[
\inf \left\{ \left| I(B^*(a, I]) \right| : \mu \right\} = \left\{ x \in X : \sum_{b=1}^m \frac{|\langle x, e_b \rangle|^2}{\lambda_b^2} < \infty \right\}
\]

where \( 1/a + 1/\beta = 1 \). Hence (in this case) \( \inf \left\{ \left| I(B^*(a, I]) \right| : \mu \right\} \neq A_\mu \) unless \( a = 2 \).

**Definition 5.** A stochastic process \( \{ X_t : 0 \leq t \leq 1 \} \) is said to be a stable process of index \( a \) if the finite-dimensional distributions of \( \{ X_t \} \) are all stable (of index \( a \)).

Let \( \{ X_t : 0 \leq t \leq 1 \} \) be the stable process of type \( a \) such that
1. \( \{ X_t \} \) has stationary and independent increments and
2. \( E[e^{it\xi}] = e^{-|\alpha|^2} \).

For the remainder of this paper \( \{ X_t \} \) will always denote such a process.

Let \( D[I] \) be the Skorohod space of real-valued function on the square \( I' = [0, 1] \times [0, 1] \), which has been studied by Straf [17] and Neuhans [11]. Similarly \( D[I] \) be the usual Skorohod space (again, see, e.g., [17]). For a function \( g \in L^a = L^a([0, 1], \mathbb{R}) \) the stochastic integral
\[
\int g(t) dX(t)
\]
has been defined by Schilder [14]. Hence for \( f \in D[I] \) we may define the process \( \{ Y(t) : 0 \leq t \leq 1 \} \) by the formula
\[
Y(t) = \int_0^t f(t, s) dX(s).
\]

It is not hard to see that \( Y \) is a symmetric stable process with sample paths in \( L^1[I] \). We now prove a Fubini-type result.

**Proposition 9.** Let \( \{ X_t : t \in I \} \) be as above. Then if \( f \in D[I] \), we have
\[
\int_0^t f(t, s) dX(s) = \int_0^t \left[ \int_0^t f(t, s) dX(s) \right] dt.
\]

The proof is similar to that of Proposition 8.
Proof. Choose \( \{f_n\} \subseteq D[I^0] \) such that

\[
|f_n(t, s)| = \sum_{k=1}^{N} a_k \lambda(t, s) \lambda(t, s)
\]

and (see Straf [17]) \( f_n \) converges to \( f \) uniformly. Since \( (\ast) \) holds trivially for \( f_n \), we need only show:

(i) \[
\int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt \to \int_0^1 \left[ \int f(t, s) dX(s) \right] dt
\]
in probability and

(ii) \[
\int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt \to \int_0^1 \left[ \int f(t, s) dX(s) \right] dt
\]
in probability.

To show (i) we shall compute the characteristic function of

\[
\int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt
\]
and show that it converges to 1. But

\[
\sum_{j=1}^{N} \int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt = \int_0^1 \left[ \int f(t, s) dX(s) \right] dt
\]

has the characteristic function

\[
\Phi(\mathbf{u}) = \exp \left[ -|\mathbf{u}|^2 \int_0^1 \left[ \int f(t, s) dX(s) \right] dt \right]
\]

Now since \( f_n \) and \( f \) are bounded, we have (by approximating the integrals and taking limits):

\[
\int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt
\]
has the characteristic function

\[
\Psi(\mathbf{u}) = \exp \left[ -|\mathbf{u}|^2 \int_0^1 \left[ \int f(t, s) dX(s) \right] dt \right]
\]

Therefore, since \( f_n \to f \) uniformly,

\[
\int_0^1 \left| \int f_n(t, s) dX(s) \right| \to \int_0^1 \left| \int f(t, s) dX(s) \right| dt
\]

This yields

\[
\int_0^1 \left[ \int f_n(t, s) dX(s) \right] dt \to \int_0^1 \left[ \int f(t, s) dX(s) \right] dt,
\]
in probability. (ii) follows even more simply.

Since the paths of \( Y \) for \( f \in D[I^0] \) are in \( L^1[I^0] \), \( Y \) induces a measure \( \mu \) on \( D[I^0] \) which is symmetric stable of index \( \alpha \). Hence \( \mu = [\alpha, \Gamma] \). We shall now describe \( \Gamma \) in terms of the given \( f \) if \( \alpha < 2 \).

For \( \sigma \in \mathcal{U}(I) \) the characteristic function of

\[
\int_0^1 \left[ \int f(t, s) dX(s) \right] dt = \int_0^1 \left[ \int f(t, s) dX(s) \right] dt
\]
is

\[
\Phi(\mathbf{u}) = \exp \left[ -|\mathbf{u}|^2 \int_0^1 \left[ \int f(t, s) dX(s) \right] dt \right]
\]
(apply Proposition 9).

Now define \( \Phi : I \to \mathcal{S} \) by

\[
\Phi(\mathbf{u}) = \frac{f}{||f||_2}, \quad \text{where} \quad f(\mathbf{u}) = f(\mathbf{u}, s)
\]
(we will also define \( f(\mathbf{u}, s) = f(\mathbf{u}, s) \) and \( ||f||_2 \) is the \( L^2 \)-norm). For \( A \in \mathcal{S}(\mathcal{S}) \), let

\[
\Gamma(\mathbf{u}) = \int_{e^{-i\mathbf{u}}} ||f||_2 d\mathbf{u}
\]
Then

\[
\int_{\mathbf{u}} \langle \mathbf{u}, \mathbf{w} \rangle^2 \Gamma(\mathbf{u}) d\mathbf{w} = \frac{1}{2} \int_0^1 \left[ \int f(t, s) dX(s) \right] \left[ ||f||_2^2 \right] dt d\mathbf{w}
\]

Hence, since the symmetric measure on the sphere is uniquely determined by \( \mu \), we have \( \Gamma = [\Gamma \wedge + \Gamma \wedge +] \).

For the rest of the paper we make the following assumptions:

(i) \( \sigma \in D[I^0] \),

(ii) span \( \{f_1 : \sigma \in I\} \) is dense in \( D[I^0] \)

(iii) span \( \{f^\alpha : \sigma \in I\} \) is dense in \( D[I^0] \).
Consider the map $A: L^2 \to L^2$ defined by

$$(Ax)(s) = \int_0^s x(t) f(t, s) \, dt.$$ 

(i) and (iii) imply that $A$ is an injection. (Note that $A$ is clearly continuous.) By (i) and (ii) we have that the range of $A$ is dense in $L^2$ and hence in $L^p$. Since $\|e\|_{L^p} = \|Ae\|_{L^p}$, we obtain $B(\mu, \Gamma) = L^2[0, 1].$

Now if $a = s^*(b^*)$, then

$$\int_0^1 b^*(s) \left[ \int_0^s x(t) f(t, s) \, dt \right] \, ds = \int_0^1 x(t) \left[ \int_0^1 b^*(s) f(t, s) \, ds \right] \, dt. $$

Hence

$$s^*(b^*) = \int_0^1 b^*(s) f(t, s) \, ds.$$

We record the above remarks as

**Proposition 10.** If $f$ satisfies (i), (ii) and (iii) (above), then $B(\mu, \Gamma) = L^2[I]$ and

$$s^*[B^*(\mu, \Gamma)] = \{x \in L^2[I]: x(t) = \int_0^1 b^*(s) f(t, s) \, ds \text{ for some } b^* \in L^2[I]^*\}.$$

**Corollary 10.1.** If $f$ satisfies (i), (ii) and (iii) and $0 < a < 1$, then $A_a = (0)$ and moreover, $a \neq 0$ implies $\mu_a \nleq_\mu$.

Proof. $(L^2[I]^*) = (0)$ for $0 < a < 1$. Now apply Proposition 6.

**Corollary 10.2.** Let $(X^0)$ and $(X^0)$ be stable processes with indices $\alpha$ and $\beta$ $(\neq a)$, respectively, such that $(X^0)$ and $(X^0)$ have stationary, independent increments. Let

$$Y(t) = \int_0^t f(s) \, dX^0(s), \quad \text{and} \quad Z(t) = \int_0^t f(s) \, dX^0(s),$$

where $f$ and $g$ satisfy (i), (ii) and (iii). Then the measures $\mu$ and $\nu$ induced by $Y$ and $Z$, respectively, are singular.

Proof. Apply Proposition 7.

7. In this section we will show that the set of admissible translates of the measure associated with the process $X_t$ is trivial. Note that

$$X(t) = \int_{\Gamma_0} x(t, s) \, d\Gamma(s), \quad \text{and} \quad f(t, s) = \Gamma_0(x, s).$$

Therefore $x \in s^*[B^*(\mu, \Gamma)]$ if and only if $x(t) = \int_0^t g(s) \, ds$ where $g \in L^2[I]^*$. By Corollary 10.1, $s^*[B^*(\mu, \Gamma)] = (0)$ if $0 < a < 1.$

For $1 \leq a < 2$ we must do a little more work. For $t \in (0, 1)$ and $t_i \in (0, 1)$ define the map $A : D(I) \to \mathbb{R}^\omega$ by

$$A(x) = \lim_{i \to \infty} \frac{x(t_i) - x(t_{i+1})}{t_i - t_{i+1}}.$$ 

By Proposition 8, $A_{\mu} = \{x \in \mathbb{R}^\omega: \sum_{i=1}^\infty a_i^2 < \infty\}$. By Kakutani [7], $A_{\nu} = (B_{\nu})^*$ and therefore, by Proposition 4, $A_{\mu} \subseteq (A_{\nu})^*.$

We now conclude that $x \in A_{\mu}$ implies that

$$\sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})^a} < \infty$$

for all sequences $(t_i)_{i=1}^\infty \subseteq I$, which are strictly decreasing.

If

$$x(t_i) - x(t_{i+1})$$

converges to a non-zero constant, and

$$\sum_{i=1}^\infty \frac{(t_i - t_{i+1})^{\beta}}{(t_i - t_{i+1})^{\alpha}} = \infty \quad \left(\frac{\alpha}{\beta} = 1\right),$$

then

$$\sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})^a} = \infty \quad \sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})^{\alpha}} \leq \sum_{i=1}^\infty \left(\frac{\beta}{\alpha}\right)^2 (t_i - t_{i+1})^{\beta} = \infty.$$ 

This would contradict $x \in A_{\mu}$. However,

$$\frac{x(t_i) - x(t_{i+1})}{t_i - t_{i+1}} = \frac{1}{t_i - t_{i+1}} \int_{t_{i+1}}^{t_i} g(s) \, ds \quad (g \in L^2[I]),$$

If $g \neq 0$, it is not hard to construct a sequence $t_i \in (0, 1)$ such that (*) and (**) hold. Hence $g = 0$ a.e. and $w = 0$.

**Remark.** Let $(Y_t: 0 \leq t \leq 1)$ be a symmetric stable process with independent increments. Suppose also that $(Y_t)$ is stochastically continuous. Then $\mathcal{E}[e^{\alpha X(t)}] = \exp(-\gamma(t) |a|)$, where

$\gamma = \text{Studia Mathematica LIX}.$
(1) $\gamma(0) = 0$.
(2) $\gamma$ is non-decreasing.
(3) $\gamma$ is continuous.

If we let $\delta(t) = \inf \{s : \gamma(s) \geq t\}$, then $Z(t) = Y(\delta(t))$ is equivalent to $X(t)$. Define

$$A : D(I) \to D([0, \delta(1)]) \quad \text{by} \quad A(y)(t) = y(\delta(t)) \quad \text{for} \quad y \in D(I).$$

By Proposition 4, $A^{-1}(A_\gamma) = A_\mu$, where $\mu$ is the measure on $D(I)$ induced by $Y$. By the above results, $A_\gamma = 0$ and hence $A_\mu = 0$.

Note that the non-existence of non-trivial admissible translates of $X_1$ or $Y_1$ also follows from Theorem 7.5 [5].

8. We end this paper with some questions and remarks.

We can always write $I = I_1 + I_\mu$, where $I_1$ sits on finite-dimensional sets, and $I_\mu = \{a, I_1\}$ and $\mu_\mu = \{a, I_\mu\}$. Then $\mu = \mu_1 + \mu_\mu$. If $I_1(F) > 0$ for some finite-dimensional set $F$, then $I_\mu = I_\mu^0 + I_\mu^0$, where $I_\mu^0$ is $I_\mu$ restricted to $F$ and $I_\mu^0 = I_\mu - I_\mu^0$. Hence $\mu_\mu = \mu_1^0 + \mu_\mu^0$ and certainly $A_\mu^0 = 0$. Therefore $A_\mu = 0$. Also, $A_\mu = A_\mu^0 + A_\mu^0$.

**Question 1.** Is $A_\mu = A_\mu^0 + A_\mu^0$?

Note that in the case of $(X(t))$, $I = 0$ and $A_\mu = 0$.

**Question 2.** Is $A_\mu$ always trivial?

Recall that via Theorem 5 [10], if $\sigma \in A_\mu$, then $\mu = \sigma P \times \mu^0 = \nu$, where $P$ is the projection of $X$ onto the one-dimensional subspace generated by $\sigma$, and $Q = I - P$ (I is the identity). Hence the measure on the sphere $\sigma$ associated with $\nu$ has an atom, and the rest of its support is contained in the orthogonal complement of the span of $\sigma$. Assume that $\sigma$ could show that $\mu = \sigma P \times \mu^0 = \nu$, and $\mu_\mu = \mu_\mu^0 \sim (I_\mu)^{n_\mu}$, then since $\sigma \in A_\mu$, $\mu = \sigma P \times \mu^0 = \nu$, we would have $(I_\mu)^{n_\mu} \sim (I_\mu)^{n_\mu}$, which is impossible. Hence we would have $A_\mu = 0$.

It is easy to see that Theorem 6 is directly related to Theorem 1 [2]. In [2] Dudley applies Theorem 1 to obtain a better bound on $A_\mu$ in the case where $I$ sits on an orthonormal set. However, in the proof (Theorem 2) Dudley uses some non-linear functionals. It would be interesting to know if one could prove Theorem 2 using only linear functionals.

**References**


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