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Products of group-valued measures

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Abstract. Limit theorems for group-valued integrals are established, and applied to derive conditions guaranteeing the existence of a product of two group-valued measures. Fubini-type theorems for both sets and functions are given. Also presented is a construction of a Radon product measure of two Radon measures. Applications to the case of vector-valued measures yield results on the ε- and projective tensor products of such measures.

0. Preliminaries. Throughout this paper, $\emptyset$ denotes the empty set, ω the set $\{0, 1, \ldots\}$ and for any sets A, B, $A\setminus B$ the set-theoretic difference. For any family $\mathscr{A}$ of sets let

$$\bigcup \mathscr{A} = \bigcup_{A\in\mathscr{A}} A, \quad \text{and} \quad \bigcap \mathscr{A} = \bigcap_{A\in\mathscr{A}} A.$$

For any set X let $\mathscr{S}X$ denote the family of all non-empty subsets of X. We shall often denote a sequence $(x_n)_{n\in\omega}$ simply by x.

Let X be a commutative group, with addition represented by $+$. For any subsets A, B of X, and $n\in\omega$, let

$$A \pm B = \{x\pm y\colon x\in A, y\in B\},$$
$$nA = A + \ldots + A, \ n \text{ times}.$$

The identity will always be denoted by 0.

For any topological space X and $x\in X$, nbhd$x$ *in* X will be the family of all neighbourhoods of x. Our topologies will be always Hausdorff.

0.1. Definitions. Let X be a commutative, topological group, and S an abstract space. For any X-valued function u on all subsets of S:

$A\subset S$ is *u-measurable* iff $u(B) = u(B\cap A)+u(B\setminus A)$ for all $B\subset S$.

A is *u-null* iff $u(B) = 0$ for all $B\subset A$.

M_u is the family of all u-measurable subsets of S. For any family $\mathscr{A}$ of subsets of S, u is *σ-additive* on $\mathscr{A}$ iff for each countable, disjoint $P\subset\mathscr{A}$ with $\bigcup P\in\mathscr{A}$,

$$u\left(\bigcup P\right) = \sum_{A\in P} u(A),$$

in the sense of unordered summation ([12], p. 120), u is *monotonely convergent* on $\mathscr{A}$ iff for any increasing or decreasing sequence A in $\mathscr{A}$, $\lim_n g(A_n)$ in X always exists (G. Fox, Can. J. Math. 20, M. Sion [13]).

u is an *outer measure* iff M_u is a σ-algebra on which u is σ-additive, and for any $A \subset S$, $V \in \text{nbhd}\, u(A)$ in X, there exists an $A' \in M_u$ containing A such that

$$A \subset a \subset A' \Rightarrow u(a) \in V.$$

When u is an outer measure, u-null sets are u-measurable and any countable union of u-null sets is still u-null. Also true for outer measures is the following, which we state without proof.

0.2. Lemma. *Let X be a commutative topological group, S an abstract space, and u an X-valued outer measure on S. For any decreasing sequence B in M_u with empty intersection, and $U \in \text{nbhd}\, 0$ in X, there exists $n \in \omega$ such that*

$$a \subset B_n \Rightarrow u(a) \in U.$$

1. Limit theorems. Our definition of the integral is essentially that given by M. Sion in [14]. Let S be an abstract space, X, Y, Z topological groups, and let $\cdot : (x, y) \in X \times Y \to x \cdot y \in Z$ be bi-additive.

Let v be a Y-valued outer measure on S. P is a *partition* iff P is a countable disjoint subfamily of M_v. For any $A \subset S$, $\mathscr{P}(A)$ is the family of all partitions covering A, and

$$\mathscr{D}(A) = \{(P, \Delta): P \in \mathscr{P}(A),\ \Delta: Q \in \mathscr{P}(A) \to \Delta(Q), \text{ a finite subset of } Q\}.$$

For any $f: s \in S \to f(s) \in \mathscr{S}X$,

$$f[A] = \bigcup_{s \in A} f(s),$$

and

$$\mathscr{D}(A, f) = \{(P, \Delta, l): (P, \Delta) \in \mathscr{D}(A), \text{ and } l: a \in \Delta(P) \to l_a \in f[a]\},$$

directed by $(P, \Delta, l) \dashv (P', \Delta', l')$ iff P' is *finer* than P (i.e. each $a' \in P'$ is contained in some $a \in P$), and $\Delta(Q) \subset \Delta'(Q)$ for all Q.

1.1. Definition.

$$\int_A f \cdot dv = \lim_{(P,\Delta,l) \in \mathscr{D}(A,f)} \sum_{a \in \Delta(P)} l_a \cdot v(a) \in Z$$

whenever the limit exists.

For any $W \in \text{nbhd}\, 0$ in Z, $\mathscr{D}_W(A, f)$ consists of those (P, Δ, h) in $\mathscr{D}(A, f)$ for which $(P, \Delta, l) \dashv (Q, \Gamma, r)$ implies that

$$\sum_{a \in \Gamma(Q)} r_a \cdot v(a) \in \int_A f \cdot dv + W.$$

In the event that $x \cdot y$ is denoted by $\varphi(x, y)$ we shall write

$$\int_A \varphi(f, dv) \quad \text{for} \quad \int_A f \cdot dv.$$

1.2. Remarks. For fixed A the integral is additive on the family of functions f for which $\int_A f \cdot dv$ is defined ([11], [15]). For fixed f, if $\int_S f \cdot dv$ is defined then $\int_A f \cdot dv$ is defined for all subsets A of S, and

$$A \subset S \to \int_A f \cdot dv \in Z$$

is an outer measure ([15]).

1.3. Definitions. With the above notation:

(1) ([14]) An $\mathscr{S}X$-valued function f on S is *v-partitionable* iff for each $U \in \text{nbhd}\, 0$ in X there exists a partition P such that $S \setminus \bigcup P$ is v-null and for every $a \in P$,

$$f[a] - f[a] \subset U.$$

(2) $A \subset X$ is *bounded* w.r.t. $(v, \cdot)$ iff for each $W \in \text{nbhd}\, 0$ in Z and each partition P there exists a finite $\Delta(P) \subset P$ such that for every finite partition F finer than $P \setminus \Delta(P)$,

$$\sum_{a \in F} A \cdot v(a) \subset W.$$

(3) A family $\mathscr{F}$ of $\mathscr{S}X$-valued functions on S is *uniformly bounded* w.r.t. $(v, \cdot)$ iff for each $W \in \text{nbhd}\, 0$ in Z there exists a partition P covering S and for each partition Q finer than P a finite $\Delta(Q) \subset Q$, such that for every finite partition F finer than $Q \setminus \Delta(Q)$,

$$\sum_{a \in F} f[a] \cdot v(a) \subset W \quad \text{for all } f \in \mathscr{F}.$$

When $\mathscr{F}$ is a singleton $\{f\}$ we say simply that f is *bounded* w.r.t. $(v, \cdot)$.

(4) ([14]) v is *bounded* w.r.t. $\cdot$ iff for each $W \in \text{nbhd}\, 0$ in Z there exists a $U \in \text{nbhd}\, 0$ in X such that for all finite partitions F,

$$\sum_{a \in F} U \cdot v(a) \subset W.$$

1.4. Remark. For any $\mathscr{S}X$-valued function f on S, if $f[S]$ is bounded w.r.t. $(v, \cdot)$ then f is bounded w.r.t. $(v, \cdot)$.

Concerning existence of the integral we have ([11], [15]):

1.5. Theorem. *Let f be an $\mathscr{S}X$-valued function on S. If f is v-partitionable and bounded* w.r.t. $(v, \cdot)$, *v is bounded* w.r.t. $\cdot$, *and Z is complete, then $\int_S f \cdot dv$ exists.*

Proof. One shows directly that the net

$$(P, \Delta, h) \in \mathscr{D}(S, f) \to \sum_{a \in \Delta(P)} h_a \cdot v(a) \in Z$$

is Cauchy, and by the completeness of Z therefore converges to a unique $z \in Z$.

We shall use the following convergence notions.

1.6. DEFINITIONS. For any net $(h_i)_{i \in I}$ of $\mathscr{S}X$-valued functions on S, and an $f\colon S \to \mathscr{S}X$:

1. $h_i \to f$ *quasi-uniformly* w.r.t. $(v, \cdot)$ iff for each $U \in \mathrm{nbhd}\, 0$ in X, and $W \in \mathrm{nbhd}\, 0$ in Z, there exists an $i \in I$ and a partition P such that

(a) for every finite partition F finer than P,

$$\sum_{a \in F} g[a] \cdot v(a) \subset W \quad \text{for all } g \in \{f\} \cup \{h_j\colon i < j\},$$

(b) for all $i < j$,

$$f(s) \subset h_j(s) + U, \quad h_j(s) \subset f(s) + U \quad \text{for all } s \in S \setminus \bigcup P.$$

2. $h_i \to f$ *uniformly* iff for each $U \in \mathrm{nbhd}\, 0$ in X there exists an $i \in I$ such that for all $i < j$ and $s \in S$,

$$f(s) \subset h_j(s) + U, \quad h_j(s) \subset f(s) + U.$$

3. $h_i \to f$ *pointwise* iff for each $s \in S$ and $U \in \mathrm{nbhd}\, 0$ in X there exists an $i \in I$ such that for all $i < j$,

$$f(s) \subset h_j(s) + U, \quad h_j(s) \subset f(s) + U.$$

Remark. The convergence notions above utilize in fact the topology induced on $\mathscr{S}X$ by the uniformity in $\mathscr{S}X \times \mathscr{S}X$ consisting of the sets

$$\{(A, B)\colon A \subset X, B \subset X, A \subset B + U, B \subset A + U\},$$

for all possible $U \in \mathrm{nbhd}\, 0$ in X.

The limit theorems below are of independent interest.

1.7. THEOREMS. *Let h be a sequence of $\mathscr{S}X$-valued functions on S converging pointwise to an $f\colon S \to \mathscr{S}X$.*

1. *The uniform limit of any net of $\mathscr{S}X$-valued, v-partitionable functions on S is again v-partitionable.*

2. (Sion [14].) *If h_n is v-partitionable for each $n \in \omega$ then f is v-partitionable.*

3. (Generalized Egoroff's theorem.) *If h_n is-partitionable for each $n \in \omega$, then for each $U \in \mathrm{nbhd}\, 0$ in X there exists a decreasing sequence B in M_v with v-null intersection and for every $n \in \omega$,*

$$f(s) - h_n(s) \subset U \text{ for all } s \notin B_n.$$

4. *If h is a sequence of v-partitionable functions, uniformly bounded* w.r.t. $(v, \cdot)$, *and v is bounded* w.r.t. $\cdot$, *then $h_n \to f$ quasi-uniformly* w.r.t. $(v, \cdot)$. (*Note: The boundedness condition on v can be dropped if continuity of $\cdot$ is assumed; see proof* 1.7.4.)

1.8. THEOREMS. *Assume now that v is bounded* w.r.t. $\cdot$, *and that Z is complete.*

1. *Let $(h_i)_{i \in I}$ be a net of v-partitionable $\mathscr{S}X$-valued functions on S converging to $f\colon S \to \mathscr{S}X$ quasi-uniformly* w.r.t. $(v, \cdot)$. *If the integrals $\int_S h_i \cdot dv$ all exist, then $\int_S f \cdot dv$ exists, and for all $A \subset S$,*

$$\int_A f \cdot dv = \lim_{i \in I} \int_A h_i \cdot dv.$$

2. *Let $(h_n)_{n \in \omega}$ be a sequence of $\mathscr{S}X$-valued, v-partitionable functions on S, converging pointwise to $f\colon S \to \mathscr{S}X$. If $\{h_n\colon n \in \omega\}$ is uniformly bounded* w.r.t. $(v, \cdot)$, *then the integrals $\int_S f \cdot dv$, $\int_S h_n \cdot dv$, $n \in \omega$, all exist, and for every $A \subset S$,*

$$\int_A f \cdot dv = \lim_{n \in \omega} \int_A h_n \cdot dv.$$

Remark. In the above theorems we may replace pointwise convergence by 'a.e.' pointwise convergence, i.e. pointwise convergence outside some v-null subset of S.

Proof of 1.7.1. Follows directly from the definitions.

Proof of 1.7.2. See M. Sion [14].

Proof of 1.7.3. Choose a symmetric $U' \in \mathrm{nbhd}\, 0$ in X such that $4U' \subset U$. For each $n \in \omega$, let

$$d_n = \{s \in S\colon f(s) - h_n(s) \not\subset 2U'\}.$$

By Theorem 1.7.2 the function f must be v-partitionable. Hence, for each $n \in \omega$ there exists a partition P_n such that $S \setminus \bigcup P_n$ is v-null, and

(1) $\qquad$ for all $a \in P_n$, $\quad h_n[a] - h_n[a] \subset U'$, $\quad f[a] - f[a] \subset U'$.

Let

$$B_n = S \setminus \bigcap_{k \geqslant n} \big(\bigcup \{a \in P_k\colon a \not\subset d_k\}\big).$$

Then B is clearly a decreasing sequence in M_v. Further, for any $s \notin \big(\bigcup_n (S \setminus \bigcup P_n)\big)$ there exists a $k \in \omega$ such that for all $j > k$,

(2) $$f(s) \subset h_j(s) + U'.$$

Then, for each $x \in f(s)$ and $y \in h_j(s)$, since $s \in a \in P_j$ for some a, we deduce from (1) and (2) that

$$x - y = x - y' + y' - y \in 2U',$$

for some $y'\in h_j(s)$. Thus $f(s)-h_j(s)\subset 2U'$ for all $j>k$. Hence

$$\bigcap_n B_n \subset \bigcup_n (S\setminus \bigcup P_n).$$

Since $\bigcup_n (S\setminus\bigcup P_n)$ is v-null then so also is $\bigcap_n B_n$ (cf. §0).

Finally, if $s\notin B_n$ then there exists $\alpha\in P_n$ with $s\in\alpha$ and $\alpha\not\subset d_n$. In which case, for any $s'\in\alpha\setminus d_n$,

$$f(s)-h_n(s)\subset f(s)-f(s')+f(s')-h_n(s')+h_n(s')-h_n(s)\subset 4U'\subset U.$$

Proof of 1.7.4. Let $U, W\in \mathrm{nbhd}\,0$ in X, Z, respectively. Choose $W'\in\mathrm{nbhd}\,0$ in Z with $2W'\subset W$. Let B be a decreasing sequence in M_v chosen w.r.t. U as in Theorem 1.7.3. Then $\{B_n\setminus B_{n+1}\colon n\in\omega\}$ is a partition. Since h is uniformly bounded w.r.t. $(v,\cdot)$, there exists a $k\in\omega$ and a refinement P of $\{B_n\setminus B_{n+1}\colon n>k\}$ such that for all finite partitions F finer than P,

$$\sum_{\alpha\in F} h_j[\alpha]\cdot v(\alpha)\subset W' \quad \text{for all } j\in\omega.$$

If v is bounded w.r.t. $\cdot$, or $\cdot$ is continuous, then also

$$\sum_{\alpha\in F} f[\alpha]\cdot v(\alpha)\subset 2W'.$$

P clearly satisfies the hypotheses of Definition 1.6.1.

Proof of 1.8.1. Let $W\in\mathrm{nbhd}\,0$ in Z. Choose a closed symmetric W_1 from nbhd 0 in Z such that $6W_1\subset W$. Since v is bounded w.r.t. $\cdot$, there exists $U\in\mathrm{nbhd}\,0$ in X such that for every finite partition F

$$\sum_{\alpha\in F} U\cdot v(\alpha)\subset W_1.$$

Since $h_i\to f$ quasi-uniformly w.r.t. $(v,\cdot)$, there exists a partition P_0 and an $i_0\in I$ such that for every finite partition F finer than P_0,

$$\sum_{\alpha\in F} g[\alpha]\cdot v(\alpha)\subset W \quad \text{for all } g\in\{f\}\cup\{h_j\colon i_0<j\}$$

and for every $i_0<j$

$$f(s)\subset h_j(s)+U \quad \text{for all } s\in S\setminus\bigcup P_0.$$

Let $A\subset S$. By the hypotheses and Remarks 1.2, $\int_A h_j\cdot dv$ exists for all j. Given any $i_0<j$, choose $(P, \Delta), l, l'$, such that P is finer than $\{S\setminus\bigcup P_0\}\cup \cup P_0$, for each $\alpha\in P$ either α is v-null or $h_j[\alpha]-h_j[\alpha]\subset U$, $(P,\Delta,l)\in\mathscr{D}(A,f)$, and $(P,\Delta,l')\in\mathscr{D}_{W_1}(A,h_j)$. Then for each $\alpha\in P$ with $\alpha\subset S\setminus\bigcup P_0$,

(1) either α is v-null or $f[\alpha]-h_j[\alpha]\subset 2U$.

Thus for any $(Q,\Gamma,r)\in\mathscr{D}(A,f)$ such that $(P,\Delta,l)\dashv(Q,\Gamma,r)$, and any $r'\colon \alpha\in\Gamma(Q)\to r'_\alpha\in h_j[\alpha]$,

$$\sum_{\alpha\in\Gamma(Q)} r_\alpha\cdot v(\alpha)-\sum_{\alpha\in\Gamma(Q)} r'_\alpha\cdot v(\alpha)\in \sum_{\alpha\in\Gamma(Q),\alpha\subset\bigcup P_0} f[\alpha]\cdot v(\alpha)-$$
$$-\sum_{\alpha\in\Gamma(Q),\alpha\subset\bigcup P_0} h_j[\alpha]\cdot v(\alpha)+\sum_{\alpha\in\Gamma(Q),\alpha\subset S\setminus\bigcup P_0}(f[\alpha]-h_j[\alpha])\cdot v(\alpha)$$
$$\subset 2W_1-W_1+2W_1=5W_1.$$

Since $(P,\Delta,l')\dashv(Q,\Gamma,r')$ and $(P,\Delta,l')\in\mathscr{D}_{W_1}(A,h_j)$, we have

(2) $$\sum_{\alpha\in\Gamma(Q)} r_\alpha\cdot v(\alpha)-\int_A h_j\cdot dv\in 6W_1\subset W.$$

Considering (2) for fixed $j=i_0$, we see that

$$\Big(\sum_{\alpha\in\Gamma(Q)} r_\alpha\cdot v(\alpha)\Big)_{(Q,\Gamma,r)\in\mathscr{D}(A,f)}$$

is a Cauchy net in Z, and therefore converges to $\int_A f\cdot dv$ since Z is complete. Again using (2) it then follows that

$$\int_A h_j\cdot dv\to\int_A f\cdot dv.$$

Proof of 1.8.2. By Theorems 1.5, 1.7.2, 1.7.4 and 1.8.1.

2. Product measures. Throughout this section let X, Y, Z be commutative, topological groups, with Z being complete,

$$\cdot\colon X\times Y\to Z$$

a separately continuous bi-additive map, S and T abstract spaces, u an X-valued outer measure on S, and v a Y-valued outer measure on T. For any $A\subset S\times T$, $s\in S$, $t\in T$, let

$$A_s=\{t'\in T\colon (s,t')\in A\},\qquad A^t=\{s'\in S\colon (s',t)\in A\}.$$

Let

$$\mathrm{Rect}(uv)=\{a\times b\colon a\in M_u,\ b\in M_v\}.$$

We shall denote by $\mathfrak{R}$, $\mathfrak{S}$, respectively the smallest algebra and smallest σ-algebra containing $\mathrm{Rect}(uv)$, and by g the unique finitely-additive function on $\mathfrak{R}$ such that

$$g(a\times b)=u(a)\cdot v(b)$$

for all $a\times b\in\mathrm{Rect}(uv)$ ([1], p. 57).

2.1. Definition. μ is a *product measure* of u and v w.r.t. $\cdot$ iff μ is a Z-valued outer measure on $X\times Y$ such that $\mathrm{Rect}(uv)\subset M_v$, and

$$\mu(a\times b)=u(a)\cdot v(b)$$

for all $a\times b\in\mathrm{Rect}(uv)$.

Hereafter we shall omit explicit reference to $\cdot$ when speaking about product measures of u and v.

Using the monotone class theorem (Halmos [5]), one shows easily

2.2. PROPOSITION. *If μ, η are any two product measures of u and v, then μ and η agree on $\mathfrak{S}$.*

2.3. Remarks. If there exists any product measure of u and v then g is σ-additive and monotonely convergent on $\mathfrak{R}$. Conversely, if g satisfies these latter conditions then there exists a Z-valued outer measure μ on $S\times T$ which extends g ([13], Theorem 3.3). μ is clearly a product measure of u and v. The theorems below establish conditions under which g is σ-additive and monotonely convergent.

2.4. THEOREMS. 1. *For all $A\in\mathfrak{S}$: $t\in T\to u(A^t)\in X$ is v-partitionable, and A^t is u-measurable for all $t\in T$.*

2. *For all $A\in\mathfrak{R}$,*

$$g(A) = \int_T u(A^t)\cdot dv(t).$$

If range u is bounded w.r.t. $(v,\cdot)$ *and v is bounded* w.r.t. $\cdot$, *then*

3. *g is σ-additive and monotonely convergent on $\mathfrak{R}$.*

4. *The Z-valued outer measure μ on $S\times T$ generated by g ([13]) is a product measure of u and v. Further, for every $A\in\mathfrak{S}$,*

$$\mu(A) = \int_T u(A^t)\cdot dv(t).$$

5. *If, also, range v is bounded* w.r.t. $(u,\cdot)$ *and u is bounded* w.r.t. $\cdot$, *then for all $A\in\mathfrak{S}$,*

$$\mu(A) = \int_T u(A^t)\cdot dv(t) = \int_S du(s)\cdot v(A_s).$$

Proof of 2.4.1. Let $\mathscr{H}$ consist of those subsets A of $S\times T$ for which A^t is u-measurable for all $t\in T$, and $t\in T\to u(A^t)\in X$ is v-partitionable. Then $\mathscr{H}$ is a monotone class containing $\mathfrak{R}$. In particular, the smallest monotone class containing $\mathfrak{R}$ must be a subfamily of $\mathscr{H}$. Since $\mathfrak{R}$ is a ring, the smallest monotone class containing it must coincide with $\mathfrak{S}$, i.e. $\mathfrak{S}\subset\mathscr{H}$ (cf. Halmos [5], Monotone class theorem).

Proof of 2.4.2. By 2.4.1, 1.2, and 1.5

$$h\colon A\in\mathfrak{R}\to\int_T u(A^t)\cdot dv(t)\in Z$$

is well-defined and is additive. One shows readily that for each $a\times b\in\operatorname{Rect}(uv)$ we have $h(a\times b) = u(a)\cdot v(b)$. Hence, by the uniqueness of g, $h = g$.

Proof of 2.4.3. Let A be a monotone sequence in $\mathfrak{R}$. Then for all $t\in T$, using 2.4.1,

$$\lim_{n\in\omega} u(A_n^t) = u(\lim_{n\in\omega} A_n^t) = u\big((\lim_{n\in\omega} A_n)^t\big).$$

Hence, by the hypotheses, and theorems 1.8.2, 2.4.1, 2.4.2,

$$(1)\qquad \lim_{n\in\omega} g(A_n) = \lim_{n\in\omega}\int u(A_n^t)\cdot dv(t) = \int u\big((\lim_{n\in\omega} A_n)^t\big)\cdot dv(t)$$

which equals $g(\lim_{n\in\omega} A_n)$ if $\lim_{n\in\omega} A_n\in\mathfrak{R}$.

In particular, g is monotonely convergent on $\mathfrak{R}$. The finite additivity of g taken together with (1) implies that g is σ-additive.

Proof of 2.4.4. By Theorem 3.3 of [13] and Theorem 2.4.3 we have that μ is a product measure of u and v. Let $\mathscr{H}$ be as in the proof of 2.4.1 and let $\mathscr{H}'$ be the family of all $A\in\mathscr{H}$ for which

$$\mu(A) = \int_T u(A^t)\cdot dv(t).$$

Then by Theorems 1.8.2, 2.4.1, and 2.4.2, $\mathscr{H}'$ is a monotone class containing $\mathfrak{R}$. Thus, as in the proof of 2.4.1, $\mathfrak{S}\subset\mathscr{H}'$.

Proof of 2.4.5. Immediate from 2.4.4.

3. Fubini's Theorem. Together with the notation of the previous section: X_1, Z_1, Z_2 are commutative topological groups,

$$\varphi\colon X_1\times X\to Z_1,\qquad \eta\colon Z_1\times Y\to Z_2,\qquad \eta_1\colon X_1\times Z\to Z_2,$$

are bi-additive such that for each $(x^1, x, y)\in X_1\times X\times Y$

$$\eta_1(x^1, x\cdot y) = \eta\big(\varphi(x^1, x), y\big).$$

Let μ be some product measure of u and v.

3.1. THEOREM. *Suppose that Z_1, Z_2 are complete, and that μ, v, u, are bounded* w.r.t., *respectively, η_1, η, φ. Let $f\colon S\times T\to\mathscr{S}X_1$ be the uniform limit of a net of $\mathscr{S}X_1$-valued functions on $S\times T$, $(h_i)_{i\in I}$, so that for all $i\in I$:*

1. *h_i is μ-partitionable, $t\in T\to\int_S\varphi\big(h_i(s,t), du(s)\big)\in Z_1$ is v-partitionable, for each $t\in T$ the function $s\in S\to h_i(s,t)\in X_1$ is u-partitionable, and*

2. $$\int_{S\times T}\eta_1(h_i, d\mu) = \int_T\eta\Big(\int_S\varphi\big(h_i(s,t), du(s)\big), dv(t)\Big).$$

Then f is μ-partitionable, $t\in T\to\int_S\varphi\big(f(s,t), du(s)\big)\in Z_1$ is well-defined and v-partitionable, for each $t\in T$ the function $s\in S\to f(s,t)\in X_1$ is u-partitionable, and

$$\int_{S\times T}\eta_1(f, d\mu) = \int_T\eta\Big(\int_S\varphi\big(f(s,t), du(s)\big), dv(t)\Big).$$

Proof of Theorem 3.1. Let $W^1 \in \text{nbhd}\,0$ in Z_1 be closed. Since u is bounded w.r.t. φ, there exists $U^1 \in \text{nbhd}\,0$ in X_1 such that for all finite partitions $F \subset M_u$,

$$\sum_{a \in F} \varphi\big(U^1, u(a)\big) \subset W^1.$$

Choose $i \in I$ such that for all $i < j$

$$f(s, t) \subset h_j(s, t) + U^1.$$

For each $i < j$ and $t \in T$ there exists a partition $P_t \subset M_u$ such that $S \setminus \bigcup P_t$ is u-null and for every $a \in P_t$

$$h_j[a, t] - h_j[a, t] \subset U^1.$$

Then for each $s \in \bigcup P_t$ we have $s \in a$ for some $a \in P_t$, and therefore

$$f(s, t) - h_j(s, t) \subset h_j(s, t) - h_j(s, t) + U^1 \subset 2U^1.$$

Thus for any finite partition F finer than $P_t \cup \{S \setminus \bigcup P_t\}$

$$(1) \qquad \sum_{a \in F} \varphi\big(\{f(s, t) - h_j(s, t) \colon s \in a\}, u(a)\big) \subset 2W^1.$$

By Theorem 1.8.1 $\int\limits_S \varphi\big(f(s, t), du(s)\big)$ is defined. Then by (1) and Remarks 1.2

$$\int\limits_S \varphi\big(f(s, t), du(s)\big) - \int\limits_S \varphi\big(h_j(s, t), du(s)\big) = \int\limits_S \varphi\big(f(s, t) - h_j(s, t), du(s)\big) \in \mathrm{Cl}\, 2W^1 = 2W^1.$$

Since the closed neighbourhoods of 0 in Z_1 constitute a base for nbhd 0 in Z_1, it is now clear that

$$\int\limits_S \varphi\big(h_i(s, t), du(s)\big) \to \int\limits_S \varphi\big(f(s, t), du(s)\big)$$

uniformly for $t \in T$. Hence, by Theorem 1.8.1 and the hypotheses on the h_i, all integrals below are defined and

$$\begin{aligned}\int\limits_{S\times T} \eta_1(f, d\mu) &= \lim_{i \in I} \int\limits_{S\times T} \eta_1(h_i, d\mu) \\ &= \lim_{i \in I} \int\limits_T \eta\Big(\int\limits_S \varphi\big(h_i(s, t), du(s)\big), dv(t)\Big) \\ &= \int\limits_T \lim_{i \in I} \eta\Big(\int\limits_S \varphi\big(h_i(s, t), du(s)\big), dv(t)\Big) \\ &= \int\limits_T \eta\Big(\int\limits_S \varphi\big(f(s, t), du(s)\big), dv(t)\Big).\end{aligned}$$

The assertions on partitionability are a consequence of Theorem 1.7.1.

By imposing enough conditions on f and the functions h_i, we can obtain theorems similar to the above, using quasi-uniform or pointwise convergence instead of uniform convergence. However, the technique used below is also of interest — we approximate the integrals under discussion by finite sums, and impose conditions which insure that these finite sums can in turn be made close.

3.2. Definition. An $\mathscr{S}X_1$-valued function f on $S \times T$ is *approximately product partitionable* iff for each $U_1 \in \text{nbhd}\,0$ in X_1 and each $W_2 \in \text{nbhd}\,0$ in Z_2 there *exist* a countable disjoint $P_v \subset M_v$ covering S, a countable disjoint $P_v \subset M_v$ covering T, and a finite $E \subset \{a \times b \colon a \in P_u,\ b \in P_v\}$, such that

1. for each $a \in E$, $f[a] - f[a] \subset U_1$,
2. for every finite disjoint $F \subset \mathrm{Rect}(uv)$ and finer than $\{a \times b \notin E \colon a \in P_u,\ b \in P_v\}$ we have that

$$\sum_{a \in F} \eta_1\big(f[a], \mu(a)\big) \subset W_2.$$

3.3. Theorem. *Suppose that μ is bounded* w.r.t. η_1, *and that η_1, η are continuous in each variable. If f is an $\mathscr{S}X_1$-valued function on $S \times T$ which is approximately product partitionable, then the equality below holds whenever the integrals all exist,*

$$\int\limits_{S\times T} \eta_1(f, d\mu) = \int\limits_T \eta\Big(\int\limits_S \varphi\big(f(s, t), du(s)\big), dv(t)\Big).$$

Proof. We shall show that given any closed symmetric $W_2 \in \text{nbhd}\,0$ in Z_2

$$(1) \qquad \int\limits_{S\times T} \eta_1(f, d\mu) - \int\limits_T \eta\Big(\int\limits_S \varphi\big(f(s, t), du(s)\big), dv(t)\Big) \in 8W_2.$$

Let π_S, π_T denote the canonical projections from $S \times T$ onto S and T respectively. Define

$$h \colon t \in T \to \int\limits_S \varphi\big(f(s, t), du(s)\big) \in Z_1.$$

Since μ is bounded w.r.t. η_1, choose $U_1 \in \text{nbhd}\,0$ in X_1 such that

$$\sum_{a \in F} \eta_1\big(U_1, \mu(a)\big) \subset W_2$$

for every finite disjoint $F \subset M_\mu$. Since f is approximately product partitionable, there exist P_u, P_v, E as in Definition 3.2. Choose $x_a \in f[a]$ for each $a \in E$. It is clear that

$$\int\limits_{S\times T \setminus \bigcup E} \eta(f, d\mu) \in W_2.$$

Hence, defining

$$g\colon r\in S\times T\to\begin{cases}x_a & \text{if } r\in a\in E,\\ 0 & \text{otherwise,}\end{cases}$$

then by the continuity property of η_1,

$$\int_{S\times T}\eta(g, d\mu) = \sum_{a\in E}\eta\big(x_a, \mu(a)\big).$$

Further, $f(r)-g(r)\in U_1$, for all $r\in\bigcup E$. Hence

$$(2)\qquad \int_{S\times T}\eta_1(f, d\mu)-\sum_{a\in E}\eta_1\big(x_a, \mu(a)\big) = \int_{\bigcup E}\eta_1(f-g, d\mu)+\int_{S\times T\setminus\bigcup E}\eta_1(f, d\mu)\in 2W_2.$$

Since $\int_T\eta(h, dv)$ exists, we can find a finite disjoint $\Lambda_v\subset M_v$ finer than P_v such that

$$(3)\qquad \int_T\eta(h, dv)-\sum_{b\in\Lambda_v}\eta(h[b], dv)\subset W_2.$$

Moreover, by the continuity property of η_1, the finiteness of E, and Lemma 0.2, we may assume that

$$(4)\qquad \sum_{a\in E}\eta_1\big(x_a, \mu(\beta_a)\big)\subset W_2$$

for every $\{\beta_a\colon a\in E\}\subset M_\mu$ with $\beta_a\subset a\setminus\pi_T^{-1}[\bigcup\Lambda_v]$. For each $b\in\Lambda_v$ choose a $t_b\in b$. Similarly, there exists a finite disjoint $\Lambda_u\subset M_u$ finer than P_u such that

$$(5)\qquad \sum_{b\in\Lambda_v}\eta\Big(h(t_b)-\sum_{a\in\Lambda_u}\varphi\big(f[a, t_b], u(a)\big), v(b)\Big)\subset W_2,$$

and for every $\{\beta_a\colon a\in E\}\subset M_\mu$ with $\beta_a\subset a\setminus\pi_S^{-1}[\bigcup\Lambda_u]$,

$$(6)\qquad \sum_{a\in E}\eta_1\big(x_a, \mu(\beta_a)\big)\subset W_2.$$

Let $x_{a,b}\in f[a, t_b]$ for each $(a, b)\in\Lambda_u\times\Lambda_v$,

$$I = \{a\times b\subset\bigcup E\colon (a, b)\in\Lambda_u\times\Lambda_v\},$$

$$J = \{a\times b\colon (a, b)\in\Lambda_u\times\Lambda_v\}\setminus I.$$

Then J is finer than $\{a\times b\notin E\colon (a, b)\in P_u\times P_v\}$, and for each $a\in E$,

$$a\setminus\bigcup I\subset\big(a\setminus\pi_S^{-1}[\bigcup\Lambda_u]\big)\cup\big(a\setminus\pi_T^{-1}[\bigcup\Lambda_v]\big).$$

From the commutativity of the maps, the choices of U_1 and E, (4) and (6), we now deduce that

$$\sum_{a\in E}\eta_1\big(x_a, \mu(a)\big)-\sum_{(a,b)\in\Lambda_u\times\Lambda_v}\eta\Big(\varphi\big(x_{a,b}, u(a)\big), v(b)\Big)$$

$$= \sum_{a\in E}\eta_1\big(x_a, \mu(a)\big)-\sum_{(a,b)\in\Lambda_u\times\Lambda_v}\eta_1\big(x_{a,b}, \mu(a\times b)\big)$$

$$= \sum_{a\in E}\eta_1\big(x_a, \mu(a\setminus\bigcup I)\big)+\sum_{a\in E, a\times b\in I, a\times b\subset a}\eta_1\big(x_a-x_{a,b}, \mu(a\times b)\big)-$$

$$-\sum_{a\times b\in J}\eta_1\big(x_{a,b}, \mu(a\times b)\big)\subset 3W_2+W_2-W_2 = 5W_2.$$

Taken together with (2), (3) and (5), the above implies that (1) holds.

3.4. Remarks. Symmetrization of the hypotheses (e.g. if η_1, η and φ all coincide with the multiplication in a topological algebra) clearly yields 'symmetric' Fubini theorems, i.e. sufficient conditions for both iterated integrals to be equal to that w.r.t. the product measure. Of further interest would be a study of functions f whose sections f_s (f^t) are v-partitionable (resp. u-partitionable) for all $s\in S$ (resp. all $t\in T$).

3.5. Examples. Let S, T be topological spaces, u, v be such that the corresponding open sets are measurable, and assume that the maps η_1, η, φ are continuous in each variable. Let f be an X_1-valued function on $S\times T$ continuous w.r.t. the product topology.

1. If f has compact support then it satisfies the hypotheses of Theorem 3.1.

2. If open sets of $S\times T$ are in $\mathfrak{S}$, range u is bounded w.r.t. $(v,\cdot)$, v is bounded w.r.t. $\cdot$, and f has precompact range (i.e. for each $U_1\in$ nbhd 0 in X_1 finitely many translates of U_1 cover $f[S\times T]$), then f satisfies the hypotheses of Theorem 3.1. (Use Theorem 2.4.4.)

3. If $S\times T$ is Lindelöf and $f[S\times T]$ is bounded w.r.t. μ, then f satisfies the hypotheses of Theorem 3.3.

4. If S, T are Lindelöf, the families $\{f^t\colon t\in T\}$, $\{f_s\colon s\in S\}$ are equicontinuous, and μ is bounded w.r.t. η_1, then f is approximately product partitionable.

4. Radon product measures. Let X be a commutative, topological group, $\mathscr{H}$ a family of subsets of a space S, and γ an X-valued function on $\mathscr{H}$.

For any subfamilies $\mathscr{A}$, $\mathscr{B}$ of $\mathscr{H}$, γ is *$\mathscr{A}$-inner ($\mathscr{A}$-outer) regular* on $\mathscr{B}$ iff for each $B\in\mathscr{B}$ and $U\in$ nbhd 0 in X there exists $A\in\mathscr{A}$ with $A\subset B$ ($B\subset A$) such that

$$a\in\mathscr{A},\ A\subset a\subset B\ (B\subset a\subset A)\ \Rightarrow\ \gamma(B)-\gamma(a)\in U.$$

When S is a topological space, denoting by $\mathscr{G}(S)$, $\mathscr{K}(S)$ the families of open, respectively compact, subsets of S, we call γ a *Radon measure* iff γ

is defined on all subsets of S, M_γ is a σ-field on which γ is σ-additive, $\mathcal{G}(S) \subset M_\gamma$, γ is $\mathcal{G}(S)$-outer regular on its domain and $\mathcal{K}(S)$-inner regular on $\mathcal{G}(S)$.

A family $\mathcal{A}$ of subsets is *ω-compact* iff each countable subfamily of $\mathcal{A}$ with empty intersection necessarily contains a finite subfamily with empty intersection.

The first lemma below extends a result of E. Marczewski [8]. We state the second without proof.

4.1. Lemma. *Let $\mathcal{H}$ be an algebra and γ be finitely additive. If γ is $\mathcal{A}$-inner regular on $\mathcal{H}$ for some ω-compact $\mathcal{A} \subset \mathcal{H}$ which is closed under finite unions, then γ is σ-additive. If γ is also monotonely convergent on $\mathcal{H}$ then the outer measure ξ generated by γ* (Sion [13]) *is $\mathcal{A}_\delta$-inner regular on M_ξ.*

4.2. Lemma. *For any X-valued outer measure ξ on S, $A \in M_\xi$ iff for each $U \in \mathrm{nbhd}\,0$ in X, there exist α, β in M_ξ with $\alpha \subset A \subset \beta$ and $A' \subset \beta \setminus \alpha \Rightarrow \xi(A') \in U$.*

Proof of 4.1. Let H be a sequence of mutually disjoint elements in $\mathcal{H}$. Let $U \in \mathrm{nbhd}\,0$ in X, and let V be a sequence of closed symmetric elements from nbhd 0 in X with $2V_{k+1} \subset V_k \subset U$ for each $k \in \omega$. In particular, for each $k \in \omega$,

$$V_0 + \ldots + V_k \subset U.$$

Choose $A_0 \in \mathcal{A}$ with $A_0 \subset \bigcup_n H_n$ such that for all $A' \in \mathcal{A}$

$$A' \subset \bigcup_n H_n \setminus A_0 \Rightarrow \gamma(A') \in V_0.$$

for each $n \in \omega$ choose $A_{n+1} \in \mathcal{A}$ with $A_{n+1} \subset A_n \setminus H_n$ such that for all $A' \in \mathcal{A}$,

$$A' \in (A_n \setminus H_n) \setminus A_{n+1} \Rightarrow \gamma(A') \in V_{k+1}.$$

Since (A_n) is a decreasing sequence in $\mathcal{A}$ with empty intersection, there exists $m \in \omega$ with $A_m = \emptyset$. Now, since γ is $\mathcal{A}$-inner regular on $\mathcal{H}$, for any $B \in \mathcal{H}$ and $n \in \omega$ we have

$$B \subset (A_n \setminus H_n) \setminus A_{n+1} \Rightarrow \gamma(B) \in V_{k+1},$$

(Similarly for $B \subset \bigcup_n H_n \setminus A_0$).

Further, for any $B \in \mathcal{H}$ with $B \subset \bigcup_{n>m} H_n$,

$$B = \Big(B \cap \big(\bigcup_n H_n \setminus A_0\big)\Big) \cup (B \cap A_0)$$
$$= \Big(B \cap \big(\bigcup_n H_n \setminus A_0\big)\Big) \cup \bigcup_{n<m} B \cap (A_n \setminus A_{n+1}).$$

Since $B \cap (A_n \setminus A_{n+1}) \subset (A_n \setminus H_n) \setminus A_{n+1}$ for each $n < m$, we have

$$\gamma(B) \in V_0 + \ldots + V_m \subset U.$$

Thus γ is σ-additive on $\mathcal{H}$. A similar technique is used in proving the second assertion.

For the rest of this section, X, Y, Z are commutative topological groups with Z being complete, $\cdot : X \times Y \Rightarrow Z$ is bi-additive, S, T are regular topological spaces, $S \times T$ carries the product topology (which is then also regular), u is an X-valued Radon measure on S, and v is a Y-valued Radon measure on T.

$\mathrm{Rect}(u, v)$, $\mathfrak{R}$, $g\colon \mathfrak{R} \to Z$ are as defined in § 2, and $\mathcal{K}_0$ is the family of all finite unions of compact rectangles.

4.3. Theorems. 1. *$\mathcal{K}_0$ is an ω-compact family.*

2. *If u, v are both bounded with respect to $\cdot$, then g is $\mathcal{K}_0$-inner regular on $\mathfrak{R}$.*

3. *If u, v are both bounded with respect to $\cdot$, and g is monotonely convergent on $\mathfrak{R}$ then there exists a product measure μ of u and v which is $\mathfrak{R}_\sigma$-outer regular on its domain, and a Radon product measure ξ of u and v with $M_\mu \subset M_\xi$ such that*

$$\xi/M_\mu = \mu/M_\mu \quad \text{and} \quad \xi/\mathcal{K}(S \times T) = \mu/\mathcal{K}(S \times T).$$

Proofs 4.3. 1. See Meyer [9], a fortiori, by Tychonoff's theorem.

2. It is enough to show that g is $\mathcal{K}_0$-inner regular on $\mathrm{Rect}(u, v)$. Let $W \in \mathrm{nbhd}\,0$ in Z. By the hypotheses on u and v, there exist neighbourhoods U, V of the identities in, respectively, X, Y with

$$\sum_{a \in F} U \cdot v(a) \subset W \quad \text{for all finite disjoint } F \subset M_v,$$
$$\sum_{a \in F} u(a) \cdot V \subset W \quad \text{for all finite disjoint } F \subset M_u.$$

Let $A \times B \in \mathrm{Rect}(u, v)$. Choose $a \in \mathcal{K}(S)$, $b \in \mathcal{K}(T)$ such that $a \subset A$, $b \subset B$ and

$$a \in M_u,\ a \subset A \setminus a \Rightarrow u(a) \in U.$$
$$a \in M_v,\ a \subset B \setminus b \Rightarrow v(a) \in V.$$

Then for any finite disjoint $F \subset \mathrm{Rect}(u, v)$, $\bigcup F \subset A \times B \setminus a \times b$, we have for each $\alpha \times \beta \in F$ that

$$\alpha \subset A \setminus a \quad \text{or} \quad \beta \subset B \setminus b.$$

Hence, letting

$$C = \bigcup \{\alpha \times \beta \in F\colon \alpha \subset A \setminus a\} \quad \text{and} \quad D = \bigcup F \setminus C,$$

by Theorem 2.4 we have

$$g\big(\bigcup F\big) = g(C) + g(D)$$
$$= \int_T u(C^t) \cdot dv(t) + \int_S du(s) \cdot v(D_s) \in 2(\mathrm{Cl}\, W) \subset 3W$$

since

$$C^t \in M_u, \quad C^t \subset A \setminus a \quad \text{for all } t \in T,$$

$$D_s \in M_v, \quad D_s \subset B \setminus b \quad \text{for all } s \in S.$$

3. Letting μ be the outer measure on $S \times T$ generated by g (by 4.1, 4.3.1, 4.3.2, and Theorem 3.3 of [13]), then μ has the required properties. Now let $\mathfrak{x}$ consist of all open rectangles in $S \times T$, and let $\mathfrak{x}_0$ be the family of all unions of finite subfamilies from $\mathfrak{x}$. Arguing as in Proof 4.3.2 above one can show readily that

(1) μ is $\mathfrak{x}_0$-outer regular on its domain

and therefore that

(2) μ is $\mathfrak{x}_0$-outer regular on $\mathscr{K}(S \times T)$.

Using (2), and Theorem 3.2 of [13], we can easily check that $\mu/\mathscr{K}(S \times T)$ satisfies the hypotheses of Definition 6.1.3 of [13]. Let ξ be the Radon measure on $S \times T$ generated by $\mu/\mathscr{K}(S \times T)$ (Theorem 6.3 of [13]). Then

(3) $$\xi/\mathscr{K}(S \times T) = \mu/\mathscr{K}(S \times T).$$

It remains for us to show that $M_\mu \subset M_\xi$ and that ξ, μ coincide on M_μ. Let $A \in M_\mu$. For any closed, symmetric $W \in \text{nbhd}\,0$ in Z, by 4.3.1, 4.3.2, Lemma 4.1, and (1) above, there exist $K \in (\mathscr{K}_0)_\delta$ and $G \in \mathfrak{x}_\sigma$ such that $K \subset A \subset G$ and

$$a \in M_\mu, \; a \subset G \setminus K \;\Rightarrow\; \mu(a) \in W.$$

By (1) and $\mathfrak{x}_\sigma \subset M_\mu$ it follows that

(4) $$a \subset G \setminus K \;\Rightarrow\; \mu(a) \in W.$$

Since $G \setminus K$ is open and ξ is Radon, (3) now implies that

(5) $$a \subset G \setminus K \;\Rightarrow\; \xi(a) \in W.$$

Since W was arbitrary and K, G are both ξ-measurable, then $A \in M_\xi$ (Lemma 4.2). Then using (4) and (5),

$$\mu(A) - \xi(A) = \mu(A) - \mu(K) + \xi(K) - \xi(A) = \mu(A \setminus K) - \xi(A \setminus K) \in 2W.$$

It follows that $\mu(A) = \xi(A)$.

Remarks. By Theorem 2.4.3, boundedness conditions on the range of u or the range of v will guarantee that g is monotonely convergent. The monotone convergence of g is also guaranted when the range of g is a subset of the positive elements of a suitable partially-ordered group (e.g. $L^p(\Omega, \nu)$, $1 \leqslant p \leqslant \infty$, with the usual topology).

5. Applications. Products of vector-valued measures. Let X, Y be (separated) locally convex spaces over C, u (resp. v) an X (resp. Y)-valued outer measure on S (resp. T).

Let $Z = X \check{\otimes} Y$, the ε-tensor product of X and Y, and $\cdot : X \times Y \to X \check{\otimes} Y$ be the canonical injection (Schaeffer [12]). We note that the ranges of u, v, are bounded subsets of respectively X and Y (Dunford–Schwartz [4]). We then have (see also [3]):

5.1. Theorem. *Range u is bounded with respect to $(v, \cdot)$, and range v with respect to $(u, \cdot)$, u is bounded with respect to $\cdot$, and v is bounded with respect to $\cdot$. Hence*

1. *u and v have a product measure μ on $S \times T$, and for each $A \in \mathfrak{S}$,*

$$\mu(a) = \int_T u(a^t) \cdot dv(t) = \int_S du(s) \cdot v(a_s).$$

2. *When S and T are regular topological spaces and u, v are Radon measures, then u and v have a product measure ξ on $S \times T$ which is Radon with respect to the product topology.*

Proof. Given the truth of the boundedness assertions, the other statements are consequences of Theorems 2.4 and 4.3. We prove only that range u is bounded with respect to $(v, \cdot)$. The other proofs are analogous.

Let A be any bounded subset of X, $P \subset M_v$ a partition, and $W \in \text{nbhd}\,0$ in Z. There exists an absolutely convex $U \in \text{nbhd}\,0$ in X and an absolutely convex $V \in \text{nbhd}\,0$ in Z such that for any $z \in Z$,

$$z = \sum_{a \in F} x_a \otimes y_a,$$

and

$$\sup\Big\{\Big|\sum_{a \in F} \langle x_a, x' \rangle \langle y_a, y' \rangle\Big| : x' \in U^0, \; y' \in V^0\Big\} \leqslant 1$$

imply

$$z \in W.$$

Choose $\lambda > 0$ such that $A \subset \lambda U$. By Proposition 0.2 there exists a finite $\Delta(P) \subset P$ such that for each finite disjoint $F \subset M_v$ with $\bigcup F \subset \bigcup (P \setminus \Delta(P))$,

$$\sum_{a \in F} v(a) \in (4\lambda)^{-1} V.$$

Then for any $\{x_a\}_{a \in F} \subset A$, $x' \in U^0$ and $y' \in 4\lambda V^0$, we have that

$$\Big|\sum_{a \in F} \langle x_a, x' \rangle \langle v(a), y' \rangle\Big| \leqslant \lambda \sum_{a \in F} |\langle v(a), y' \rangle| \leqslant 4\lambda \sup_{E \subset F} \Big|\sum_{a \in E} \langle v(a), y' \rangle\Big| \leqslant 1,$$

i.e.

$$\sum_{a\in F} A\cdot v(a) \subset W.$$

Let $Z = X\hat{\otimes} Y$ be the projective tensor product of X and Y and $\cdot$: $X\times Y\to X\hat{\otimes} Y$ the canonical injection.

5.2. THEOREM. *If v is of bounded variation it is bounded with respect to $\cdot$, and further, range u is bounded with respect to $(v,\cdot)$. Hence u and v have a product measure μ on $S\times T$ and for each $A\in\mathfrak{S}$*

$$\mu(A) = \int_T u(A^t)\cdot dv(t).$$

Proof. Saying that v is of bounded variation implies that for each continuous seminorm q on Y, the set function

$$\|v\|_q\colon\ b\in M_v\to \sup\Big\{\sum_{a\in F} q(v(a))\colon\ F\subset M_v \text{ finite, disjoint, } \bigcup F = b\Big\}\geqslant 0$$

is σ-additive on the σ-field M_v ([1], p. 41, Prop. 10). In particular,

1) $\|v\|_q(T)<\infty$.

2) For each partition $P\subset M_v$ and $\varepsilon>0$, there exists finite $\Delta(P)\subset P$ such that for every finite, disjoint $F\subset M_v$ with $\bigcup F\subset \bigcup\big(P\setminus\Delta(P)\big)$,

$$\sum_{a\in F}\|v\|_q(a)<\varepsilon.$$

Using (1) and the characterization of the projective topology in terms of seminorms (Schaeffer [12], III. 6.3), we deduce that v is bounded with respect to $\cdot$. Using (2) we check that any bounded subset of X is bounded with respect to $(v,\cdot)$. The first assertions of the theorem are thus established, and the existence of the product measure is now guaranteed by Theorem 2.4.3.

We close now with a counter example which shows that for the projective tensor product topology the product of two vector-valued measures may not exist. Another such example has been given by Kluvanek [6]. Our example further suggests an error in [2]. Notation and definitions on perfect sequence spaces are as given in Koethe [7].

5.3. PROPOSITION. *Let Y be a perfect sequence space, with its normal topology. If there exists a sequence $\{y_m\}$ in Y which is summable but not absolutely summable* (Schaeffer [12]) *then there exists a Y-valued measure v on ω, and an l^∞-valued measure u on ω such that u and v do not have a product measure on $\omega\times\omega$ with respect to the canonical bilinear map $l^\infty\times Y\to l^\infty\hat{\otimes} Y$.*

Proof. We can identify $l^\infty\hat{\otimes} Y$ with the space Z of all sequences z in l^∞ such that for each $f\in Y^\times$,

$$\Pi_f(z) = \sum_{n\in\omega}|f_n|\,\|z_n\|_\infty<\infty,$$

endowed with the locally convex topology generated by the seminorms Π_f [10].

By hypothesis there exists a sequence y in Y which is summable but not absolutely summable. In particular, for some $f\in Y^\times$,

$$\sum_m\sum_n|f_n y_{m,n}| = \sum_n\sum_m|f_n y_{m,n}| = \infty.$$

Choose a sequence λ with $\lim_{n\in\omega}\lambda_n = 0$ such that

$$\sum_n\lambda_n\sum_m|f_n y_{m,n}| = \infty. \tag{1}$$

Let $e_k\in l^\infty$ be such that $e_k(i) = 1$ for $k=i$, and 0 otherwise. Then $(\lambda_k e_k)_{k\in\omega}$ is a summable sequence in l^∞. For each $A\subset\omega$ let

$$v(A) = \sum_{m\in A} y_m,\qquad u(A) = \sum_{k\in A}\lambda_k e_k,$$

u and v are then respectively l^∞- and Y-valued outer measures on ω, both having unbounded variation (for v by choice of y above). Further, M_u and M_v coincide with the family of all subsets of ω. For any $A\subset\omega$, $B\subset\omega$, let

$$g(A\times B) = u(A)\otimes v(B).$$

In particular, for any $(k,m)\in\omega\times\omega$,

$$g(\{(k,m)\}) = \lambda_k e_k\otimes y_m = (\lambda_k y_{m,n}e_k)_{n\in\omega}. \tag{2}$$

We extend g (uniquely) to a finitely additive set function on the ring $\mathfrak{R}$ generated by $\mathrm{Rect}(uv)$ (§ 2). We shall now show that g is not monotone convergent and therefore cannot be extended to a σ-additive function on $\mathfrak{S}$, the σ-algebra generated by $\mathfrak{R}$. First we note that for any finite family $I\subset C$ (A. Pietsch, *Nukleare lokalkonvexe Räume*, p. 18)

$$\sum_{i\in I}|i|\leqslant 4\sup_{F\subset I}\Big|\sum_{i\in F} i\Big|. \tag{3}$$

Since y is a summable sequence in Y, there exists non-negative $C<\infty$ such that for every finite $F\subset\omega$

$$\sum_n|f_n|\Big|\sum_{m\in F}y_{m,n}\Big|<C.$$

Therefore for each $n\in\omega$,

$$\Big|\sum_{m\in F}y_{m,n}\Big|<|f_n|^{-1}C,$$

and thus by (3),

$$\sum_{m\in F}|y_{m,n}|<4|f_n|^{-1}C.$$

Hence

$$\sup_{n\in\omega}\sum_{m\in\omega}|f_n y_{m,n}|<4C<\infty. \tag{4}$$

Let $\varepsilon>0$. By (4), for each $n\in\omega$ there exists a finite $A_n\subset\omega$, and by (3) a finite $B_n\subset A_n$ such that

$$\sum_{m\in\omega}|f_n y_{m,n}|-\sum_{m\in A_n}|f_n y_{m,n}|<2^{-n}\lambda_n^{-1}\varepsilon \tag{5}$$

and

$$\Big|\sum_{m\in B_n}y_{m,n}\Big|\geqslant 4^{-1}\sum_{m\in A_n}|y_{m,n}|. \tag{6}$$

Now for each $N\in\omega$ let

$$B^N=\bigcup_{n\leqslant N}\{n\}\times B_n.$$

Then for each $N\in\omega$, $B^N\subset B^{N+1}$. Also, using the additivity of g on $\mathfrak{R}$ and the representation (2),

$$\begin{aligned}\Pi_f(g(B^N)) &= \sum_{n\in\omega}|f_n|\sup_{i\in\omega}\Big|\sum_{k\leqslant N,m\in B_k}\lambda_k y_{m,n}e_k(i)\Big|\\ &\geqslant \sum_{n\in\omega}|f_n|\Big|\sum_{k\leqslant N,m\in B_k}\lambda_k y_{m,n}e_k(n)\Big| = \sum_{n\leqslant N}|f_n\lambda_n|\Big|\sum_{m\in B_n}y_{m,n}\Big|\\ &> 4^{-1}\sum_{n\leqslant N}\lambda_n\sum_{m\in A_n}|f_n y_{m,n}| > 4^{-1}\sum_{n\leqslant N}\lambda_n\sum_{m\in\omega}|f_n y_{m,n}|-\tfrac{1}{2}\varepsilon\\ &\to\infty \quad \text{as} \quad N\to\infty.\end{aligned}$$

Remark. The norm topology of l^1 coincides with its normal topology (Koethe [7]) and by the Theorem of Dvoretsky-Rogers there exists a summable sequence in l^1 which is not absolutely summable. Proposition 5.3 then implies that l^1 is not an "admissible factor", in the terminology of [2].

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