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A mean ergodic theorem for a contraction semigroup in Lebesgue space

by

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Abstract. Let $(X, \boldsymbol{A}, m)$ be a σ-finite measure space and let $\Gamma = \{T_t;\ t > 0\}$ be a strongly continuous semigroup of linear contraction operators on $L_1(X, \boldsymbol{A}, m)$. The main purpose of this paper is to prove the following: T_t converges weakly as $t \to \infty$ if and only if $\int_0^\infty a_n(t) T_t\, dt$ converges strongly as $n \to \infty$ for any sequence (a_n) of Lebesgue measurable complex-valued functions on $(0, \infty)$ satisfying

$$\sup_{n\geq 1} \int_0^\infty |a_n(t)|\, dt < \infty, \quad \lim_{n\to\infty} \int_0^\infty a_n(t)\, dt = 1, \quad \text{and} \quad \lim_{n\to\infty} \|a_n\|_\infty = 0.$$

This is a continuous parameter version of Akcoglu and Sucheston's mean ergodic theorem [1] in Lebesgue space.

1. Introduction. In [2] Blum and Hanson proved that if a linear operator T on the L_1-space of a finite measure space is induced by a measure preserving transformation of the measure space, then the following two conditions are equivalent:

(I) *T^n converges weakly;*

(II) $\frac{1}{n}\sum_{i=1}^{n} T^{k_i}$ *converges strongly for any strictly increasing sequence* (k_i) *of nonnegative integers.*

Later Akcoglu and Sucheston [1] generalized this result as follows. If T is a linear contraction operator on the L_1-space of a σ-finite measure space, then the equivalence of (I) and (II) still holds. Condition (I) corresponds to *mixing*, or more generally, *stability* in applications to Ergodic Theory. In the present paper we intend to extend the result to semigroups of operators on the L_1-space.

Let $\Gamma = \{T_t;\ t > 0\}$ be a strongly continuous semigroup of linear contraction operators on the L_1-space, and consider the following two conditions:

(i) T_t *converges weakly as* $t\to\infty$;

(ii) $\int_0^\infty a_n(t)T_t\,dt$ *converges strongly as* $n\to\infty$ *for any sequence* (a_n) *of Lebesgue measurable complex-valued functions on* $(0,\infty)$ *satisfying*

$$\sup_{n\geqslant 1}\|a_n\|_1<\infty,\qquad \lim_{n\to\infty}\int_0^\infty a_n\,dt=1,\qquad \textit{and}\qquad \lim_{n\to\infty}\|a_n\|_\infty=0.$$

Theorem 1 below states that (i) and (ii) are equivalent. Applying Theorem 1 we obtain that if all the T_t are positive and T_tf converges weakly as $t\to\infty$ for any integrable f with zero integral, then for any such f and any sequence (a_n) of Lebesgue measurable complex-valued functions on $(0,\infty)$ as in (ii) above, $\int_0^\infty a_n(t)T_tf\,dt$ converges strongly as $n\to\infty$.

2. Definitions and theorems. Let $(X,\boldsymbol{A},m)$ be a σ-finite measure space with positive measure m and let $L_p(X,m)=L_p(X,\boldsymbol{A},m)$, $1\leqslant p\leqslant\infty$, be the usual (complex) Banach spaces. If $A\in\boldsymbol{A}$ then 1_A is the indicator function of A and $L_p(A,m)$ denotes the Banach space of all $L_p(X,m)$-functions that vanish a.e. on $X-A$. Also, $L_p^+(A,m)$ denotes the positive cone of $L_p(A,m)$ consisting of nonnegative $L_p(A,m)$-functions. A linear operator T on $L_p(X,m)$ is called *positive* if $T\big(L_p^+(X,m)\big)\subset L_p^+(X,m)$ and a *contraction* if $\|T\|_p\leqslant 1$. A set $A\in\boldsymbol{A}$ is called *T-closed* if $T\big(L_p(A,m)\big)\subset L_p(A,m)$. The adjoint of T is denoted by T^*. It is known [3] that given a contraction T on $L_1(X,m)$, there exists a unique positive contraction τ on $L_1(X,m)$, called the *linear modulus* of T, such that

$$\tau g=\sup\{|Tf|;\ f\in L_1(X,m)\ \text{and}\ |f|\leqslant g\}$$

for any $g\in L_1^+(X,m)$. It is easy to see that a set $A\in\boldsymbol{A}$ is T-closed if and only if it is τ-closed.

Let $\Gamma=\{T_t;\ t>0\}$ be a semigroup of linear contraction operators on $L_1(X,m)$, i.e., $T_tT_s=T_{t+s}$ for all $t,s>0$ and all the T_t are linear contraction operators on $L_1(X,m)$. Throughout this paper we shall assume that Γ is strongly continuous. This means that for any $s>0$ and any $f\in L_1(X,m)$ we have $\lim_{t\to s}\|T_tf-T_sf\|_1=0$. It follows that if $a(t)$ is a Lebesgue integrable complex-valued function on $(0,\infty)$, then for any $f\in L_1(X,m)$, the vector-valued function $t\to a(t)T_tf$ on $(0,\infty)$ is also Lebesgue integrable.

We are now in a position to state our results.

THEOREM 1. *The following statements are equivalent.*

(i) *If* $f\in L_1(X,m)$ *then* T_tf *converges weakly in* $L_1(X,m)$ *as* $t\to\infty$.

(ii) *If* $f\in L_1(X,m)$ *then* $\int_0^\infty a_n(t)T_tf\,dt$ *converges strongly in* $L_1(X,m)$ *as* $n\to\infty$ *for any sequence* (a_n) *of Lebesgue measurable complex-valued functions on* $(0,\infty)$ *satisfying*

$$\text{(1)}\qquad \sup_{n\geqslant 1}\int_0^\infty |a_n(t)|\,dt<\infty,$$

$$\text{(2)}\qquad \lim_{n\to\infty}\int_0^\infty a_n(t)\,dt=1,$$

$$\text{(3)}\qquad \lim_{n\to\infty}\|a_n\|_\infty=0.$$

THEOREM 2. *Suppose that all the* T_t *are positive. Then the following statements are equivalent.*

(i) *If* $f\in L_1(X,m)$ *and* $\int f\,dm=0$, *then* T_tf *converges weakly in* $L_1(X,m)$ *as* $t\to\infty$.

(ii) *If* $f\in L_1(X,m)$ *and* $\int f\,dm=0$, *then* $\int_0^\infty a_n(t)T_tf\,dt$ *converges strongly in* $L_1(X,m)$ *as* $n\to\infty$ *for any sequence* (a_n) *of Lebesgue measurable complex-valued functions on* $(0,\infty)$ *satisfying conditions* (1), (2), *and* (3) *in Theorem* 1.

For the proof of Theorem 1 we need two lemmas which are given in the next section, and Theorem 2 follows from Theorem 1.

3. Lemmas.

LEMMA 1. *If the semigroup* $\Gamma=\{T_t;\ t>0\}$ *satisfies, in addition, that* $\|T_tf\|_\infty\leqslant\|f\|_\infty$ *for all* $t>0$ *and all* $f\in L_1(X,m)\cap L_\infty(X,m)$, *then the statement* (i) *of Theorem* 1 *implies the statement* (ii) *of Theorem* 1.

Proof. It follows from the Riesz convexity theorem ([4], Theorem VI. 10. 11) that $\|T_t\|_2\leqslant 1$ for all $t>0$, and an approximation argument shows that the semigroup $\Gamma=\{T_t;\ t>0\}$ is a strongly continuous semigroup of linear contractions on $L_2(X,m)$ and that T_tf converges weakly in $L_2(X,m)$ as $t\to\infty$ for all $f\in L_2(X,m)$. Thus a slight modification of [6] implies that $\int_0^\infty a_n(t)T_tf\,dt$ converges strongly in $L_2(X,m)$ as $n\to\infty$ for any $f\in L_2(X,m)$ and any sequence (a_n) of Lebesgue measurable complex-valued functions on $(0,\infty)$ satisfying conditions (1), (2), and (3).

Suppose $f\in L_1(X,m)\cap L_\infty(X,m)$, and let $\varepsilon>0$ be given. Then, since T_tf converges weakly in $L_1(X,m)$ as $t\to\infty$, using the Vitali–Hahn–Saks theorem ([4], Theorem III. 7. 2) we can choose a set $A\in\boldsymbol{A}$ such that $m(A)<\infty$ and $\int_{X-A}|T_tf|\,dm<\varepsilon$ for all $t\geqslant 1$. Let $f_n=\int_0^\infty a_n(t)T_tf\,dt$ for

$n = 1, 2, \ldots$, and let g be a function in $L_2(X, m)$ such that $\lim_{n\to\infty} \|f_n - g\|_2 = 0$. Then we have

$$\|(f_n - g)1_A\|_1 \leqslant \|f_n - g\|_2 \sqrt{m(A)} \to 0$$

as $n\to\infty$, and

$$\|f_n 1_{X-A}\|_1 = \Big\| \int_0^\infty a_n(t) 1_{X-A} T_t f\,dt \Big\|_1 \leqslant \int_0^\infty |a_n(t)|\, \|1_{X-A} T_t f\|_1\,dt \leqslant \|a_n\|_\infty \|f\|_1 + \varepsilon \Big(\sup_{n\geqslant 1} \int_0^\infty |a_n(t)|\,dt\Big).$$

Hence, by conditions (1) and (3), we observe that (f_n) is a Cauchy sequence in $L_1(X, m)$, and f_n converges strongly in $L_1(X, m)$ as $n\to\infty$. Since $L_1(X, m) \cap L_\infty(X, m)$ is a dense subspace of $L_1(X, m)$ in the strong topology and $\sup_{n\geqslant 1} \|\int_0^\infty a_n(t)\, T_t\,dt\|_1 < \infty$ by condition (1), this completes the proof of the present lemma.

For each $t > 0$, let us denote by τ_t the linear modulus of T_t.

Lemma 2. *Let $f \in L_1(X, m)$ and $Y \in \boldsymbol{A}$. Suppose $T_t f$ converges weakly in $L_1(X, m)$ as $t\to\infty$ and $X - Y$ is T_t-closed for all $t > 0$. Then either $\lim_{t\to\infty} \int_Y |T_t f|\,dm = 0$ or there exists a function $g \in L_1^+(Y, m)$ with $\|g\|_1 > 0$ and $\tau_t g = g$ for all $t > 0$.*

Proof. Let η be any invariant mean on the additive semigroup $(0, \infty)$ (see, for example, [5], Sections 3.3–3.5) and define

$$\mu(A) = \eta\Big(\int_{A\cap Y} |T_t f|\,dm \Big)$$

for all $A \in \boldsymbol{A}$. Since the set $\{T_t f;\ t \geqslant 1\}$ is weakly sequentially compact in $L_1(X, m)$, it follows from the Vitali–Hahn–Saks theorem that $\lim_{n\to\infty} \big(\sup_{t\geqslant 1} \int_{A_n\cap Y} |T_t f|\,dm\big) = 0$ for any decreasing sequence (A_n) of measurable sets with $\lim_{n\to\infty} A_n = \emptyset$, from which it may be readily seen that μ is a finite measure on $(X, \boldsymbol{A})$ absolutely continuous with respect to m. Let $g = d\mu/dm$. Then clearly $g \in L_1^+(Y, m)$, and since $X - Y$ is T_t-closed for all $t > 0$, for any $s > 0$ and any $A \in \boldsymbol{A}$ we have

$$\int_A \tau_s g\,dm = \int g(\tau_s^* 1_A)\,dm = \eta\Big(\int_Y |T_t f|\, \tau_s^* 1_A\,dm \Big) = \eta\Big(\int_A \tau_s(1_Y |T_t f|)\,dm \Big)$$
$$\geqslant \eta\Big(\int_{A\cap Y} \tau_s |T_t f|\,dm \Big) \geqslant \eta\Big(\int_{A\cap Y} |T_{s+t} f|\,dm \Big) = \eta\Big(\int_{A\cap Y} |T_t f|\,dm \Big) = \int_A g\,dm.$$

Therefore $\tau_s g \geqslant g$, and hence $\tau_s g = g$ since $\|\tau_s\|_1 \leqslant 1$. On the other hand, it may be readily seen that $\|g\|_1 = \lim_{t\to\infty} \int_Y |T_t f|\,dm$. Hence the lemma is proved.

4. Proof of Theorem 1. (i)$\Rightarrow$(ii). We can choose a set $P \in \boldsymbol{A}$ such that there exists a function $h \in L_1^+(P, m)$ with $h > 0$ a.e. on P and $\tau_t h = h$ for all $t > 0$ and also such that $g \in L_1^+(X, m)$ and $\tau_t g = g$ for all $t > 0$ imply $g \in L_1^+(P, m)$. Clearly, $T_t\big(L_1(P, m)\big) \subset L_1(P, m)$ for all $t > 0$. So P is T_t-closed for all $t > 0$. Let λ be the finite measure on $(X, \boldsymbol{A})$ defined by $\lambda(A) = \int_A h\,dm$ for all $A \in \boldsymbol{A}$. Then it follows that $f \in L_1(P, \lambda)$ if and only if $fh \in L_1(P, m)$, and that $\int_P |f|\,d\lambda = \int_P |fh|\,dm$ for all $f \in L_1(P, \lambda)$. Thus, if we set for all $t > 0$ and all $f \in L_1(P, \lambda)$,

$$S_t f = \frac{1}{h} T_t(fh),$$

then $\Delta = \{S_t;\ t > 0\}$ is a strongly continuous semigroup of linear contraction operators on $L_1(P, \lambda)$. Clearly, (i) implies that $S_t f$ converges weakly in $L_1(P, \lambda)$ as $t\to\infty$ for any $f \in L_1(P, \lambda)$. Moreover, since $\tau_t h = h$ for all $t > 0$, it follows that $\|S_t f\|_\infty \leqslant \|f\|_\infty$ for all $t > 0$ and all $f \in L_\infty(P, \lambda)$. Hence we may apply Lemma 1 to infer that for any $f \in L_1(P, \lambda)$ and any sequence (a_n) of Lebesgue measurable complex-valued functions on $(0, \infty)$ satisfying conditions (1), (2), and (3), $\int_0^\infty a_n(t) S_t f\,dt$ converges strongly in $L_1(P, \lambda)$ as $n\to\infty$. Since

$$\int_0^\infty a_n(t) S_t f\,dt = \frac{1}{h} \int_0^\infty a_n(t) T_t(fh)\,dt$$

and

$$L_1(P, m) = \{fh;\ f \in L_1(P, \lambda)\},$$

this shows that $\int_0^\infty a_n(t) T_t f\,dt$ converges strongly in $L_1(P, m)$ as $n\to\infty$ for any $f \in L_1(P, m)$.

Next let $f \in L_1(X - P, m)$. Then, since P is T_t-closed for all $t > 0$, Lemma 2 implies that for any given $\varepsilon > 0$ we can choose a positive real c such that $\int_{X-P} |T_t f|\,dm < \varepsilon$ for all $t \geqslant c$. It follows that

$$\int_0^\infty a_n(t) T_t f\,dt = \int_0^c a_n(t) T_t f\,dt + \int_c^\infty a_n(t) T_t f\,dt,$$

$$\Big\| \int_0^c a_n(t) T_t f\,dt \Big\|_1 \leqslant \|a_n\|_\infty \|f\|_1 c \to 0$$

as $n\to\infty$ by condition (3), and

$$\int_c^\infty a_n(t)T_tf\,dt = \int_0^\infty a_n'(t)T_tf_1\,dt + \int_0^\infty a_n'(t)T_tf_2\,dt,$$

where $a_n'(t) = a_n(t+c)$ for all $t>0$, $f_1 = (T_cf)1_P$, and $f_2 = (T_cf)1_{X-P}$. Therefore

$$\Big\|\int_0^\infty a_n'(t)T_tf_2\,dt\Big\|_1 \leqslant \int_0^\infty |a_n'(t)|\,\|f_2\|_1\,dt < \varepsilon\Big(\sup_{n\geqslant 1}\int_0^\infty |a_n(t)|\,dt\Big),$$

and $\int_0^\infty a_n'(t)T_tf_1\,dt$ converges strongly in $L_1(X, m)$, since $f_1\in L_1(P, m)$ and the sequence (a_n') satisfies conditions (1), (2), and (3). Consequently, we observe that the sequence $\big(\int_0^\infty a_n(t)T_tf\,dt\big)$ is a Cauchy sequence in $L_1(X, m)$, which completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). It suffices to show that $\lim_{t\to\infty}\langle T_tf, u\rangle$ exists for any $f\in L_1(X, m)$ and any $u\in L_\infty(X, m)$. Let $\varepsilon>0$ be given, and choose a positive integer N such that $1\leqslant t, s\leqslant 2$ and $|s-t|\leqslant 1/N$ imply $\|T_tf-T_sf\|_1<\varepsilon$. Then for any real s with $1\leqslant n/N\leqslant s\leqslant (n+1)/N$ we have

$$\Big\|T_sf - N\Big(\int_{n/N}^{(n+1)/N} T_tf\,dt\Big)\Big\|_1 = \Big\|N\Big(\int_{n/N}^{(n+1)/N}(T_sf-T_tf)\,dt\Big)\Big\|_1 \tag{4}$$

$$\leqslant N\Big(\int_{n/N}^{(n+1)/N}\|T_sf-T_tf\|_1\,dt\Big) < \varepsilon.$$

Let us write $T = T_{1/N}$ and $f' = N\big(\int_0^{1/N} T_tf\,dt\big)$. Then (ii) implies that $\frac{1}{n}\sum_{i=1}^n T^{k_i}f'$ converges strongly in $L_1(X, m)$ as $n\to\infty$ for any strictly increasing sequence (k_i) of nonnegative integers. Thus it follows from [1], Theorem 2.1, that for any $u\in L_\infty(X, m)$ the limit

$$\lim_{n\to\infty}\langle T^nf', u\rangle = \lim_{n\to\infty}\Big\langle N\Big(\int_{n/N}^{(n+1)/N} T_tf\,dt\Big), u\Big\rangle$$

exists. This together with (4) shows that

$$\lim_{t,s\to\infty}|\langle T_tf, u\rangle - \langle T_sf, u\rangle| = 0,$$

and hence the proof of (ii) $\Rightarrow$ (i) is completed.

5. Proof of Theorem 2. Suppose (i) holds. If there does not exist a function $g\in L_1^+(X, m)$ such that $\|g\|_1>0$ and $T_tg = g$ for all $t>0$, then (ii) follows readily from Lemma 2. On the other hand, if there exists a function $g\in L_1^+(X, m)$ such that $\|g\|_1>0$ and $T_tg = g$ for all $t>0$, then we can see that T_tf converges weakly in $L_1(X, m)$ as $t\to\infty$ for any $f\in L_1(X, m)$. In fact, any $f\in L_1(X, m)$ can be written as $f = f_1+f_2$, where $f_1 = \big(\int f\,dm/\int g\,dm\big)g$ and $f_2 = f-f_1$ (cf. [1]). Clearly, $T_tf_1 = f_1$ for all $t>0$ and $\int f_2\,dm = 0$. Hence, in this case, (ii) follows from Theorem 1.

The proof of (ii) $\Rightarrow$ (i) is similar to that of (ii) $\Rightarrow$ (i) in Theorem 1.

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