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On the spectrum of the Laplacian on the affine group of the real line

by

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Abstract. If G is the affine group of the real line and X^2+Y^2 is the Laplacian on it regarded as a densely defined operator on all $L_p(G)$, then it has the same spectrum for all p, $1 \leqslant p < \infty$.

Let G be a Lie group and LG the Lie algebra of G. Let $X_1, \ldots, X_n$ be a basis of LG and let

$$L = -X_1^2 - \ldots - X_n^2.$$

If m is a left-invariant Haar measure on G, then L is a densely defined operator on each of $L_p(G, m)$ which is non-negative, essentially self-adjoint on $L_2(G,m)$.

Let

$$\mathrm{Sp}_p L = \{\lambda \in \boldsymbol{C}: (\lambda I - L)^{-1} \text{ is bounded on } L_p(G, m)\}^c.$$

It is well known that $-L$ is the infinitesimal generator of a one-parameter semigroup of convolution operators whose kernels p_t, $t > 0$, are $L_1(G, m)$ functions. In [2] a commutative Banach algebra $\boldsymbol{A}$, defined as the $L_1(G, m)$ closure of $\mathrm{lin}\{p_t: t > 0\}$, is studied. If G is of polynomial growth (e.g. G is a compact extension of nilpotent Lie group), then $\boldsymbol{A}$ is symmetric and hence

$$(*) \qquad \mathrm{Sp}_p L = \mathrm{Sp}_2 L \quad \text{for all} \quad 1 \leqslant p < \infty;$$

cf. [2] and [3].

In this note we consider a Lie group which is not of polynomial growth; namely, the group of affine transformations of the real line. As it has been recently proved by R. Aravamudhan [1], the whole $L_1(G, m)$ is not symmetric.(1) However, as we shall show here, the algebra $\boldsymbol{A}$ is symmetric and equality $(*)$ holds also for this group. The method of the proof used here is quite different from the ones of [2] and [3]. First we establish

(1) *Added in proof*: Aravamudhan's proof appears to be wrong, thus the question about symmetry of $L_1(G, m)$ remains open.

the fact that $\mathrm{Sp}_1 L$ is real. This is done by using the spectral properties of an ordinary differential operator studied in [6]. From this we shall easily infer that A is symmetric. Now Proposition 5.3 in [2] says that if G is amenable and A is symmetric, then (*) holds.

1. Let G be the group of affine transformations of the real line, i.e.

$$G = \{g = (s, t): s, t \in \boldsymbol{R}\}$$

and the multiplication rule is

$$(s, t)(u, v) = (s + ue^t, t + v).$$

The left-invariant Haar measure m on G is given by

$$dm(s, t) = e^{-t} ds\, dt.$$

The group G has two infinitely many dimensional irreducible unitary representations σ^+ and σ^- defined as follows (cf. e.g. [5]):

Both are realized on $L_2(\boldsymbol{R})$ and for an f in $L_2(\boldsymbol{R})$

$$[\sigma^{\pm}(s, t) f](x) = \exp(\pm ise^x) f(x + t).$$

Let X, Y be the basis for the Lie algebra of G defined by

$$\exp sX = (s, 0), \qquad \exp tY = (0, t).$$

Then, for an f in $C_c^\infty(\boldsymbol{R})$ we have

$$[d\sigma^{\pm}(X) f](x) = \frac{d}{ds} \exp(\pm ise^x) f(x)|_{s=0} = \pm ie^x f(x),$$

$$[d\sigma^{\pm}(Y) f](x) = \frac{d}{dt} f(x + t)|_{t=0} = \frac{d}{dx} f(x).$$

Let

(1) $$L = -X^2 - Y^2.$$

THEOREM 1. *The spectrum* $\mathrm{Sp}_1 L$ *of* L *on* $L_1(G, m)$ *is real.*

Proof. We have

(2) $$d\sigma^{\pm}(L) = -\frac{d^2}{dx^2} + e^{2x} = A.$$

A is a non-negative essentially self-adjoint operator on $L_2(\boldsymbol{R})$ whose spectral properties are described in [6], pp. 95–96. Let λ be a complex number such that

$$\operatorname{Im} \lambda \neq 0$$

and let

$$\lambda = (\alpha + i\beta)^2 \quad \text{with} \quad \beta > 0.$$

We put

$$\nu = i(\alpha + i\beta) = -\beta + i\alpha.$$

Let $I_{-\nu}(z)$ and $K_{-\nu}(z)$ be the modified Bessel functions corresponding to the parameter $-\nu$ and let

(3) $$k_\lambda(s, t) = \begin{cases} e^t \pi^{-1} \int\limits_{-\infty}^{+\infty} I_{-\nu}(u) K_{-\nu}(e^t u) \cos(us)\, du & \text{for} \quad t > 0, \\ e^t \pi^{-1} \int\limits_{-\infty}^{+\infty} I_{-\nu}(e^t u) K_{-\nu}(u) \cos(us)\, du & \text{for} \quad t < 0. \end{cases}$$

Then, by [4], p. 413,

$$e^{-t} k_\lambda(s, t) = e^{-t/2} \mathcal{Q}_{-\nu-1/2}[(1 + e^{2t} + s^2)\tfrac{1}{2} e^{-t}],$$

where $\mathcal{Q}_\nu(z)$ is the Legendre function of the second kind with the parameter $\mu = 0$.

We are going to show that

(4) $$k_\lambda \in L_1(G, m),$$

or equivalently that

$$\int\limits_{\boldsymbol{R}^2} = \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} e^{-t/2} |\mathcal{Q}_{-\nu-1/2}[\tfrac{1}{2}(1 + e^{2t} + s^2) e^{-t}]|\, ds\, dt < +\infty.$$

By [4], pp. 196–197, the function $\mathcal{Q}_{-\nu-1/2}(z)$, $\operatorname{Re}(-\nu) > 0$, has two singular points: $z = 1$ and $z = \infty$.

Clearly,

$$\tfrac{1}{2}(1 + e^{2t} + s^2) e^{-t} = 1 \quad \text{implies} \quad t = 0 \text{ and } s = 0.$$

We then write

$$\int\limits_{\boldsymbol{R}^2} = \int\limits_{\varepsilon I^2} + \int\limits_{\boldsymbol{R}^2 \setminus nI^2} + \int\limits_{nI^2 \setminus \varepsilon I^2},$$

where I^2 denotes the square $(-1, 1) \times (-1, 1)$ in $\boldsymbol{R}^2$, and we note that only first and the second summands could be infinite. For an ε small enough and constants a and b, by [4], p. 196, we have

$$\begin{aligned} &\int\limits_{\varepsilon I^2} e^{-t/2} |\mathcal{Q}_{-\nu-1/2}[\tfrac{1}{2}(1 + e^{2t} + s^2) e^{-t}]|\, ds\, dt \\ &\quad \leqslant 1 - a \int\limits_{\varepsilon I^2} \log[\tfrac{1}{2}(1 + e^{2t} + s^2) e^{-t} - 1]\, ds\, dt \\ &\quad \leqslant 1 - b \int\limits_{\varepsilon I^2} \log[(1 - e^t)^2 + s^2]\, ds\, dt < +\infty. \end{aligned}$$

Now, if s or t tends to $\pm\infty$, then $\frac{1}{2}(1+e^{2t}+s^2)e^{-t}\to+\infty$ and, accordingly, by [4], p. 197, for a constant c (depending on n)

$$|\mathcal{Q}_{-\nu-1/2}[\tfrac{1}{2}(1+e^{2t}+s^2)e^{-t}]|\leqslant c[(1+e^{2t}+s^2)e^{-t}]^{-1/2}$$

for all $(s,t)\notin nI^2$.

Thus

$$\int\limits_{\mathbf{R}^2\setminus nI^2}\leqslant c\int\limits_{\mathbf{R}^2\setminus nI^2}e^{\beta t}(1+e^{2t}+s^2)^{-\beta-1/2}\,ds\,dt$$
$$\leqslant c\int_0^{+\infty}du\int\limits_{\mathbf{R}}u^{\beta-1}(1+u^2+s^2)^{-\beta-1/2}\,ds$$
$$\leqslant c'\int_{-\pi/2}^{\pi/2}\cos^{-1}\theta\,d\theta\int_0^{+\infty}(1+r^2)^{-(1+\beta)/2}\,dr<+\infty,$$

which completes the proof of (4).

Now for a function φ in $L_1(G,m)$ we write

$$\sigma^{\pm}(\varphi)=\int\limits_G\varphi(g)\sigma^{\pm}(g)m(dg),$$

whence

(5) $$[\sigma^{\pm}(\varphi)f](x)=\int_{-\infty}^{+\infty}G_{\pm}(x,y)f(y)\,dy,$$

where

$$G_{\pm}(x,y)=\int_{-\infty}^{+\infty}\varphi(s,y-x)\exp(\pm ise^{x})e^{x-y}\,ds,$$

i.e.

(6) $$G_{+}(x,y)=(2\pi)^{1/2}e^{x-y}\widehat{\varphi(-e^{x}},y-x),$$
$$G_{-}(x,y)=(2\pi)^{1/2}e^{x-y}\widehat{\varphi(e^{x}},y-x),$$

where for a function $\varphi(s,t)$ which is Lebesgue integrable with respect to s for almost all t we write

$$\varphi(\hat{\xi},t)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\varphi(s,t)\exp(-i\xi s)\,ds.$$

Now for a complex λ such that $\operatorname{Im}\lambda\neq0$, let k_λ be defined as in (3). Let

(7) $$[\sigma^{\pm}(k_\lambda)f](x)=\int_{-\infty}^{+\infty}\Phi_\lambda^{\pm}(x,y)f(y)\,dy,$$

where, by (5),

$$\Phi_\lambda^{\pm}(x,y)=(2\pi)^{1/2}e^{x-y}\widehat{k_\lambda(\mp e^{x}},y-x).$$

But, by (3),

$$k_\lambda(-s,t)=k_\lambda(s,t),$$

whence

$$\Phi_\lambda^{+}(x,y)=\Phi_\lambda^{-}(x,y)=\Phi_\lambda(x,y).$$

Now, comparing

$$k_\lambda(s,t)=\sqrt{\frac{2}{\pi}}\int_0^{+\infty}k_\lambda(\hat{u},t)\cos(us)\,du$$

and (3), we get

$$\Phi_\lambda(x,y)=\begin{cases}I_{-\nu}(e^{x})K_{-\nu}(e^{y}) & \text{for}\quad y>x,\\ I_{-\nu}(e^{y})K_{-\nu}(e^{x}) & \text{for}\quad y<x.\end{cases}$$

If for a function f in $L_2(\mathbf{R})$ we put

(8) $$\Phi(x,\lambda)=\int_{-\infty}^{+\infty}\Phi_\lambda(x,y)f(y)\,dy,$$

then, by [6], p. 96, and the identity $K_{-\nu}(x)=K_\nu(x)$, we get

$$(\lambda-A)\Phi(x,\lambda)=f(x).$$

Hence, by (7), (8) and (2), we get

(9) $$[\lambda-d\sigma^{\pm}(L)]\sigma^{\pm}(k_\lambda)f=f\quad\text{for all } f\text{ in } C_c^\infty(\mathbf{R}).$$

Now for a function k in $L_1(G,m)$ let $T(k)$ denote the left convolution operator it defines on G. For a φ in $C_c^\infty(G)$, let

$$\psi=(\lambda-L)(k_\lambda*\varphi)-\varphi.$$

Since $k_\lambda\in L_1(G,m)$, we have $\psi\in L_1(G,m)\cap C^\infty(G)$. Consequently, on each C^∞-vector ξ of the representation $\sigma^{\pm}$ we have

$$\sigma^{\pm}(\psi)=[d\sigma^{\pm}(\lambda-L)\sigma^{\pm}(k_\lambda)-I]\sigma^{\pm}(\varphi),$$

which, by (9), shows that $\sigma^{\pm}(\psi)=0$, and, consequently, by (5) and (6), $\psi=0$.

This proves

(10) $$(\lambda-L)T(k_\lambda)=I.$$

But since L is an essentially self-adjoint operator on $L_2(G,m)$ and $\operatorname{Im}\lambda\neq0$, then $(\lambda-L)^{-1}$ is a bounded operator on $L_2(G,m)$, whence, by (10),

$$T(k_\lambda)=(\lambda-L)^{-1},$$

which shows that $(\lambda-L)^{-1}$ is a bounded operator on $L_1(G,m)$ and so $\lambda\notin\operatorname{Sp}_1(L)$. This completes the proof of Theorem 1.

THEOREM 2. *The algebra $\mathbf{A}$ is symmetric.*

Proof. First we show that if $\lambda<0$, then $\operatorname{Sp}_1T(k_\lambda)$ is real. In fact, suppose $\operatorname{Im}\lambda\neq0$. Then, if $\nu=\lambda-\mu^{-1}$, $\operatorname{Im}\nu\neq0$ and so

$$[\mu I-T(k_\lambda)]^{-1}=\mu^{-1}-\mu^{-2}(\nu I-L)^{-1},$$

whence, by Theorem 1,

$$[\mu I-T(k_\lambda)]^{-1}$$

is a bounded operator on $L_1(G,m)$.

Since for $\lambda < 0$ we have

$$\|\lambda k_\lambda\|_1 = 1$$

(cf. e.g. [2]) an easy application of Hille–Yoshida theorem (cf. e.g. [2]) shows that

$$\lim_{n\to-\infty} \left\| \left(\frac{n}{t} k_{n/t}\right)^{*-n} * f - p_t * f \right\|_1 = 0 \tag{11}$$

for all f in $L_1(G, m)$. Therefore, since $\{p_t\}_{t\to 0}$ is a bounded approximate identity in $L_1(G, m)$,

$$\left\{\left(\frac{n}{t} k_{n/t}\right)^{*-n}\right\}_{t\to 0,\ n\to-\infty}$$

is an approximate identity in $L_1(G, m)$. Putting $f = k_\lambda$, $\lambda < 0$, in (11), we see that the real algebra generated by the k_λ's, $\lambda < 0$, is dense in the real algebra generated by p_t, $t > 0$. Thus we see that $\mathrm{Sp}_1\, p_t$ is real and so, since $p_t = p_{t/2} * p_{t/2}$, it is non-negative. From this we easily infer that for each f in A $\mathrm{Sp}_1 f^* * f$ is real non-negative, which completes the proof of Theorem 2.

Since G is amenable, Proposition 5.3 of [2] thus yields our main result.

Corollary. *If L is the Laplacian on G defined by* (1), *then*

$$\mathrm{Sp}_p L = \mathrm{Sp}_2 L \quad \textit{for all } 1 \leqslant p < \infty.$$

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On Kadec–Klee norms on Banach spaces

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Abstract. If E is a non-reflexive Banach space with a Kadec–Klee norm, then "many" (in particular, if E^* is separable, then all) proper total subspaces V of E^* have characteristic $r(V) < 1$. Application: Every non-reflexive Banach space E admits an equivalent norm for which there exists no projection of norm 1 of E^{**} onto $\varkappa(E)$.

0. Definition of Kadec–Klee norms. Terminology and notations.

In the present paper we shall study some properties and give some applications of Kadec–Klee norms, which are defined as follows:

Definition 0.1. Let E be a Banach space and W a separable subspace (by *subspace* we shall always mean: norm-closed linear subspace) of the conjugate space E^*. We shall say that the norm of E is a *Kadec–Klee norm* (or, briefly, a (KK)-*norm*) *with respect to* W if for every net $\{g_d\}_{d\in D} \subset E^*$ and every $g \in W$ such that $g_d \xrightarrow{w^*} g$ and $\|g_d\| \to \|g\|$ we have $\|g_d - g\| \to 0$.

In the particular case when E^* is separable, a (KK)-norm with respect to $W = E^*$ will be simply called a (KK)-*norm* (in this case, clearly, the above nets can be replaced by sequences).

M. I. Kadec [5] and V. Klee [7] have proved that every Banach space E with separable conjugate space admits an equivalent (KK)-norm (for other proofs see also [6], [9]). More generally, W. J. Davis and W. B. Johnson have proved ([2], lemma 1) that if E is a Banach space and W a separable subspace of E^*, then E admits a (KK)-norm with respect to W, equivalent to the initial norm (actually, their result is slightly stronger, but we shall use only this version of it).

We recall (see [3]) that the *characteristic* of a subspace V of a conjugate Banach space E^* is the greatest number $r = r(V)$ such that the

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