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On functors from compact pairs to Banach algebras

by

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Abstract. Three natural properties of the functor $\mathscr{C}$ which carries a compact pair (X, A) to the Banach algebra $\mathscr{C}(X, A)$ of continuous scalar-valued functions on X vanishing on A are shown to characterize it.

A compact pair is an ordered pair (X, A) where X is a compact (Hausdorff) space and A is a closed subset of X. The category of compact pairs, denoted by **Comp Pr**, consists of such pairs and morphisms $f\colon (X, A)\to(Y, B)$ where $f\colon X\to Y$ is a continuous function taking A into B (i.e. $f(A)\subset B$). Given a compact pair (X, A), we let $\mathscr{C}(X, A)$ denote the set of all continuous scalar-valued functions on X that vanish on A. If f is the above morphism in **Comp Pr** we let $f^{\#}\colon \mathscr{C}(Y, B)\to\mathscr{C}(X, A)$ be the function given by $f^{\#}(a) = a\circ f$. One easily verifies that the maps $(X, A)\mapsto\mathscr{C}(X, A)$ and $f\mapsto f^{\#}$ constitute a contravariant functor (which we denote by $\mathscr{C}$) from **Comp Pr** to any of a number of important categories of functional analysis (e.g. Banach spaces, Banach lattices, Banach algebras). We refer to Semadeni [5] for more details and an extensive bibliography. It is our purpose to display certain natural properties of this functor that serve to characterize it, i.e. any contravariant functor having these properties will be naturally equivalent to $\mathscr{C}$. Similar programs are carried out in [2] and [4]. It is remarkable that the properties singled out by Eilenberg and Steenrod [1] to obtain a cohomology theory are also the properties that essentially serve to characterize $\mathscr{C}$.

Our use of categories is only to provide a convenient language, anyone familiar with the basic definitions as given, for example, in Semadeni [5] will have no difficulty reading this paper. The category of compact spaces is denoted by **Comp**, it is a full subcategory of **Comp Pr** where we identify the compact space X with the pair $(X, \emptyset)$. We also do not distinguish between $\mathscr{C}(X, \emptyset)$ and $\mathscr{C}(X)$. **Ban Alg** is the category of commutative,

* Results in this paper form a portion of my doctoral dissertation written under the guidance of Professor Ky Fan. I wish to thank him for his helpful advice.

semi-simple, complex Banach algebras. The morphisms in **Ban Alg** are the algebraic homomorphisms (automatically continuous) with the added condition that if $\mathfrak{A}$ and $\mathfrak{B}$ have identities, they are preserved by any $\eta \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$. The sets of morphisms in **Comp Pr** and **Ban Alg** will be denoted by $\mathrm{Hom}(-, -)$, since X, Y, A, B, etc. will always denote compact spaces while Banach algebras will be denoted by $\mathfrak{A}, \mathfrak{B}$, etc., no confusion will arise. Given a functor $\mathscr{F}$: **Comp Pr**$\to$**Ban Alg**, the functor obtained by restricting $\mathscr{F}$ to **Comp** will be denoted by $\mathscr{F}|$**Comp**.

It is easily verified that $\mathscr{C}$: **Comp Pr**$\to$**Ban Alg** has the following properties:

I. $\mathscr{C}|$**Comp** *is full, that is, the map* $f \to f^{\#}$ *taking* $\mathrm{Hom}(X, Y)$ *to* $\mathrm{Hom}(\mathscr{C}(Y), \mathscr{C}(X))$ *is onto.*

II. (Exactness) *Given a compact pair* (X, A), *if*

$$(*) \qquad (A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$$

are the natural maps, then the induced sequence

$$0 \leftarrow \mathscr{C}(A) \xleftarrow{i^{\#}} \mathscr{C}(X) \xleftarrow{j^{\#}} \mathscr{C}(X, A) \leftarrow 0$$

is exact.

III. (Excision) *Given a compact pair* (X, A) *and* $U \subset A$ *open in* X, *if*

$$(X \setminus U, A \setminus U) \xrightarrow{i} (X, A)$$

is the natural map, then

$$\mathscr{C}(X \setminus U, A \setminus U) \xleftarrow{i^{\#}} \mathscr{C}(X, A)$$

is an isomorphism.

Any functor naturally equivalent to $\mathscr{C}$ will also have these properties. Conversely, any functor $\mathscr{F}$ having these properties (together with a normalization property, $\mathscr{F}(\Delta) = \mathscr{C}(\Delta)$ where $\Delta = \{z \in \boldsymbol{C}: |z| \leqslant 1\}$) will be naturally equivalent to $\mathscr{C}$. This is proven below. We remark that some normalization property is unavoidable since the functor taking each compact pair to the trivial Banach algebra 0 has properties I, II and III.

THEOREM. *Let* $\mathscr{F}$: **Comp Pr**$\to$**Ban Alg** *be a contravariant functor that is full on* **Comp** *and satisfies the exactness and excision properties above. If, in addition,* $\mathscr{F}(\Delta) = \mathscr{C}(\Delta)$ *then* $\mathscr{F}$ *is naturally equivalent to* $\mathscr{C}$.

The proof will be carried out in a sequence of lemmas, the natural equivalence first being established between $\mathscr{F}|$**Comp** and $\mathscr{C}|$**Comp** then extended via exactness. Given a morphism f in **Comp Pr**, $\mathscr{F}(f)$ will be denoted by f^*.

The first lemma extends exactness to triples and is proven in the standard fashion ([1], p. 25).

LEMMA 1. *Let* X *be compact and let* $A \supset B$ *be closed in* X. *If*

$$(A, B) \xrightarrow{i} (X, B) \xrightarrow{j} (X, A)$$

are the natural maps, then the induced sequence

$$0 \leftarrow \mathscr{F}(A, B) \xleftarrow{i^*} \mathscr{F}(X, B) \xleftarrow{j^*} \mathscr{F}(X, A) \leftarrow 0$$

is exact.

For what follows, $\{p\}$ will be a fixed singleton space. Observe that since the unique morphism $\{x\} \to \{p\}$ is an isomorphism in **Comp Pr**, $\mathscr{F}(\{x\})$ and $\mathscr{F}(\{p\})$ are isomorphic.

LEMMA 2. $\mathscr{F}(\{p\}) \cong \boldsymbol{C}$.

Proof. Fix a point $z \in \Delta$, let

$$(\{z\}, \emptyset) \xrightarrow{i} (\Delta, \emptyset) \xrightarrow{j} (\Delta, \{z\})$$

denote the natural maps. The induced sequence

$$0 \leftarrow \mathscr{F}(\{z\}) \xleftarrow{i^*} \mathscr{C}(\Delta) \xleftarrow{j^*} \mathscr{F}(\Delta, \{z\}) \leftarrow 0$$

is exact embedding $\mathscr{F}(\Delta, \{z\})$ as a closed ideal of the form $\mathscr{C}(\Delta, K)$ in $\mathscr{C}(\Delta)$, K closed in Δ. Hence $\mathscr{F}(\{z\}) \cong \mathscr{C}(\Delta)/\mathscr{C}(\Delta, K) \cong \mathscr{C}(K)$. Since $\mathrm{Hom}(\mathscr{F}(\{z\}), \mathscr{F}(\{z\}))$ is a singleton ($\mathscr{F}|$**Comp** is full), so also is $\mathrm{Hom}(\mathscr{C}(K), \mathscr{C}(K))$ implying that K is a singleton too. End of proof. ∎

There is one isomorphism from $\mathscr{F}(\{p\})$ to $\boldsymbol{C}$; denote it by ν.

LEMMA 3. *Given a compact pair* (X, A), $X = A$ *if and only if* $\mathscr{F}(X, A) = 0$.

Proof. Necessity follows from exactness. On the other hand, suppose $A \neq X$ and choose $x \in X \setminus A$. The diagram

$$\begin{array}{ccc} (A \cup \{x\}, A) & \xrightarrow{i} (X, A) \xrightarrow{j} & (X, A \cup \{x\}) \\ \uparrow k & & \\ (\{x\}, \varphi) & & \end{array}$$

consisting of natural maps induces

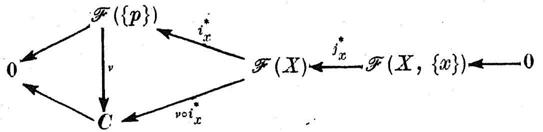
$$\begin{array}{c} 0 \leftarrow \mathscr{F}(A \cup \{x\}, A) \xleftarrow{i^*} \mathscr{F}(X, A) \xleftarrow{j^*} \mathscr{F}(X, A \cup \{x\}) \leftarrow 0 \\ \downarrow k^* \\ \mathscr{F}(\{x\}). \end{array}$$

The row is exact by Lemma 1 and k^* is an isomorphism by excision. Since $\mathscr{F}(\{x\}) \cong \boldsymbol{C}$, it follows that $\mathscr{F}(X, A) \neq 0$. ∎

The following notation will be employed. Given a compact pair (X, A) and natural maps i, j as in $(*)$, the closed ideal $j^*[\mathscr{F}(X, A)]$ in $\mathscr{F}(X)$ will be denoted by $I(X, A)$. When A reduces to a singleton $\{x\}$ we let

$$(\{p\}, \emptyset) \xrightarrow{i_x} (X, \emptyset) \xrightarrow{j_x} (X, \{x\})$$

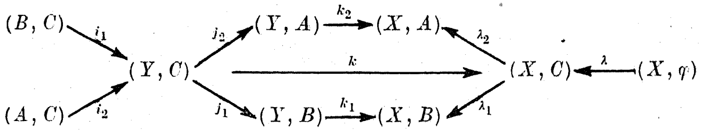
denote the obvious maps (fixing the initial space). The induced maps are still exact, in fact in the diagram



both "rows" are exact and the map $i_x^* \mapsto \nu \circ i_x^*$ is a bijection between $\mathrm{Hom}(\mathscr{F}(X), \mathscr{F}(\{p\}))$ and $\mathrm{Hom}(\mathscr{F}(X), C)$. (It is onto because $\mathscr{F}|$Comp is full.)

LEMMA 4. *Let X be compact and let A, B be closed in X. Then $A \subset B$ if and only if $I(X, A) \supset I(X, B)$.*

Proof. Let $Y = A \cup B$ and $C = A \cap B$, the proof is based on the diagram in **Ban Alg** induced from



which consists of natural maps in **Comp Pr**. Note that

$$I(X, A) = \lambda^* \circ \lambda_2^* (\mathscr{F}(X, A)) \quad \text{and} \quad I(X, B) = \lambda^* \circ \lambda_1^* (\mathscr{F}(X, B)).$$

If $A \subset B$ then $C = A$, λ_2^* is the identity map and $I(X, A) \supset I(X, B)$ follows easily. Assume now that $I(X, A) \supset I(X, B)$. Since λ^* is injective, we have $\lambda_2^*(\mathscr{F}(X, A)) \supset \lambda_1^*(\mathscr{F}(X, B))$. Using the fact that both k_1^* and k_2^* are surjective we have

$$j_1^*(\mathscr{F}(Y, B)) = j_1^* \circ k_1^* (\mathscr{F}(X, B)) = k^* \circ \lambda_1^* (\mathscr{F}(X, B)) \subset k^* \circ \lambda_2^* (\mathscr{F}(X, A))$$
$$= j_2^* \circ k_2^* (\mathscr{F}(X, A)) = j_2^* (\mathscr{F}(Y, A)).$$

Therefore,

$$i_2^* \circ j_1^* (\mathscr{F}(Y, B)) \subset i_2^* \circ j_2^* (\mathscr{F}(Y, A)) = 0$$

(by Lemma 1). But $i_2^* \circ j_1^*$ is an isomorphism by excision so that $\mathscr{F}(Y, B) = 0$, implying that $Y = B$ by Lemma 3. ■

COROLLARY. *The map $x \mapsto i_x^*$ is a bijection between X and* $\mathrm{Hom}(\mathscr{F}(X), \mathscr{F}(\{p\}))$.

In view of the remarks preceding Lemma 4 we obtain a bijection $x \mapsto \nu \circ i_x^*$ between X and $\mathrm{Hom}(\mathscr{F}(X), C)$, we identify these two sets. Let $\mathscr{T}_\omega$ denote the Gelfand topology on X, i.e. the weakest topology making all maps $\hat{a}: X \to C$ given by $\hat{a}(x) = \nu \circ i_x^*(a)$ continuous, where a ranges over $\mathscr{F}(X)$. This is a locally compact topology on X and each $\hat{a}$ vanishes at infinity. We wish to show that $\mathscr{T}_\omega$ coincides with the original topology $\mathscr{T}$ on X. For Lemmas 5 and 6, $X[\mathscr{T}]$ is a fixed compact space.

LEMMA 5. *$\mathscr{T}_\omega$ is a compact topology on X.*

Proof. This will follow if we can show that $\widehat{\mathscr{F}(X)}$ contains the constant functions. Let $j: X \to \{p\}$ be the unique morphism in **Comp Pr** and $e = j^* \circ \nu^{-1}(1)$; we claim that $\hat{e}$ is the function identically equal to 1 on X. In fact, given $x \in X$, $j \circ i_x = Id_{\{p\}}$ so $i_x^* \circ j^* = Id_{\mathscr{F}(\{p\})}$ and $\hat{e}(x) = \nu \circ i_x^*(e) = \nu \circ i_x^* \circ j^* \circ \nu^{-1}(1) = \nu \circ \nu^{-1}(1) = 1$.

LEMMA 6. *$\mathscr{T} = \mathscr{T}_\omega$, hence given $a \in \mathscr{F}(X)$, $\hat{a} \in \mathscr{C}(X)$.*

Proof. Since both are compact topologies, we need only to show that $\mathscr{T} \subset \mathscr{T}_\omega$. Let S be a $\mathscr{T}$-closed set in X and let $\bar{S}$ denote its $\mathscr{T}_\omega$-closure; we prove that $\bar{S} = S$. Let $x \in \bar{S}$.

If $a \in \mathscr{F}(X)$ and $\hat{a}|S = 0$, then $\hat{a}(x) = 0$ by continuity. This shows that

$$I(X, \{x\}) \supset \bigcap_{s \in S} I(X, \{s\}).$$

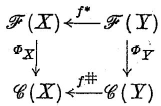
But Lemma 4 implies that $I(X, \{s\}) \supset I(X, S)$ for each $s \in S$ and so

$$I(X, \{x\}) \supset I(X, S).$$

Once more Lemma 4 applies to allow us to assert that $x \in S$. ■

Given X compact, let $\Phi_X: \mathscr{F}(X) \to \mathscr{C}(X)$ be the map $\Phi_X(a) = a$ (essentially the Gelfand transform). The family $\Phi = \{\Phi_X: X \text{ compact}\}$ will be shown to be a natural equivalence from $\mathscr{F}|$**Comp** to $\mathscr{C}|$**Comp**. We begin by showing that it constitutes a natural transformation.

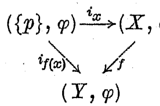
Let $f: X \to Y$ be a morphism in **Comp**; we wish to show that the diagram



commutes. In other words, given $b \in \mathscr{F}(Y)$ and $x \in X$,

$$[f^\# \circ \Phi_Y(b)](x) = [\Phi_X \circ f^*(b)](x).$$

This follows because



commutes, so $i_x^* \circ f^* = i_{f(x)}^*$. Thus

$$[f^\# \circ \Phi_Y(b)](x) = \Phi_Y(b)(f(x)) = \hat{b}(f(x)) = \nu \circ i_{f(x)}^*(b)$$
$$= \nu \circ i_x^* \circ f^*(b) = \widehat{f^*(b)} = [\Phi_X \circ f^*(b)](x).$$

THEOREM. *The family of maps* $\Phi = \{\Phi_X: X \text{ compact}\}$ *is a natural equivalence from* $\mathscr{F}|\mathbf{Comp}$ *to* $\mathscr{C}|\mathbf{Comp}$.

Proof. We need only show that each morphism Φ_X is onto (it is one-to-one by semi-simplicity). Normality (i.e. $\mathscr{F}(\Delta) = \mathscr{C}(\Delta)$) shows that Φ_Δ is onto. If K is a closed subset of Δ and $i: K \to \Delta$ is the natural map then

$$\begin{array}{ccc} \mathscr{F}(K) & \xleftarrow{i^*} & \mathscr{F}(\Delta) \\ {\scriptstyle \Phi_K}\downarrow & & \downarrow{\scriptstyle \Phi_\Delta} \\ \mathscr{C}(K) & \xleftarrow{i^\#} & \mathscr{C}(\Delta) \end{array}$$

commutes and both $i^\#$ and Φ_Δ are onto implying that Φ_K is also. Because any compact subset of C is homeomorphic to a subset of Δ, Φ_K will be onto for every $K \subset C$.

Now let X be an arbitrary compact space and $\alpha \in \mathscr{C}(X)$. Define $f: X \to \alpha(X)$ as $f(x) = \alpha(x)$ for all $x \in X$. The induced diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \xleftarrow{f^*} & \mathscr{F}(\alpha(X)) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_{\alpha(X)}} \\ \mathscr{C}(X) & \xleftarrow{f^\#} & \mathscr{C}(\alpha(X)) \end{array}$$

commutes. Since $\Phi_{\alpha(X)}$ is onto, there is some $b \in \mathscr{F}(\alpha(X))$ such that $\Phi_{\alpha(X)}(b)(z) = z$ for all $z \in \alpha(X)$. We claim that $\Phi_X(f^*(b)) = \alpha$. In fact, given $x \in X$,

$$\begin{aligned} [\Phi_X \circ f^*(b)](x) &= [f^\# \circ \Phi_{\alpha(X)}(b)](x) \\ &= \Phi_{\alpha(X)}(b)(f(x)) = f(x) = \alpha(x). \ \blacksquare \end{aligned}$$

We conclude with the theorem that extends Φ to a natural equivalence from $\mathscr{F}$ to $\mathscr{C}$. Since this result is of independent interest, we state it in a more general context.

THEOREM. *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be contravariant functors from* $\mathbf{Comp\ Pr}$ *to* $\mathbf{Ban\ Alg}$. *Let* $\Phi = \{\Phi_X: X \text{ compact}\}$ *be a natural transformation from* $\mathscr{F}|\mathbf{Comp}$ *to* $\mathscr{G}|\mathbf{Comp}$. *If both* $\mathscr{F}$ *and* $\mathscr{G}$ *satisfy the exactness condition then* Φ *has a unique extension to a natural transformation* $\Psi = \{\Phi_{(X,A)}: (X, A)$ a compact pair$\}$ *from* $\mathscr{F}$ *to* $\mathscr{G}$, *i.e.* $\Phi_{(X,\emptyset)} = \Phi_X$. *If* Φ *is a natural equivalence, so also is* Ψ.

Proof. Given a compact pair (X, A) and natural maps i, j as in (*), the induced diagram

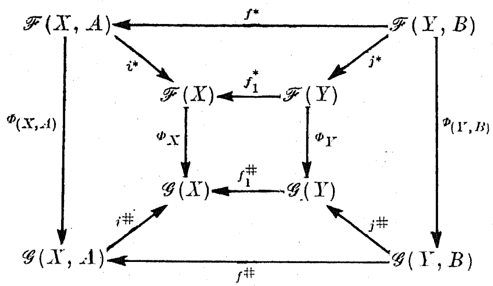
$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathscr{F}(A, \emptyset) & \xleftarrow{i^*} & \mathscr{F}(X, \emptyset) & \xleftarrow{j^*} & \mathscr{F}(X, A) & \leftarrow & 0 \\ & & \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_X} & & & & \\ 0 & \leftarrow & \mathscr{G}(A, \emptyset) & \xleftarrow{i^\#} & \mathscr{G}(X, \emptyset) & \xleftarrow{j^\#} & \mathscr{G}(X, A) & \leftarrow & 0 \end{array}$$

commutes and the rows are exact. (We let $i^\#$ and $j^\#$ denote $\mathscr{G}(i)$ and $\mathscr{G}(j)$.) It follows easily that there exists a unique morphism $\Phi_{(X,A)}: \mathscr{F}(X, A) \to \mathscr{G}(X, A)$ making the resulting diagram commute. Moreover, if both Φ_A and Φ_X are isomorphisms so also is $\Phi_{(X,A)}$. (The five lemma, [3], p. 201, may be applied.) This proves uniqueness and it remains to show that $\Psi = \{\Phi_{(X,A)}\}$ has the commutativity property of a natural transformation.

Thus, let $f: (X, A) \to (Y, B)$ be a morphism in $\mathbf{Comp\ Pr}$. The diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{f} & (Y, B) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ (X, \emptyset) & \xrightarrow{f_1} & (Y, \emptyset) \end{array}$$

where i and j are the natural maps and $f_1(x) = f(x)$ for all $x \in X$ commutes. Therefore in the diagram below all "rectangles" will commute except possibly the outer one. It is this rectangle that we wish to show is commutative.



Using each rectangle in turn one easily shows that $i^\# \circ \Phi_{(X,A)} \circ f^* = i^\# \circ f^\# \circ \Phi_{(Y,B)}$. Because $i^\#$ is injective we conclude that

$$\Phi_{(X,A)} \circ f^* = f^\# \circ \Phi_{(Y,B)}. \ \blacksquare$$

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On the spectrum of the Laplacian on the affine group of the real line

by

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Abstract. If G is the affine group of the real line and X^2+Y^2 is the Laplacian on it regarded as a densely defined operator on all $L_p(G)$, then it has the same spectrum for all p, $1 \leqslant p < \infty$.

Let G be a Lie group and LG the Lie algebra of G. Let $X_1, \ldots, X_n$ be a basis of LG and let

$$L = -X_1^2 - \ldots - X_n^2.$$

If m is a left-invariant Haar measure on G, then L is a densely defined operator on each of $L_p(G, m)$ which is non-negative, essentially self-adjoint on $L_2(G,m)$.

Let

$$\mathrm{Sp}_p L = \{\lambda \in \boldsymbol{C}\colon (\lambda I - L)^{-1} \text{ is bounded on } L_p(G, m)\}^c.$$

It is well known that $-L$ is the infinitesimal generator of a one-parameter semigroup of convolution operators whose kernels p_t, $t > 0$, are $L_1(G, m)$ functions. In [2] a commutative Banach algebra $\boldsymbol{A}$, defined as the $L_1(G, m)$ closure of $\mathrm{lin}\{p_t\colon t > 0\}$, is studied. If G is of polynomial growth (e.g. G is a compact extension of nilpotent Lie group), then $\boldsymbol{A}$ is symmetric and hence

$$(*) \qquad \mathrm{Sp}_p L = \mathrm{Sp}_2 L \quad \text{for all} \quad 1 \leqslant p < \infty;$$

cf. [2] and [3].

In this note we consider a Lie group which is not of polynomial growth; namely, the group of affine transformations of the real line. As it has been recently proved by R. Aravamudhan [1], the whole $L_1(G, m)$ is not symmetric.(1) However, as we shall show here, the algebra $\boldsymbol{A}$ is symmetric and equality $(*)$ holds also for this group. The method of the proof used here is quite different from the ones of [2] and [3]. First we establish

(1) *Added in proof*: Aravamudhan's proof appears to be wrong, thus the question about symmetry of $L_1(G, m)$ remains open.