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A unified approach to Riesz type representation theorems

by

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Abstract. We establish abstract versions of the Riesz representation theorem. Necessary and sufficient conditions for the existence of regular finitely additive, σ-additive and τ-additive representing measures are found. A methodological simplification is obtained by constructing the measures directly, rather than *via* a preliminary extension of the linear functional. Thus our approach is in agreement with the viewpoints of Alexandroff rather than with those of Bourbaki. We are able to easily deduce the Daniell extension theorem as well as numerous topological representation theorems such as those developed by Radon, Markoff, Alexandroff, Hewitt, LeCam, Mařík and Varadarajan. Indeed, these results are sometimes strengthened. Our method is based on the theory developed by the second author; hopefully, our results demonstrate the usefulness of this theory.

1. A common problem in Functional Analysis is whether a given bounded linear functional defined on a vector lattice of real valued functions is representable as an integral with respect to some suitable regular measure. By well-known techniques this problem can be reduced to the following situation:

On a set X there is given a convex cone $\mathscr{C}$ of non-negative real functions, closed under the finite lattice operations and containing the zero function. A non-negative, monotone, linear functional T is defined on $\mathscr{C}$. That is, our basic assumptions are:

A1. *$\mathscr{C}$ is a $(0, \bigvee f, \bigwedge f)$ convex cone in $[0, \infty[^X$;*

A2. *$T\colon \mathscr{C} \to [0, \infty[$,*
$T(a_1h_1 + a_2h_2) = a_1Th_1 + a_2Th_2$ for $a_1, a_2 \geqslant 0$ and $h_1, h_2 \in \mathscr{C}$, $h_1 \leqslant h_2$ and $h_1, h_2 \in \mathscr{C}$ implies that $Th_1 \leqslant Th_2$.

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For the positive cone of a vector lattice of functions we also have closure under the operation $\setminus$ defined by

$$f\setminus g = (f-g)^+ (= f - f\wedge g).$$

Another frequently satisfied condition is Stone's condition that $h\wedge 1 \in \mathscr{C}$ for all $h \in \mathscr{C}$. Thus we consider:

A3. *$\mathscr{C}$ is closed under $\setminus$ and satisfies Stone's condition.*

A3 is not needed to prove one of the basic results, but if it is satisfied then the subsequent analysis is greatly simplified. Notice that if A3 is not satisfied then the monotonicity of T does not follow from its non-negativity alone.

The regularity properties we desire for our representing measures are expressible in terms of a paving $\mathscr{K}$ of subsets of X. We shall always assume that this $\mathscr{K}$ is closed under finite unions and intersections and that it contains the empty set:

A4. *$\mathscr{K}$ is a $(\emptyset, \bigcup f, \bigcap f)$ paving.*

Following Topsøe [11], we define the associated pavings

$$\mathscr{F}(\mathscr{K}) = \{F\colon K\cap F \in \mathscr{K} \text{ for all } K\in\mathscr{K}\},$$

$$\mathscr{G}(\mathscr{K}) = \{G\colon K\setminus G \in \mathscr{K} \text{ for all } K\in\mathscr{K}\},$$

$$\mathscr{A}(\mathscr{K}) = \text{the field spanned by } \mathscr{G}(\mathscr{K}),$$

$$\mathscr{B}(\mathscr{K}) = \text{the } \sigma\text{-field spanned by } \mathscr{G}(\mathscr{K}).$$

A non-negative set function μ, on a domain containing $\mathscr{K}$, is said to be *$\mathscr{K}$-regular* if $\mu K < \infty$ for all $K \in \mathscr{K}$ and

(1) $\mu(A) = \sup\{\mu K\colon K \subseteq A,\ K\in\mathscr{K}\}$ for every A in the domain of μ.

If μ is defined on $\mathscr{A}(\mathscr{K})$, is finitely additive, and $\mathscr{K}$-regular, it is called a *$\mathscr{K}$-regular finitely additive measure*. If $\mathscr{K}$ is also closed under countable intersections $(\bigcap c)$ and μ is defined on $\mathscr{B}(\mathscr{K})$, is countably additive, and satisfies (1) then it is called a *$\mathscr{K}$-regular σ-additive measure*. Finally, if $\mathscr{K}$ is also closed under arbitrary intersections $(\bigcap a)$, a $\mathscr{K}$-regular σ-additive measure is said to be a *$\mathscr{K}$-regular τ-additive measure* if it satisfies the further smoothness condition: *for every downward filtering family $\{F_a\}$ of $\mathscr{F}(\mathscr{K})$-sets, for which $\mu F_a < \infty$ for some a, we have*

$$\mu\left(\textstyle\bigcap_a F_a\right) = \inf_a \mu(F_a).$$

The following theorem on the construction of such measures, starting from a μ defined only on $\mathscr{K}$, is basic to our whole method.

THEOREM A (cf. Topsøe [11]; Theorem 2.2, Lemma 2.3, Theorem 4.1.). *Let $\mathscr{K}$ be a $(\emptyset, \bigcup f, \bigcap f)$ paving and μ a map from $\mathscr{K}$ into $[0, \infty[$ such that, for every pair K_1, K_2 in $\mathscr{K}$ with $K_1 \subseteq K_2$, we have*

(2) $$\mu K_1 + \sup\{\mu K\colon K \subseteq K_2\setminus K_1,\ K\in\mathscr{K}\} = \mu K_2.$$

Define μ_* on 2^X by

(3) $$\mu_* E = \sup\{\mu K\colon K\subseteq E,\ K\in\mathscr{K}\}.$$

Then

(i) *The restriction μ_* to $\mathscr{A}(\mathscr{K})$ is an extension of μ to a $\mathscr{K}$-regular finitely additive measure.*

(ii) *If $\mathscr{K}$ is closed under $\bigcap c$ and μ is σ-smooth at $\emptyset$ (i.e. for any sequence in $\mathscr{K}$, $K_n\downarrow\emptyset$ implies $\mu K_n \downarrow 0$), then the restriction of μ_* to $\mathscr{B}(\mathscr{K})$ is an extension of μ to a $\mathscr{K}$-regular σ-additive measure.*

(iii) *If $\mathscr{K}$ is closed under $\bigcap a$ and μ is τ-smooth at $\emptyset$ (i.e. for any family in $\mathscr{K}$, $K_a\downarrow\emptyset$ implies $\mu K_a\downarrow 0$) then the restriction of μ_* to $\mathscr{B}(\mathscr{K})$ is an extension of μ to a $\mathscr{K}$-regular τ-additive measure.*

In each case, the extension will also be denoted by μ.

The history of this result goes back at least to Alexandroff ([1], Theorem 3.2).

We shall apply Theorem A to the set function μ defined on $\mathscr{K}$ by

(4) $$\mu K = \inf\{Th\colon h \geqslant 1_K,\ h\in\mathscr{C}\}.$$

To ensure that this μ satisfies the conditions of Theorem A we of course require some further assumptions about the relationship between $\mathscr{C}$, $\mathscr{K}$ and T. Specifically, we shall work with a type of lower semi-continuity requirement

A5. *For every $h\in\mathscr{C}$ and $a > 0$, $\{h\leqslant a\}\in\mathscr{F}(\mathscr{K})$,*

and also a separation assumption

A6. *If K_1 and K_2 are disjoint $\mathscr{K}$ sets and $\varepsilon > 0$, then there are $\mathscr{C}$ functions $h_1 \geqslant 1_{K_1}$, $h_2 \geqslant 1_{K_2}$ for which $T(h_1\wedge h_2) < \varepsilon$.*

Observe that A6 implies that the μ defined by (4) is finite valued.

In applications though we often have another form of separation

A6′. *If K_1 and K_2 are disjoint $\mathscr{K}$ sets, then there is an $h\in\mathscr{C}$ taking the value 1 on K_1 and 0 on K_2.*

When combined with A3, A6′ leads to the stronger form of A6: *if K_1 and K_2 are disjoint $\mathscr{K}$ sets, then there are $\mathscr{C}$ functions $h_i \geqslant 1_{K_i}$ for which $h_1\wedge h_2 = 0$.* [Choose h' taking the value 1 on $K_1\cup K_2$ and take h as in A6′. Then $h_1 = h\setminus(h'\setminus h)$ and $h_2 = (h'\setminus h)\setminus h$ are the required functions.]

We prefer however to retain A6 since it is this weaker form of separation which allows us to obtain the Daniell extension theorem as a corollary to our results.

Whenever Theorem A is applicable we define, for a *simple function* $k = \sum_1^n a_i 1_{A_i}$ (with the A_i's in the domain of definition of μ),

$$\mu(k) = \sum_1^n a_i \mu A_i. \tag{5}$$

Further, we define, for every $f \in [0, \infty]^X$, the *inner integral*

$$\mu_*(f) = \sup\{\mu(k)\colon k \leqslant f,\ k \text{ a simple function}\}. \tag{6}$$

Notice that for simple functions $\mu_*(k) = \mu(k)$, and also that for any subset E of X, $\mu_*(1_E) = \mu_*(E)$ (as defined by (3)). Since μ is $\mathscr{K}$-regular we have, for every $f \in [0, \infty]^X$, the more useful expression

$$\mu_*(f) = \sup\Big\{\sum_1^n a_i \mu K_i\colon \sum_1^n a_i 1_{K_i} \leqslant f, \text{ the } K_i \text{ disjoint } \mathscr{K} \text{ sets}\Big\}. \tag{7}$$

In general, μ_* is only superadditive and positively homogeneous on $\mathscr{C}$. In those cases where it is in fact linear on $\mathscr{C}$ it is customary to write $\mu(h)$ or $\int h d\mu$ for $\mu_*(h)$. We shall say that μ is *dominated* by T if $\mu_*(h) \leqslant Th$ for all $h \in \mathscr{C}$, and that μ is a *representation* of T if $\mu_*(h) = Th$ for all $h \in \mathscr{C}$. Of course, when μ is a representation of T, it must be linear on $\mathscr{C}$.

In Section 2 we establish the existence of the largest $\mathscr{K}$-regular finitely additive measure dominated by T, and then deduce necessary and sufficient conditions for this to be a representation. From these the Markoff and Alexandroff representation theorems follow immediately.

In Section 3 we extend these results to find necessary and sufficient conditions for representations in terms of $\mathscr{K}$-regular σ-additive and τ-additive measures. From these we deduce the Daniell extension theorem, as well as a number of topological representation theorems associated with the names: Alexandroff, LeCam, Hewitt, Mařík, Radon and Varadarajan. In fact, we obtain stronger versions of some of these results.

For Section 3, further closure assumptions have to be made on $\mathscr{K}$. In Section 4 we try to avoid this. The difficulties involved are illustrated by simple examples.

All of the results in this paper can be generalized to the case where T is allowed to take the value $+\infty$. Apart from obvious modifications we only need the extra assumptions that for every K there is an $h \geqslant 1_K$ for which $Th < \infty$, and also that Th is the supremum of Th' with $h' \leqslant h$, $Th' < \infty$.

In particular, Examples 4 and 5 can be extended to the case of Radon measures on general topological spaces without the restriction of total finiteness.

For historical background to these results we refer the reader to Batt [2], Bourbaki [3] and Dunford and Schwartz [4].

2. Henceforth $h, h_1, \ldots$ will denote functions in $\mathscr{C}$, and $K, K_1, \ldots$ sets in $\mathscr{K}$.

THEOREM 1. *Assume that* A1, A2, A4, A5 *and* A6 *hold.*

Then there exists a largest $\mathscr{K}$-regular finitely additive measure μ dominated by T, and this finitely additive measure is determined by the values of μ on $\mathscr{K}$, given by (4).

Proof. Let μ be the set function on $\mathscr{K}$ defined by (4). Clearly, μ is non-negative finite valued, monotone and subadditive $(\mu(K_1 \cup K_2) \leqslant \mu K_1 + \mu K_2)$. To prove that μ is additive, let K_1, K_2 with $K_1 \cap K_2 = \emptyset$ and $\varepsilon > 0$ be given. Choose $h \geqslant 1_{K_1 \cup K_2}$ such that $\mu(K_1 \cup K_2) \geqslant Th - \varepsilon$. Then choose, according to A6, $h_1 \geqslant 1_{K_1}$ and $h_2 \geqslant 1_{K_2}$ such that $T(h_1 \wedge h_2) < \varepsilon$. We may also assume that $h_1 \leqslant h$, $h_2 \leqslant h$. Then

$$\begin{aligned} \mu K_1 + \mu K_2 \leqslant Th_1 + Th_2 &= T(h_1 + h_2) \\ &= T(h_1 \vee h_2 + h_1 \wedge h_2) \\ &= T(h_1 \vee h_2) + T(h_1 \wedge h_2) \\ &\leqslant Th + \varepsilon \leqslant \mu(K_1 \cup K_2) + 2\varepsilon, \end{aligned}$$

which proves additivity.

Define μ_* on 2^X as in Theorem A. To enable us to apply Theorem A, we must verify (2). Note that by monotonicity and additivity of μ, "$\leqslant$" holds in (2), so it remains to prove that, for $K_1 \subseteq K_2$,

$$\mu K_1 + \mu_*(K_2 \setminus K_1) \geqslant \mu K_2. \tag{8}$$

To do this, let $\varepsilon > 0$ and choose $h_1 \geqslant 1_{K_1}$ such that $\mu K_1 \geqslant Th_1 - \varepsilon$. For $0 < a < 1$ put $K_a = K_2 \cap \{h_1 \leqslant a\}$. Then $K_a \in \mathscr{K}$ by A5. Since $h \geqslant 1_{K_a}$ implies $h + a^{-1}h_1 \geqslant 1_{K_2}$, we obtain $Th + a^{-1}Th_1 \geqslant \mu K_2$ for all such h and hence $\mu K_a + a^{-1}Th_1 \geqslant \mu K_2$. Since $K_a \subseteq K_2 \setminus K_1$, it follows that $\mu_*(K_2 \setminus K_1) + a^{-1}Th_1 \geqslant \mu K_2$, hence $\mu_*(K_2 \setminus K_1) + Th_1 \geqslant \mu K_2$ and (8) follows since $Th_1 \leqslant \mu K_1 + \varepsilon$.

By Theorem A, μ has an extension to a $\mathscr{K}$-regular finitely additive measure which we also denote by μ.

In order to prove that T dominates μ, let $h \in \mathscr{C}$ be given. According to (7), it suffices to show that if $\sum_1^n a_i 1_{K_i} \leqslant h$ and if the K_i's are disjoint, then

$$\sum_1^n a_i \mu K_i \leqslant Th. \tag{9}$$

By A6, we can, to a given $\varepsilon > 0$, find h_{ij}, $i, j = 1, 2, \ldots, n$, $i \neq j$, such that, for all these i, j,

$$h_{ij} \geqslant 1_{K_i}, \quad T(h_{ij} \wedge h_{ji}) < \varepsilon.$$

Now define

$$h_i = h \wedge a_i \bigwedge_{j\neq i} h_{ij}; \qquad i = 1, 2, \ldots, n.$$

Notice that $h_i \geqslant a_i 1_{K_i}$ so that $Th_i \geqslant a_i \mu K_i$. Also $\bigvee_1^n h_i \leqslant h$ and $h_i \wedge h_j \leqslant M h_{ij} \wedge h_{ji}$ for $i \neq j$ where $M = \max a_i$. By this, and the general inequality

$$\sum_i h_i \leqslant \bigvee_i h_i + \sum_{i<j} h_i \wedge h_j,$$

we have

$$\sum_i a_i \mu K_i \leqslant \sum_i Th_i = T\Big(\sum_i h_i\Big) \leqslant T\Big(h + M\sum_{i<j} h_{ij}\wedge h_{ji}\Big) \leqslant Th + Mn^2\varepsilon.$$

It follows that (9) holds, and hence T dominates μ.

If ν is any other $\mathscr{K}$-regular finitely additive measure dominated by T, then, for any K,

$$\nu K \leqslant \inf\{\nu_* h\colon\ h \geqslant 1_K\} \leqslant \inf\{Th\colon\ h \geqslant 1_K\} = \mu K.$$

By $\mathscr{K}$-regularity, $\nu A \leqslant \mu A$ for any $A \in \mathscr{A}(\mathscr{K})$ follows. ∎

Remarks. 1. It follows from (4) and A5 that for $K \in \mathscr{K}$

$$\mu K = \inf\{\mu G \mid G \supseteq K,\ G \in \mathscr{G}(\mathscr{K})\}.$$

2. If each function in $\mathscr{C}$ is bounded(¹), then μ_* is additive on $\mathscr{C}$. Let us briefly indicate a proof of this. Given h_1, h_2 and $\varepsilon > 0$, choose $\sum_1^n a_i 1_{A_i} \leqslant h_1 + h_2$ with the A_i's disjoint in $\mathscr{A}(\mathscr{K})$ and $\mu(\bigcup_1^n A_i) < \infty$, such that $\mu_*(h_1+h_2) \leqslant \sum a_i \mu A_i + \varepsilon$. By A5, and the boundedness of h_1, we may assume that, for each i, $M_i - m_i < \varepsilon$ where M_i $[m_i]$ is the supremum [infimum] of h_1 on A_i. Since

$$\sum m_i 1_{A_i} \leqslant h_1, \qquad \sum (a_i - M_i)^+ 1_{A_i} \leqslant h_2,$$

we find that

$$\mu_* h_1 + \mu_* h_2 \geqslant \sum [m_i + (a_i - M_i)^+]\mu A_i \geqslant \mu_*(h_1+h_2) - \varepsilon - \varepsilon\mu\big(\bigcup A_i\big),$$

and the result follows.

Of course, if μ is σ-additive, μ_* is additive on $\mathscr{C}$ even without the boundedness assumption.

3. It is not difficult to prove that A6 is necessary and sufficient for μ defined by (4) to be additive on $\mathscr{K}$.

(¹) Note added in proof: Actually, this assumption is not necessary.

4. Note that, for any non-negative function f on X,

$$\mu_*(f) = \sup_{k\leqslant f}\inf_{h\geqslant k} Th,$$

where k denotes a $\mathscr{K}$-simple function.

5. An important consequence of Theorem 1 is that if there is a $\mathscr{K}$-regular finitely additive measure representing T, then it is unique.

The conditions of Theorem 1 are not sufficient to ensure that μ represents T (consider for instance the trivial case when $\mathscr{K} = \{\emptyset\}$). Thus, in order to obtain a representation, we need a condition to ensure that there are enough $\mathscr{K}$ sets relative to T. The following property turns out to be appropriate: We say that $\mathscr{K}$ *exhausts* T if, to every $h \in \mathscr{C}$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}$ such that $Th' < \varepsilon$ whenever $h' \in \mathscr{C}$, $h' \leqslant h$ and $h' = 0$ on K. For example, this property is satisfied if to any h there exists a K such that $\{h > 0\} \subset K$.

THEOREM 2. *Assume that A1-A6 hold. Then a necessary and sufficient condition that there exists a $\mathscr{K}$-regular finitely additive measure representing T, is that $\mathscr{K}$ exhausts T, and that*

$$Th = \sup_n T(h \wedge n) \quad \textit{for all } h \in \mathscr{C}. \tag{10}$$

Proof. In the proof, μ refers to the finitely additive measure constructed in Theorem 1.

To prove sufficiency, assume that $\mathscr{K}$ exhausts T. We know that $\mu_* h \leqslant Th$ for all h and must prove the reverse inequality. To a given h and $\varepsilon > 0$, choose K as specified by the exhaustion property. By (10), we may assume that $h < 1$. Also choose some fixed $h_K \geqslant 1_K$ and a natural number n such that $n^{-1} Th_K < \varepsilon$. By Stone's condition we may assume that $h_K = 1$ on K. Put

$$K_\nu = K \cap \{h \geqslant \nu/n\}; \qquad \nu = 1, 2, \ldots, n.$$

Since

$$K_\nu = K \cap \{h_K \setminus h \leqslant 1 - \nu/n\},$$

$\mathscr{K}_\nu \in \mathscr{K}$ by A5. Now

$$(h - n^{-1}) 1_K \leqslant \sum_1^n n^{-1} 1_{K_\nu} \leqslant h. \tag{11}$$

The right inequality gives

$$\mu_* h \geqslant \sum_1^n n^{-1}\mu K_\nu = \inf\Big\{Th_1\colon\ h_1 \geqslant \sum_1^n n^{-1} 1_{K_\nu}\Big\}. \tag{12}$$

Consider any $h_1 \geqslant \sum n^{-1} 1_{K_\nu}$. We may assume that $h_1 \leqslant h$. Then $h-h_1 \leqslant n^{-1}$ on the set K, by the left inequality in (11), so it follows that the function

$$h' = (h-h_1)\setminus n^{-1} h_K$$

satisfies $h' \leqslant h$ and $h' = 0$ on K. Thus $Th' < \varepsilon$. Since

$$h-h_1 = h' + (h-h_1)\wedge n^{-1} h_K \leqslant h' + n^{-1} h_K,$$

we obtain $T(h-h_1) \leqslant Th' + n^{-1} Th_K \leqslant 2\varepsilon$. $Th_1 \geqslant Th - 2\varepsilon$ follows. By (12), this argument shows that $\mu_* h \geqslant Th - 2\varepsilon$. ε being arbitrary, this gives the desired conclusion $\mu_* h \geqslant Th$.

To prove necessity, assume that $\mu h = Th$ for all h. To a given h and $\varepsilon > 0$, choose $\sum_1^n a_i 1_{K_i} \leqslant h$ such that $Th = \mu_* h \leqslant \sum a_i \mu K_i + \varepsilon$. Put $K = \bigcup_1^n K_i$. To verify the exhaustion property, assume that $h' \leqslant h$ and $h' = 0$ on K. Then $h - h' \geqslant \sum a_i 1_{K_i}$. Thus we have

$$Th - \varepsilon \leqslant \sum a_i \mu K_i \leqslant \mu_*(h-h') = T(h-h') = Th - Th',$$

so that $Th' \leqslant \varepsilon$, and hence the exhaustion property holds.

Clearly, (10) is also necessary. ■

Remark. For the proof of necessity A3 is not needed. [Employ that $\mu_*(h' + \sum a_i 1_{K_i}) = \mu_* h' + \mu_*(\sum a_i 1_{K_i})$; cf. the proof given in Remark 2 to Theorem 1].

Example 1 (Alexandroff [1]; see also Varadarajan [12], part I) Let X be an arbitrary topological space, $\mathscr{C}$ the cone of bounded continuous non-negative functions on X, and T a positive linear functional on $\mathscr{C}$. We take $\mathscr{K}$ as the paving of *zero sets*, i.e. sets of the form $h^{-1}(0)$ with $h \in \mathscr{C}$.

Assumptions A1–A5 and the exhaustion property are then easily verified. As is well known, A6′ is also satisfied [if $K_i = h_i^{-1}(0)$, consider $h = h_2(h_1+h_2)^{-1}$].

Thus T has a unique representation by a finitely additive measure, regular w.r.t. the zero sets. The field $\mathscr{A}(\mathscr{K})$ is usually called the *Baire field* and hence, the representing measure is a *finitely additive regular Baire measure.*

Example 2 (Markoff [9]; see also Dunford and Schwartz [4], IV. 6). Let X be a normal (not necessarily Hausdorff) topological space, $\mathscr{C}$ and T as in Example 1. This time take $\mathscr{K}$ to be the paving of closed subsets.

Proceeding as in Example 1 (noting that this time A6′ is just the Urysohn lemma), we obtain a unique representation of T by a finitely additive measure regular w.r.t. the closed sets. $\mathscr{A}(\mathscr{K})$ is the *Borel field*; The measure is called a *finitely additive regular Borel measure.*

3. Consider the problem of representing T by σ-additive or τ-additive measures. The following smoothness conditions are needed. T is *σ-smooth at* 0 if $h_n \downarrow 0$ implies $Th_n \downarrow 0$. T is *σ-smooth* at $\emptyset$ w.r.t. $\mathscr{K}$ if $K_n \downarrow \emptyset$ implies that

$$\inf\{Th:\ h \geqslant 1_{K_n} \text{ for some } n\} = 0.$$

T is *τ-smooth at* 0 if $h_a \downarrow 0$ implies $Th_a \downarrow 0$. Here (h_a) is any downward filtering collection of functions in $\mathscr{C}$ with $\inf h_a = 0$. T is *τ-smooth at* $\emptyset$ w.r.t. $\mathscr{K}$ if $K_a \downarrow \emptyset$ implies

$$\inf\{Th:\ h \geqslant 1_{K_a} \text{ for some } a\} = 0.$$

Theorem 3. *Assume that* A1–A5, *and* A6 *hold.*

If $\mathscr{K}$ is closed under $\bigcap c$, then a necessary and sufficient condition that there exists a $\mathscr{K}$-regular σ-additive measure representing T is that $\mathscr{K}$ exhausts T, that T be σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ *and that* (10) *holds.*

If $\mathscr{K}$ is closed under $\bigcap a$, then a necessary and sufficient condition that there exists a $\mathscr{K}$-regular τ-additive measure representing T is that $\mathscr{K}$ exhausts T, that T be τ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ *and that* (10) *holds.*

In both cases, the representing measure is unique and determined by its values on $\mathscr{K}$-sets which are those given by (4).

Proof. We will deal only with the case in which $\mathscr{K}$ is closed under countable intersections (the other case is handled analogously). Throughout the proof, μ denotes the $\mathscr{K}$-regular finitely additive measure constructed in Theorem 1. Note that the condition that T be σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ is equivalent to the condition that μ be σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$.

To prove sufficiency, assume that $\mathscr{K}$ exhausts T, that T is σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$, and that (10) holds. Then, by Theorem A, μ has an extension to a $\mathscr{K}$-regular σ-additive measure. This measure we shall here denote by μ_σ. By A5, each h is $\mathscr{B}(\mathscr{K})$-measurable; indeed, there exists an increasing sequence of $\mathscr{A}(\mathscr{K})$-simple functions converging pointwise to h. It follows from this that

$$\begin{aligned}\mu_\sigma(h) &= \sup\{\mu_\sigma(k):\ k\ \mathscr{A}(\mathscr{K})\text{-simple},\ k \leqslant h\}\\ &= \sup\{\mu(k):\ k\ \mathscr{A}(\mathscr{K})\text{-simple},\ k \leqslant h\}\\ &= \mu_*(h).\end{aligned}$$

Since the exhaustion property holds, it follows from Theorem 2 that $\mu_*(h) = Th$ for every bounded function in $\mathscr{C}$. Employing (10), we thus get, for any $h \in \mathscr{C}$:

$$\begin{aligned}\mu_\sigma(h) &= \sup_n \mu_\sigma(h \wedge n)\\ &= \sup_n T(h\wedge n)\\ &= Th.\end{aligned}$$

To prove necessity, assume that μ_σ is a $\mathscr{K}$-regular σ-additive representing measure. Denote by μ' the restriction of μ_σ to $\mathscr{A}(\mathscr{K})$. Then μ' is a $\mathscr{K}$-regular finitely additive measure. Employing A5 in the same way as in the proof of sufficiency, we obtain, for every $h \in \mathscr{C}$, $\mu_\sigma(h) = \mu'_*(h)$. Since μ_σ is a representation, so is μ'. By Theorem 2 it follows that $\mathscr{K}$ exhausts T and that $\mu' = \mu$. Since μ_σ is σ-additive, μ must be σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$, i.e. T is σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$. Clearly, (10) must hold.

This proves necessity as well as uniqueness. ■

Remarks. 1. For the proof of necessity, $\mathscr{C}$ need not be closed under $\setminus$.

2. Assume that μ is a $\mathscr{K}$-regular τ-additive representing measure. It is not difficult to show that for every upward filtering system of $\mathscr{C}$ functions, with supremum g: $h_\alpha \uparrow g$, g is $\mathscr{B}(\mathscr{K})$-measurable and

$$\int g\,d\mu = \sup_\alpha \int h_\alpha\,d\mu \quad (\leqslant \infty).$$

Similarly, if $h_\alpha \downarrow f$, then f is $\mathscr{B}(\mathscr{K})$-measurable and

$$\int f\,d\mu = \inf_\alpha \int h_\alpha\,d\mu.$$

It was remarked earlier that to get a representation we need "enough" $\mathscr{K}$ sets. The following results show that if only $\mathscr{C}$ and T are specified then by a suitable choice of $\mathscr{K}$, this can be achieved even in an abstract setting.

THEOREM 4. *Let $\mathscr{C}$ and T satisfy* A1, A2 *and* A3. *Define* $\operatorname{tr}\mathscr{C}$ *(the trace of $\mathscr{C}$) as the paving of sets of the form $\{h \geqslant a\}$ where $h \in \mathscr{C}$ and $a > 0$. Let $\mathscr{K}_\sigma$ be the paving of countable intersections of sets in $\operatorname{tr}\mathscr{C}$, $\mathscr{K}_\tau$ the paving of arbitrary intersections of sets in $\operatorname{tr}\mathscr{C}$.*

Then T has a representation by a $\mathscr{K}_\sigma$-regular σ-additive measure (with domain $\mathscr{B}(\mathscr{K}_\sigma)$) iff T is σ-smooth at 0; and T has a representation by a $\mathscr{K}_\tau$-regular τ-additive measure (with domain $\mathscr{B}(\mathscr{K}_\tau)$) iff T is τ-smooth at 0.

Proof. The proof of necessity is simple, cf. Remark 2 to Theorem 3. We now prove sufficiency by verifying the conditions for Theorem 3 to be applicable. A1, A2, and A3 are satisfied by assumption and also, $\mathscr{K}_\sigma$ is a $(\emptyset, \bigcup f, \bigcap c)$ paving, $\mathscr{K}_\tau$ a $(\emptyset, \bigcup f, \bigcap a)$ paving. A5 follows in both cases from the identity

$$\{h \leqslant \beta\} \cap \bigcap_i \{h_i \geqslant a_i\} = \bigcap_i \{(h_i \wedge a_i) \setminus (h \setminus \beta) \geqslant a_i\}.$$

That (10) is true follows from the fact that $h \setminus n \downarrow 0$ as $n \to \infty$. To prove the exhaustion property, we first choose an n such that $T(h \wedge n^{-1}) < \varepsilon$, and then put $K = \{h \geqslant n^{-1}\}$. If $h' \leqslant h$ and $h' = 0$ on K then $h' \leqslant h \wedge n^{-1}$, thus $Th' < \varepsilon$.

Now, if $K \in \operatorname{tr}\mathscr{C}$, then by Stone's condition we can find an h for which $h \leqslant 1$ and $K = h^{-1}(1)$. The sequence $h_n = n\big(h \setminus (1 - n^{-1})\big)$ decreases to 1_K.

Thus for every $K \in \mathscr{K}_\sigma$ $[K \in \mathscr{K}_\tau]$, we can find $h_n \downarrow 1_K$ $[h_\alpha \downarrow 1_K]$, and so the σ-smoothness [τ-smoothness] at $\emptyset$ w.r.t. $\mathscr{K}_\sigma$ $[\mathscr{K}_\tau]$ follows from the σ-smoothness [τ-smoothness] of T at 0.

Lastly, to verify A6, suppose $K' \cap K'' = \emptyset$. For the case $K', K'' \in \mathscr{K}_\sigma$ choose $h'_n \downarrow 1_{K'}$, $h''_n \downarrow 1_{K''}$. As $h'_n \wedge h''_n \downarrow 0$, we can find an n such that $T(h'_n \wedge h''_n) < \varepsilon$. The $\mathscr{K}_\tau$ case is handled analogously. ■

Remark. It is easy to show that if $1 \in \mathscr{C}$ then every totally finite σ-additive measure on $\mathscr{B}(\mathscr{K}_\sigma)$ is automatically $\mathscr{K}_\sigma$-regular. This is not true in general.

COROLLARY (*Daniell's extension theorem*). *Let $\mathscr{C}$ be a $(0, \vee f, \wedge f, \setminus)$ convex cone of non-negative functions satisfying Stone's condition, on a set X. If T is a positive linear functional on $\mathscr{C}$, which is σ-smooth at 0, then T has a representation as an integral* w.r.t. *a σ-additive measure.*

Of course, this representation immediately leads to the usual extension of T.

Recall that the paving $\mathscr{K}$ is said to be *semicompact* [*compact*] if every countable [arbitrary] family of $\mathscr{K}$ sets with empty intersection contains a finite family with empty intersection. For such pavings the σ-smoothness [τ-smoothness] of T at $\emptyset$ w.r.t. $\mathscr{K}$ appearing in Theorem 3 is trivially satisfied.

EXAMPLE 3 (*Radon measures*). We consider the classical setting of a positive linear functional defined on the cone of non-negative continuous functions with compact support on a locally compact Hausdorff space X. $\mathscr{K}$ is taken as the paving of compact sets.

A1–A5 and A6′ are easily verified. The exhaustion property as well as (10) are trivial. $\mathscr{K}$ is of course the architypical compact paving.

Theorem 3 then gives us a representation of T by a σ-additive (in fact τ-additive) Borel measure μ, regular w.r.t. the compact sets. This representation is not the one always given.

Let M denote the set of σ-additive measures defined on the Borel σ-field $\mathscr{B}(\mathscr{K})$ for which $\mu K < \infty$ for all $K \in \mathscr{K}$, and for which

$$\mu K = \inf\{\mu G\colon G \supseteq K,\ G \text{ open}\}; \qquad K \in \mathscr{K},$$
$$\mu G = \sup\{\mu K\colon K \subseteq G,\ K \in \mathscr{K}\}; \qquad G \text{ open}.$$

Let M_1 be the set of $\mu \in M$ for which

$$\mu A = \sup\{\mu K\colon K \subseteq A,\ K \in \mathscr{K}\}; \qquad A \in \mathscr{B}(\mathscr{K}),$$

and let M_2 be the set of $\mu \in M$ for which

$$\mu A = \inf\{\mu G\colon G \supseteq A,\ G \text{ open}\}; \qquad A \in \mathscr{B}(\mathscr{K}).$$

Then our method gives the unique representing measure in M_1, whereas an approach based on Bourbaki [3] will give the unique representing measure in M_2. This shows indirectly that there must be a 1-1 correspondence between M_1 and M_2. More directly, we can proceed as follows.

Associate with every $\mu \in M$, μ_1 and μ_2 defined by

$$\mu_1 A = \sup\{\mu K\colon K \subseteq A,\ K \in \mathscr{K}\}; \qquad A \in \mathscr{B}(\mathscr{K}),$$

$$\mu_2 A = \inf\{\mu G\colon G \supseteq A,\ G \text{ open}\}; \qquad A \in \mathscr{B}(\mathscr{K}).$$

Then $\mu_1 \in M_1$, $\mu_2 \in M_2$ and $\mu_1 \leqslant \mu \leqslant \mu_2$. Furthermore, μ_1 and μ_2 agree on open sets, on compact sets and, more generally, on all $A \in \mathscr{B}(\mathscr{K})$ with $\mu_2 A < \infty$. Hence

$$\int h d\mu_1 = \int h d\mu = \int h d\mu_2 \quad \text{for all } h \in \mathscr{C}.$$

Thus μ_1 is the smallest member of M representing T and μ_2 the largest.

The classical definition of Radon measures as linear functionals has been further developed by LeCam [7] and Varadarajan [12]. We now show how their results can also be obtained by a simple application of our results.

EXAMPLE 4 (cf. Example 1). Let T be a positive linear functional on the cone $\mathscr{C}$ of bounded continuous non-negative functions on an arbitrary topological space X. We consider the following sets of totally finite measures defined on the Baire σ-field, i.e. the σ-field generated by the zero sets.

M_σ = the set of countably additive Baire measures (notice that these are automatically regular w.r.t. the zero sets),

M_τ = the set of $\mu \in M_\sigma$ which are also τ-additive, i.e. if $Z_a \downarrow Z$, and Z_a, Z are zero sets, then $\mu Z_a \downarrow \mu Z$,

M_t = the set of $\mu \in M_\sigma$ for which, given $\varepsilon > 0$, there is a compact set C such that $\mu_*(\complement C) < \varepsilon$.

M_c = the set of $\mu \in M_\sigma$ for which there is a compact set C such that $\mu_*(\complement C) = 0$.

Notice that

$$M_c \subseteq M_t \subseteq M_\tau \subseteq M_\sigma.$$

We prove that these measures correspond exactly with the linear functionals having certain smoothness properties.

(i) *T is represented by an M_σ measure iff T is σ-smooth at* 0,

(ii) *T is represented by an M_τ measure iff T is τ-smooth at* 0,

(iii) *T is represented by an M_t measure iff $Th_a \to 0$ for every net (h_a) tending uniformly to 0 on compacta, with $h_a \leqslant 1$,*

(iv) *T is represented by an M_c measure iff $Th_a \to 0$ for every net (h_a) tending uniformly to 0 on compacta.*

Necessity is easy to establish in each case.

(i) follows immediately from Theorem 4 by noticing that the paving $\mathscr{K}_\sigma$ in that theorem is precisely the paving of zero sets.

Applying Theorem 4 to the τ-smooth functional in (ii) leads to a representation by a $\mathscr{K}_\tau$-regular τ-additive measure on $\mathscr{B}(\mathscr{K}_\tau)$. The restriction of this to the Baire σ-field is the required measure. Notice that when X is completely regular $\mathscr{K}_\tau$ is just the paving of all closed subsets. In this case then, Theorem 4 gives a representation by a σ-additive μ defined on the Borel σ-field, regular w.r.t. the closed sets and having the τ-smoothness property:

$$\mu F_a \downarrow \mu F \quad \textit{for every family } (F_a) \textit{ of closed sets filtering down to } F.$$

To prove sufficiency in (iii), let μ be the representing measure given by (i). Assume that there is some $\varepsilon > 0$ such that, for every compact set C, there is a zero set Z_C contained in $\complement C$ for which $\mu(Z_C) \geqslant \varepsilon$. As any continuous function achieves its maximum on a compact set, it follows that we can find continuous functions h_C with $h_C \leqslant 1$ and taking the value 1 on Z_C, 0 on C. The net (h_C) tends uniformly to 0 on compacta, but

$$T(h_C) = \mu(h_C) \geqslant \mu(Z_C) \geqslant \varepsilon,$$

which contradicts the assumed smoothness.

A similar construction can be used to prove the sufficiency of (iv). For, if there are zero sets $Z_C \subseteq \complement C$ for which $\mu(Z_C) > 0$ then, by taking suitable multiples of the above h_C's we obtain a net (h'_C) tending uniformly to 0 on compacta (but not necessarily uniformly bounded) for which $T(h'_C) \not\to 0$.

The Baire representations in (iii) and (iv) above can be strengthened to Borel representations by merely adding the weak separation property that X be completely Hausdorff.

EXAMPLE 5 (cf. Fremlin, Garling and Haydon [5]). Let T be a positive linear functional on the cone $\mathscr{C}$ of bounded continuous non-negative functions on a completely Hausdorff topological space X. Take $\mathscr{K}$ to be the paving of compact sets.

Consider the set M'_t of all totally finite $\mathscr{K}$-regular measures defined on the Borel σ-field and the set M'_c of all $\mu \in M'_t$ with $\mu(\complement K) = 0$ for some $K \in \mathscr{K}$. We show that

(i) *T is represented by an M'_t measure iff $Th_a \to 0$ for every net (h_a) tending uniformly to 0 on compacta, with $h_a \leqslant 1$;*

(ii) *T is represented by an M'_c measure iff $Th_a \to 0$ for every net (h_a) tending uniformly to 0 on compacta.*

Again, necessity is easy to prove. To prove sufficiency we use Theorem 3. All the conditions of this theorem are trivially true, except for the exhaustion property. So consider any h, which we may without loss of generality take to be $\leqslant 1$. If, for every K, we could find an h'_K which took the value 0 on K, was $\leqslant h$, and satisfied $T(h'_K) \geqslant \varepsilon$, then the resulting net (h'_K) would tend uniformly to 0 on compacta with $h'_K \leqslant 1$, but have $T(h'_K) \not\to 0$.

The rest of the proof is analogous to that in Example 4.

It is sometimes easier to verify the exhaustion property directly rather than the equivalent smoothness conditions. In case (i) this amounts to verifying that to each $\varepsilon > 0$ there exists K such that $Th \leqslant \varepsilon$ for all $h \leqslant 1$ vanishing on K; whereas in case (ii) we need a K with $Th = 0$ for every h vanishing on K. ∎

The functions in $\mathscr{C}$ have always been bounded in the examples considered so far. Theorem 3 is also able to handle unbounded functions though h, as is shown by our next example. This also illustrates the importance of condition (10) in that theorem.

Example 6 (Hewitt [6]). Let T be a positive linear functional on the cone $\mathscr{C}$ of all non-negative continuous real functions (not necessarily bounded) on an arbitrary topological space X, and let $\mathscr{K}$ denote the paving of zero sets. We show that T has a representation by a $\mathscr{K}$-regular σ-additive Baire measure μ having the additional property: for every $h \in \mathscr{C}$ there is a real number N such that $\mu\{h > N\} = 0$. This is achieved by an application of Theorem 3.

The only conditions of that theorem which are not immediately evident are (10) and the σ-smoothness of T at $\emptyset$ w.r.t. $\mathscr{K}$. To verify (10), consider any $h \in \mathscr{C}$ and define the function $g = \sum_1^\infty a_n(h \setminus n)$, where (a_n) is any sequence of non-negative real numbers. Now for any $x \in X$, we can find an N such that $h(x) < N$. On the neighbourhood $\{h < N\}$ of x, $g = \sum_1^N a_n(h \setminus n)$ which is continuous. Thus $g \in \mathscr{C}$. It follows therefore that

$$\infty > Tg \geqslant \sum_1^\infty a_n T(h \setminus n).$$

Choosing $a_n = T(h\setminus n)^{-1}$ if $T(h \setminus n) \neq 0$ and 0 otherwise shows that $T(h \setminus n) \neq 0$ for only finitely many n. Thus, for some n, $Th = T(h \wedge n) + T(h \setminus n) = T(h \wedge n)$, and so (10) is a fortiori true.

For the σ-smoothness w.r.t. $\mathscr{K}$, consider any sequence $K_n \downarrow \emptyset$. By definition of $\mathscr{K}$, we can find (h_n) such that $h_n \leqslant 1$ and $K_n = h_n^{-1}(1)$. Clearly, we may assume $h_n \downarrow$. Consider the function $f = \sum_1^\infty (h_n)^n$. Given $x \in X$, there is an N such that $x \notin K_N$. Thus there is an r for which $h_N(x) < r < 1$. Now on the neighbourhood $\{h_N < r\}$ of x, the series for f is uniformly convergent, by comparison with the series $\sum_1^\infty r^n$. It follows that $f \in \mathscr{C}$ and hence $\infty > Tf \geqslant \sum_1^\infty T(h_n^n)$. Noticing that $h_n^n \geqslant 1_{K_n}$, we obtain the required σ-smoothness. Thus Theorem 3 gives the required $\mathscr{K}$-regular σ-additive representing measure μ. The essential boundedness of each $\mathscr{C}$ function follows easily from

$$0 = T(h \setminus N) \geqslant \int_{\{h>N\}} (h \setminus N)\,d\mu$$

and the countable additivity of μ. ∎

Even though we feel that Theorem 3 is in a satisfactory form, one may ask if, in general, condition $\sigma 1$ that T be σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ may be replaced by condition $\sigma 2$ that T be σ-smooth at 0. As a consequence of Theorem 3, $\sigma 1$ together with the exhaustion property (and (10)) does imply $\sigma 2$. That $\sigma 1$ alone does not imply $\sigma 2$ is obvious (construct an example with $\mathscr{K} = \{\emptyset\}$). That $\sigma 2$ even in the presence of the exhaustion property does not imply $\sigma 1$ will be shown in Section 4. However, there is one important case in which this implication is true:

If $X \in \mathscr{K}$, A6′ holds and

(13) *to $K_n \downarrow \emptyset$ there exists $G_n \downarrow \emptyset$ with $G_n \supseteq K_n$ and $G_n \in \mathscr{G}(\mathscr{K})$ for all n,*

then $\sigma 2$ does imply $\sigma 1$.

The proof is straightforward. Actually, this situation was obtained in Theorem 4 by the judicious choice of the pavings $\mathscr{K}_\sigma$, $\mathscr{K}_\tau$. A further instance in which (13) holds is now discussed.

Example 7 (cf. Example 2). Let X be normal, $\mathscr{C}$ the bounded non-negative continuous functions and T a positive linear functional on $\mathscr{C}$. Take $\mathscr{K}$ as the paving of closed subsets of X. From Theorem 3 we know that *a necessary and sufficient condition that T be representable by a $\mathscr{K}$-regular σ-additive Borel measure is that T be σ-smooth at $\emptyset$* w.r.t. $\mathscr{K}$.

Thus a necessary condition for this representation is that T be σ-smooth at 0. As observed by Mařík [8], this condition is also sufficient if X is countably paracompact. This follows from the above since countable paracompactness is equivalent to (13). It is still an open question whether T σ-smooth at 0 is sufficient in any normal space (notice that there are normal not countably paracompact spaces as shown by Rudin [10]).

The corresponding situation for $\mathscr{K}$-regular τ-additive representations (even for completely regular spaces) is far simpler, see Example 4. ∎

4. For Theorem 5 below we need the following generalization of Theorem A (Topsøe [11], Theorem 5.1, and also the notes and remarks to that result).

Theorem B. *Let $\mathscr{K}$ and μ be as in Theorem A. Let $\mathscr{K}_\sigma$ denote the paving of countable [$\mathscr{K}_\tau$ the paving of arbitrary] intersections of sets in $\mathscr{K}$.*

If μ is σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$, *then μ has an extension to a $\mathscr{K}_\sigma$-regular σ-additive measure.*

If μ is τ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$, *then μ has an extension to a $\mathscr{K}_\tau$-regular τ-additive measure.*

In both cases the extension is unique and determined by

$$\mu K = \inf\{\mu K' : K' \supseteq K,\ K' \in \mathscr{K}\}$$

for every $K \in \mathscr{K}_\sigma$ $[K \in \mathscr{K}_\tau]$.

Using Theorem B instead of Theorem A in the proof of Theorem 3, it is easy to see that the sufficiency part of Theorem 3 is true without the condition that $\mathscr{K}$ be closed under $(\bigcap c)$ $[(\bigcap a)]$, where $\mathscr{K}$-regularity is replaced by $\mathscr{K}_\sigma$-regularity $[\mathscr{K}_\tau$-regularity$]$.

The following analogue of Theorem 1 is proved by an easy adaptation of the relevant parts of the proof of Theorem 3, this time appealing to Theorem B rather than to Theorem A.

THEOREM 5. *Assume that* A1, A2, A4, A5, *and* A6 *hold. If T is σ-smooth at* $\emptyset$ w.r.t. $\mathscr{K}$, *then there exists a largest $\mathscr{K}_\sigma$-regular σ-additive measure dominated by T, and if T is τ-smooth at* $\emptyset$ w.r.t. $\mathscr{K}$, *then there exists a largest $\mathscr{K}_\tau$-regular τ-additive measure dominated by T.*

On the surface this result looks satisfactory, but in fact, as examples below indicate, it is not. In particular, it is not possible to obtain necessary and sufficient conditions for existence of σ- or τ-additive representing measures from Theorem 5 in case $\mathscr{K}$ is not closed under $(\bigcap c)$ $[(\bigcap a)]$.

EXAMPLE 8. Let $X = [0, \omega]$, ω the first infinite ordinal, $\mathscr{C}$ be the class of non-negative functions for which $h(n) = h(\omega)$ eventually, and T be defined by $Th = h(\omega)$. Take $\mathscr{K}$ to consist of sets with finite complement and of finite sets not containing ω. A1-A6 hold. T is σ-smooth (even τ-smooth) at 0, but T is not σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ (consider $K_n = [n, \omega[$).

$\mathscr{A}(\mathscr{K})$ consists of all finite sets and of all sets whose complements are finite. By Theorem 1, or directly, we find that the largest $\mathscr{K}$-regular finitely additive measure dominated by T is given by

$$\mu A = \begin{cases} 0 & A \text{ finite}, \\ 1 & \complement A \text{ finite}. \end{cases}$$

Furthermore, we see, in agreement with Theorem 2, that μ is a representation of T. μ has no countably additive extension.

Since, for any A, $A \cup [n, \omega[\downarrow A$, we find that $\mathscr{K}_\sigma = 2^X$. Thus a $\mathscr{K}_\sigma$-regular σ-additive measure is the same as a countably additive measure on 2^X. It is easy to see directly that ε_ω (a unit mass at ω) is the largest $\mathscr{K}_\sigma$-regular σ-additive measure dominated by T. In fact, ε_ω is a representation of T.

Note that ε_ω is $\mathscr{K}_\sigma$-regular, but its restriction to $\mathscr{A}(\mathscr{K})$ is not $\mathscr{K}$-regular. This shows why the $(\bigcap c)$-closure of $\mathscr{K}$ was needed in the proof of necessity in Theorem 3.

EXAMPLE 9. Let $X = [0, \omega[$, $\mathscr{C}$ be the non-negative constant functions, and $Th = h(0)$ for $h \in \mathscr{C}$. Take $\mathscr{K}$ to consist of $\emptyset$ and all sets of the form $[n, \omega[$. Again, A1-A6 hold. Note that $\mathscr{K}_\sigma = \mathscr{K}$. It is easy to see that there exists a $\mathscr{K}$-regular finitely additive measure representing T, and that the 0-measure is the largest $\mathscr{K}_\sigma$-regular σ-additive measure dominated by T. T is not σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$.

EXAMPLE 10. Let $X = [0, \Omega[$, where Ω is the first uncountable ordinal, and $\mathscr{C}$ be the non-negative constant functions. Take $Th = h(0)$ for $h \in \mathscr{C}$ and $\mathscr{K}$ to consist of $\emptyset$ and all sets of the form $[a, \Omega[$. Note that $\mathscr{K} = \mathscr{K}_\sigma = \mathscr{K}_\tau$. T is σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$, but T is not τ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$. The reader can easily verify that there exists a $\mathscr{K}$-regular finitely additive measure as well as a $\mathscr{K}_\sigma$-regular σ-additive measure representing T, and that the 0-measure is the largest $\mathscr{K}_\tau$-regular τ-additive measure dominated by T. It may also be noted that the paving $\mathscr{K}$ is semicompact but not compact. ■

The examples show that existence of a largest regular σ-additive or τ-additive measure dominated by T does not imply the smoothness conditions of Theorem 5. Also, it does not help if we in fact have a representation (Example 8) or if $\mathscr{K} = \mathscr{K}_\sigma$ $[\mathscr{K} = \mathscr{K}_\tau]$ (Examples 9, 10). If we have a representation and if also $\mathscr{K} = \mathscr{K}_\sigma$ $[\mathscr{K} = \mathscr{K}_\tau]$, the situation is far simpler as shown by Theorem 3.

Lastly, we state a result more general than Theorem 5. It explains the behaviour in Example 8 but is not general enough for Examples 9, 10.

THEOREM 6. *Assume that* A1, A2, A4, A5, *and* A6 *hold. If the set function defined on $\mathscr{K}$ by*

$$\nu K = \inf\left\{\sum_1^\infty Th_n : \sum_1^\infty h_n \geqslant 1_K\right\}$$

is σ-smooth at $\emptyset$ w.r.t. $\mathscr{K}$ $[\tau$*-smooth at* $\emptyset$ w.r.t. $\mathscr{K}]$, *then there exists a largest $\mathscr{K}_\sigma$-regular σ-additive $[\mathscr{K}_\tau$-regular τ-additive$]$ measure dominated by T.*

The proof consists in a generalisation of the proof of Theorem 1, which we shall not carry out.

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