STUDIA MATHEMATICA, T. LIV. (1975)

Banach ideals on Hilbert spaces

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Abstract. We construct a compact non-weakly nuclear operator, and estimate various constants derived from ideal norms on Hilbert spaces; for example, the 2-summing norm of any quotient map from an L_1-space onto l_2 is $\sqrt{\pi/2}$. Some operator characterizations of l_1 are presented.

Introduction. The spaces of operators on Hilbert spaces are known to lack many properties which are common to the classical Banach spaces L_p. In [7] it was shown that the common ideals of operators on l_2 lack local unconditional structure. In this paper we present some more results on these spaces.

In Section 1 we construct a compact operator T mapping the space of all compact operators on l_2 to itself which is not weakly nuclear, that is, T cannot be written as an unconditional convergent series of rank-one operators. This answers a problem by A. Pietsch. The construction is related to the fact that the unconditional basis constants of all bounded operators on the n-dimensional Hilbert space l_2^n tend to infinity with n. This result is also given another proof here.

In Section 2 we estimate some constants obtained from considerations of the ideal norms γ_p and, more generally, i_{pq} $(1 \leqslant q \leqslant p \leqslant \infty)$ and their injective and projective envelopes. Some such constants were studied in [8] and [7], e.g. the famous Grothendieck constant is none other than $/\pi_1(l_2)$, by our notation. We prove, among other results, that $/\pi_2(l_2) = \sqrt{\pi/2}$, and the fact $/\gamma_2^*\backslash(l_2) = \pi/2$ of [8] is given a new proof.

Operator characterizations of spaces isomorphic to l_1 and $\mathscr{L}_1$ are presented in Section 3, generalizing similar results obtained in [11].

We now give some basic definitions. All Banach spaces E are taken over the reals. E' denotes the dual space of E. The space of all continuous linear operators from E into F is denoted by $L(E, F)$ and $C(E, F)$ will denote the subspace of all compact operators. A *Banach ideal of operators* $[A, a]$ is a method which associates with every pair (E, F) of Banach

* Research supported by NSF-GP-34193.

spaces an algebraic subspace $A(E,F)$ of $L(E,F)$ equipped with a norm α such that the following requirements are fulfilled:

(a) $A(E,F)$ is complete under α, and $\alpha(x'\otimes y) = \|x'\|\,\|y\|$, where $x'\otimes y$ is the rank-one operator from E to F defined by

$$x'\otimes y(x) = \langle x, x'\rangle y.$$

(b) If $u\in L(X,E)$, $v\in A(E,F)$ and $w\in L(F,Y)$, then

$$wvu\in A(X,Y) \quad \text{and} \quad \alpha(wvu) \leqslant \|w\|\alpha(v)\|u\|.$$

For convenience, we write for $u\in L(E,F)$, $\alpha(u)<\infty$ iff $u\in A(E,F)$. $\alpha(E)$ will denote $\alpha(I)$, where I is the identity operator on the space E. The *trace* of a finite-rank operator $u\in L(E,E)$ with a representation $u = \sum_{i\leqslant n} x_i'\otimes x_i$ is defined as trace $(u) = \sum_{i\leqslant n}\langle x_i, x_i'\rangle$. Given a sequence $\{x_i\}_{i\geqslant 1}\subset E$, we set

$$\varepsilon_1(\{x_i\}) = \sup\Big\{\sum_{i\geqslant 1}|\langle x_i, x'\rangle|;\ \|x'\|\leqslant 1\Big\}.$$

Associated with any Banach ideal $[A,\alpha]$ is the *adjoint ideal* $[A^*,\alpha^*]$ defined as follows: $A^*(E,F)$ is the Banach space of all $u\in L(E,F)$ in which the norm $\alpha^*(u)$ is the least $\varrho>0$ such that for all finite-dimensional Banach spaces X, Y and for all $v\in L(X,E)$, $w\in L(F,Y)$ and $z\in L(Y,X)$ the following inequality holds

$$|\mathrm{trace}(wuvz)| \leqslant \varrho\|w\|\,\|v\|\,\alpha(z) \quad (\text{cf. } [6]).$$

$[A,\alpha]$ is called *perfect* if $\alpha^{**}=\alpha$.

The classical Banach ideals are: $[\Pi_p,\pi_p]$ $(1\leqslant p<\infty)$, the ideal of *p-absolutely summing operators*: $u\in\Pi_p(E,F)$ iff for any finite subset $\{x_i\}_{i\leqslant n}\subset E$,

$$\Big(\sum_{i\leqslant n}\|ux_i\|^p\Big)^{1/p}\leqslant C\sup_{\|x'\|=1}\Big(\sum_{i\leqslant n}|\langle x_i, x'\rangle|^p\Big)^{1/p}$$

with $\pi_p(u) = \inf C$ [15].

$[I_p, i_p]$ $(1\leqslant p\leqslant\infty)$, the ideal of *p-integral operators*: $u\in I_p(E,F)$ iff there is a probability measure space (Ω,μ) and operators $v\in L(E,L_\infty(\mu))$, $w\in L(L_p(\mu),F'')$ such that $wjv = iu$, where $i\colon F\to F''$ is the canonical map, $j\colon L_\infty(\mu)\to L_p(\mu)$ the formal inclusion. $i_p(u) = \inf\|v\|\,\|w\|$, the infimum is taken over all possible factorizations [14].

$[\Gamma_p,\gamma_p]$ $(1\leqslant p\leqslant\infty)$, the ideal of *$L_p$-factorizable operators*: $u\in\Gamma_p(E,F)$ iff there is a positive measure μ and operators $v\in L(E,L_p(\mu))$, $w\in L(L_p(\mu),F'')$ with $wv = iu$. $\gamma_p(u) = \inf\|w\|\,\|v\|$, taken over all possible factorizations [6], [9].

The ideals defined above are all perfect, and $\pi_p^* = i_{p'}$ $(1/p+1/p'=1)$. For more results on various ideals see [6].

Finally, a basis $B=\{b_i\}_{i\in I}$ for a Banach space E is called *unconditional* iff there is a constant C such that for every norm one vector $x = \sum_{i\in I} x_ib_i$ in E and every choice of signs $\varepsilon_i = \pm 1$ $(i\in I)$, with $\varepsilon_i = 1$ for all but finitely many i,

$$\Big\|\sum_{i\in I}\varepsilon_i x_i\Big\|\leqslant C.$$

The smallest possible C is called the *unconditional constant* of B and denoted by $x(B)$. The *unconditional constant of* E is $x(E) = \inf x(B)$, taken over all possible bases.

§ 1. Weakly nuclear operators. We recall the following definition of Pietsch [16].

Definition 1.1. An operator $u\colon E\to F$ is called *weakly nuclear* if u has a series representation

$$u = \sum_{i=1}^{\infty} x_i'\otimes y_i$$

which converges unconditionally to u in $L(E,F)$. The *weakly nuclear norm* of u is defined as

$$\eta(u) = \inf\varepsilon_1\big((x_i'\otimes y_i)_{i\geqslant 1}\big),$$

where the infimum is taken over all representations of u. For convenience we shall write $\eta(u)=\infty$ iff u is not weakly nuclear.

Proposition 1.2. *Let $u\colon E\to F$. Then $\eta(u) = \inf\|\alpha\|x(U)\|\beta\|$, where the infimum is taken over all factorizations $u=\beta\alpha$, with $\alpha\in C(E,U)$, $\beta\in L(U,F)$ and spaces U.*

Proof. Suppose $\eta(u)<1$, and choose a representation

$$u = \sum_{i=1}^{\infty} x_i'\otimes y_i, \qquad \text{each } y_i\neq 0,$$

with

$$\varepsilon_1\big((x_i'\otimes y_i)_{i\geqslant 1}\big)<1.$$

Let U be the space of all scalar sequences $a=(a_i)_{i\geqslant 1}$ for which $(a_iy_i)_{i\geqslant 1}$ is unconditionally convergent in F, and set $|a| = \varepsilon_1\big((a_iy_i)_{i\geqslant 1}\big)$. Clearly, $x(U)=1$. Define α and β by

$$\alpha(x) = (\langle x, x_i'\rangle)_{i\geqslant 1}, \qquad \beta(a) = \sum_{i=1}^{\infty} a_iy_i.$$

Then $\|\beta\|\leqslant 1$, $\|\alpha\|\leqslant\varepsilon_1\big((x_i'\otimes y_i)_{i\geqslant 1}\big)$ and α is compact. This proves

$$\inf\|\alpha\|x(U)\|\beta\|\leqslant\eta(u),$$

and the other inequality is obvious.

The w^*-closure of the extreme points of the closed unit ball of a space E' is written $K_{E'}$.

PROPOSITION 1.3. *Let $v\colon E\to F$. Then $\eta^*(v)$ is the smallest constant b with the following property:*

There is a probability measure μ on $K_{E'}\times K_{F''}$ so that

$$|\langle v(x), y'\rangle| \leqslant b\mu(|\langle x, \cdot\rangle||\langle y', \cdot\rangle|)$$

for all $x\in E$, $y'\in F'$.

Proof. Suppose first that v satisfies the integral inequality, and consider a composition

$$N\xrightarrow{a}E\xrightarrow{v}F\xrightarrow{\beta}M\xrightarrow{u}N$$

with N and M finite-dimensional spaces and $\eta(u)<1$. Choose an unconditionally convergent representation

$$u=\sum_{i=1}^{\infty} x_i'\otimes y_i$$

so that $\varepsilon_1\big((x_i'\otimes y_i)_{i\geqslant 1}\big)<1$. Then

$$\begin{aligned}|\mathrm{trace}\,(u\beta v a)| &= \Big|\sum_{i=1}^{\infty}\langle v a(y_i), \beta'(x_i)\rangle\Big|\\ &\leqslant b\int\sum_{i=1}^{\infty}|\langle a(y_i), x'\rangle\langle\beta'(x_i'), y''\rangle|\,d\mu(x', y'')\\ &\leqslant b\sup_{x'\times y''\in K_{E'}\times K_{F''}}\sum_{i=1}^{\infty}|\langle y_i, a'(x')\rangle\langle x_i', \beta''(y'')\rangle|\\ &\leqslant b\,\|a\|\,\|\beta\|,\end{aligned}$$

thus $\eta^*(v)\leqslant b$.

For the other direction, let S be the set of functions in $C(K_{E'}\times K_{F''})$ of the form

$$f(x', y'')=\sum_{i=1}^{n}|\langle x_i, x'\rangle\langle y_i', y''\rangle|,$$

where

$$\sum_{i=1}^{n}|\langle v(x_i), y_i'\rangle|\geqslant\eta^*(v).$$

We claim that the closed convex hull of S does not intersect the open convex subset C of functions f satisfying $f(x', y'')<1$ everywhere on $K_{E'}\times K_{F''}$. To see this, let $f\in S$ with $x_i\neq 0$, and write $\delta_i=\mathrm{sgn}\langle v(x_i), y_i'\rangle$. Consider the composition of the sequence

$$U\xrightarrow{\beta}E\xrightarrow{v}F\xrightarrow{a}U\xrightarrow{1_U}U,$$

where U is $\mathbf{R}^n$ equipped with the norm $|a|=\varepsilon_1\big((a_ix_i)_{i\leqslant n}\big)$, $a(y)=(\langle y, y_i'\rangle)_{i\leqslant n}$ and $\beta(a)=\sum_{i=1}^{n}a_i\delta_ix_i$. Then

$$\sum_{i=1}^{n}|\langle v(x_i), y_i'\rangle| = |\mathrm{trace}(1_U a v\beta)|\leqslant\|\beta\|\,\eta^*(v)\,\|a\|\,\eta(1_U).$$

But $\eta(1_U)=1$, as the unit vector basis for U has unconditional constant 1, $\|\beta\|\leqslant 1$ and $\|a\|\leqslant\|f\|$, so that $1\leqslant\|f\|$.

By the well-known separation theorem and the Riesz theorem there is a measure μ on $K_{E'}\times K_{F''}$ with $\mu(g)\leqslant 1\leqslant\mu(f)$, $f\in S$, $g\in C$. It is easily seen that μ is a probability measure, and setting

$$f(x', y'')=\eta^*(v)\,|\langle v(x), y'\rangle|^{-1}|\langle x, x'\rangle\langle y', y''\rangle|$$

establishes the integral formula.

For $F\subseteq E$ let j_F be the inclusion of F into E. By definition (cf. [7]) E has *local unconditional structure* iff

$$\varkappa_u(E)=\sup_F\eta(j_F)<\infty,$$

where the supremum is taken over all finite-dimensional subspaces of E. Notice that if E is finite-dimensional, $\varkappa_u(E)=\eta(E)=$ the weakly nuclear norm of the identity on E. We now give a new, more direct, proof of the following result of [7] (Theorem 3.5).

THEOREM 1.4. $\eta(L(l_2^n, l_2^n))\geqslant 2n^{1/2}/3\pi$.

Proof. Let G be the group of isometries of l_2^n, dg be the normalized Haar measure on G, S the unit sphere of l_2^n and dm the normalized $(n-1)$-dimensional rotational invariant measure on S. Define μ on the product of the closed unit balls of $L(l_2^n, l_2^n)$ and $L(l_2^n, l_2^n)'=I_1(l_2^n, l_2^n)$ by

$$\mu(f)=\int_S\int_S\int_G f(g, x\otimes y)\,dg\,dm(x)\,dm(y).$$

Then for $u\in L(l_2^n, l_2^n)$ and $v\in L(l_2^n, l_2^n)'$,

$$\begin{aligned}\mu(|\langle u, \cdot\rangle\langle v, \cdot\rangle|) &= \Big(\int_G|\mathrm{trace}(gu)|\,dg\Big)\Big(\int_S\int_S|\langle v(x), y\rangle|\,dm(x)\,dm(y)\Big)\\ &\geqslant 3^{-1}n^{-1/2}\pi_2(u)\,2(n\pi)^{-1}\pi_2(v),\end{aligned}$$

by the inequalities of Theorem 5.2 and 2.2(b) of [7]. By [14]

$$|\langle u, v\rangle| = |\mathrm{trace}(uv)|\leqslant i_1(uv)\leqslant\pi_2(u)\pi_2(v),$$

which gives

$$|\langle u, v\rangle|\leqslant(3\pi/2)n^{3/2}\mu(|\langle u, \cdot\rangle\langle v, \cdot\rangle|).$$

By Proposition 1.3 this implies that

$$\eta^*\big(L(l_2^n, l_2^n)\big)\leqslant(3\pi/2)n^{3/2},$$

and it is clear that

$$n^2 \leqslant \eta^*\big(L(l_2^n, l_2^n)\big)\,\eta\big(L(l_2^n, l_2^n)\big).$$

THEOREM 1.5. *There is a diagonal operator u on l_2 with the following property: The map s defined on $C(l_2, l_2)$ by $s(v) = u\circ v$ is compact but not weakly nuclear.*

Proof. Write the usual basis of l_2 as a doubly indexed sequence $(e_{,i})$, $1 \leqslant i \leqslant n2^{2n}$, $n = 1, 2, \ldots$, and let $u(e_{n,i}) = 2^{-n}e_{n,i}$. For each n, denote by E_n the span of $e_{n,i}$, $1 \leqslant i \leqslant n2^{2n}$, let j_n be the inclusion of E_n into l_2 and p_n the orthogonal projection onto E_n. Define s_n on $L(E_n, E_n)$ by $s_n(v) = p_n s(j_n v p_n) j_n$. Since

$$s(v) = \sum_{n=1}^{\infty} j_n s(v) p_n \quad \text{for all } v \in C(l_2, l_2),$$

s is compact and $\eta(s) \geqslant \eta(s_n)$ for each n. By Theorem 1.4

$$\eta(s_n) = 2^{-n}\eta\big(L(E_n, E_n)\big) \geqslant 2^{-n}(2/3\pi)(\dim E_n)^{1/2} = 2n^{1/2}/3\pi$$

for each $n \geqslant 1$. Therefore, $\eta(s) = \infty$.

Remark. The sequence defining the operator u is in l_p for all $p > 2$ but is not in l_2, i.e. u is not Hilbert-Schmidt. Note that for u Hilbert-Schmidt the map s defined above factors through a Hilbert space, so that $\eta^{**}(s) < \infty$. The problem of the existence of a compact, non-weakly nuclear operator was raised by A. Pietsch [16].

§ 2. Ideal norms on Hilbert spaces. Let $1 \leqslant q \leqslant p \leqslant \infty$ and $u\colon E \to F$, let $i_{pq}(u) = \inf\|\alpha\|\,\|\beta\|$, where the infimum is taken over all probability measures μ and pairs of operators $\alpha \in L\big(E, L_p(\mu)\big)$, $\beta \in L\big(L_q(\mu), F''\big)$ for which $iu = \beta j\alpha$, i is the canonical map from F into F'' and j is the inclusion of $L_p(\mu)$ into $L_q(\mu)$. The ideal norm i_{pq} has been studied in [6] and [10], and Lapreste has shown i_{pq} to be perfect. The values $i_{pq}(l_2^n)$, $i_{pq}(l_2)$ were calculated in [6].

For α an ideal norm and $u\colon E \to F$, write $/\alpha(u) = \alpha(u\eta)$, where $\eta\colon L_1(\mu) \to E$ is any quotient map. The norm $/\alpha$ so defined is called the *left injective envelope* of α. Similarly, the *right injective* of α, $\alpha\backslash$ is defined as $\alpha\backslash(u) = \alpha(\varphi u)$, where $\varphi\colon F \to L_\infty(\mu)$ is any isometric embedding. The *injective envelope* of α is $/\alpha\backslash = /(\alpha\backslash) = (/\alpha)\backslash$.

The *left projective* envelope of α, $\backslash\alpha$, is defined as $\backslash\alpha(u) = \alpha(v)$, where $j_F\colon F \to F''$ is the canonical map, $v\varphi = j_F u$ and $\varphi\colon E \to L_\infty(u)$ is any isometric embedding. Similarly, the *right projective* envelope of α, $\alpha\backslash$, is defined as $\alpha/(u) = \alpha(w)$, where $j_F u = \eta'' w$ and $\eta\colon L_1(\mu) \to F$ is any quotient map. The *projective envelope* of α is $\backslash\alpha/ = (\backslash\alpha)_/ = \backslash(\alpha/)$.

It is easy to see that $(\alpha\backslash)^* = \backslash\alpha^*$, $(/\alpha)^* = \alpha^*/$, and that if α is perfect, so are all the envelopes of α [9].

LEMMA 2.1. *For $1 \leqslant q \leqslant p \leqslant \infty$, $u\colon E \to F$, $v\colon F \to G$,*

$$\pi_{p'}\big((vu)'\big) \leqslant /i_{pq}(u)\,\pi_{q'}(v)$$

and

$$\pi_1\big((vu)'\big) \leqslant /\pi_p(u)\,i_{p'}(v).$$

Proof. For the first inequality, let φ be an isometric embedding of E' into an $L_\infty(\mu)$-space and let w be a p-absolutely summing operator into G'. First notice

$$i_1\big(\varphi(vu)'w\big) = i_1(w'v''u''\varphi') \leqslant i^*_{pq}(w'v'')\,i_{pq}(u''\varphi').$$

Since φ' is a quotient map from $L_\infty(\mu)'$ onto E'',

$$i_{pq}(u''\varphi') = /i_{pq}(u'') = /i_{pq}(u),$$

and, by Theorem 2.15 of [6],

$$i^*_{pq}(w'v'') \leqslant \pi_{q'}(v'')\,\pi_p(w'') = \pi_{q'}(v)\,\pi_p(w).$$

Since w was arbitrary, combining inequalities yields

$$i_{p'}\big(\varphi(vu)'\big) = \pi^*_p\big(\varphi(vu)'\big) \leqslant /i_{pq}(u)\,\pi_{q'}(v),$$

and finally

$$\pi_{p'}\big((vu)'\big) = i_{p'}\backslash\big((vu)'\big) = i_{p'}\big(\varphi(vu)'\big).$$

The second inequality of the lemma follows similarly, using the fact that the integral and absolutely summing norms coincide for operators into L_∞-spaces.

THEOREM 2.2. *Let $1 \leqslant q \leqslant p \leqslant \infty$ and H be a real Hilbert space.*

(a) *For $\dim(H) = n$,*

$$/i_{pq}(l_2^n) = \pi_{p'}(l_2^n)\,\pi_{q'}(l_2^n)^{-1},$$

$$/i_q(l_2^n) = \pi_1(l_2^n)\,\pi_{q'}(l_2^n)^{-1}$$

and

$$/\pi_p(l_2^n) \geqslant n^{-1}\pi_1(l_2^n)\,\pi_p(l_2^n).$$

(b) *For H infinite-dimensional and $1 < q$,*

$$/i_{pq}(H) = \pi^{(1/q-1/p)/2}\,\Gamma\Big(\frac{p'+1}{2}\Big)^{-1/p'}\Gamma\Big(\frac{q'+1}{2}\Big)^{1/q'},$$

and

$$/i_q(H) = \pi^{1/2q}\,\Gamma\Big(\frac{q'+1}{2}\Big)^{1/q'}.$$

(c) *For H infinite-dimensional,*

$$/\pi_p(H) \geqslant 2^{-1}\pi^{(1+1/p)/2}\,\Gamma\Big(\frac{p+1}{2}\Big)^{-1/p}.$$

Proof. To show the first equality in (a), let φ be an isometric embedding of l_2^n into an $L_\infty(\mu)$-space C, and let dm be the normalized $(n-1)$-dimensional rational invariant measure on the unit sphere S of l_2^n. Let α be the isometric embedding of l_2^n into $L_{p'}(S)$ given by $\alpha(x) = \pi_{p'}(l_2^n)\langle x, \cdot\rangle$ ([4]), and let $\tilde\varphi$ be a norm one extension of φ to $L_{p'}(S)$. Consider the factorization of φ

$$l_2^n \xrightarrow{\beta} L_{q'}(S) \xrightarrow{\gamma} L_{p'}(S) \xrightarrow{\tilde\varphi} C,$$

where γ is the inclusion and $\gamma\beta = \alpha$. Then

$$/i_{pq}(l_2^n) = i_{q'p'}\backslash(l_2^n) = i_{q'p'}(\tilde\varphi\alpha) \leqslant \|\beta\|.$$

Again by [4]

$$\|\beta\| = \sup_{\|x\|_2=1} \pi_{p'}(l_2^n)\Big(\int |\langle x, t\rangle|^{q'} dm(t)\Big)^{1/q'} = \pi_{p'}(l_2^n)\pi_{q'}(l_2^n)^{-1}.$$

The other inequality follows by applying Lemma 2.1 with both u and v the identity operators on l_2^n.

The second equality of (a) follows from the first by taking $p = \infty$, and the third estimate follows from the lemma.

Since i_{pq} is perfect, $i_{pq}(H) = \lim_{n\to\infty} i_{pq}(l_2^n)$ for any infinite-dimensional H. From the results of [4] and Stirling's formula, $n^{1/2}\pi_s(l_2^n)^{-1}$ has limit

$$2^{1/2}\pi^{-1/2s}\Big(\Gamma\Big(\frac{s+1}{2}\Big)\Big)^{1/s}$$

for each s, $1 \leqslant s < \infty$, so the infinite-dimensional results (b) and (c) follow from (a).

Remarks. (1) The constant $/i_{pq}(H)$ has a geometrical interpretation. For μ a probability measure and $E \subset L_q(\mu)$ a closed infinite-dimensional subspace, the same proof as above shows that

$$/i_{q'p'}(E) \leqslant \sup\{\|f\|_p \|f\|_q^{-1};\ f \in E\}$$

for each $p, q \leqslant p < \infty$. But also by Dvoretzky's theorem [2], E has quotients $(1+\varepsilon)$-isometric to l_2^n for each n, so that $/i_{q'p'}(l_2) \leqslant /i_{q'p'}(E)$ for infinite-dimensional E. Thus $/i_{q'p'}(l_2)$ is a universal lower bound for the ratio of the L_p-norm and L_q-norm on infinite-dimensional subspaces of L_q.

(2) The constants $/i\,(H)$ and $/\pi_p(H)$ represent, respectively, the p-integral and p-absolutely summing norms of any quotient map from an $L_1(\mu)$-space onto H. By the lifting property of operators on L_1-spaces it follows that $i_p(u) \leqslant /i_p(H)\|u\|$ for all $u \in L(L_1(\mu), H)$, and further the constant is the best possible. The constant $/\pi_1(H)$, for H infinite-dimensional, is known as the Grothendieck constant K_G [8], its exact value is unknown, but (c) gives the well-known inequality $/\pi_1(H) \geqslant \pi/2$ of [8].

(3) It is proved in [5] that $\alpha(l_2^n)\alpha^*(l_2^n) = n$ for any ideal norm α, thus Theorem 2.2 gives information on $(/i_{pq})^* = i_{pq}^*/$, $(/i_q)^* = i_q^*/ = \pi_{q'}/$ and $(/\pi_p)^* = \pi_p^*/ = i_{p'}/$.

Corollary 2.3. *For each* n

$$/\pi_2(l_2^n) = \pi_1(l_2^n)n^{-1/2}, \quad \textit{and} \quad /\pi_2(l_2) = (\pi/2)^{1/2}.$$

Proof. This follows from part (a) as $\pi_2 = i_2$.

From the calculations of $/\pi_2$ we easily derive the following result of Grothendieck [8].

Corollary 2.4. *For each* n,

$$/\gamma_2^*\backslash(l_2^n) = \big(\pi_1(l_2^n)\big)^2 n^{-1}, \quad \textit{and} \quad /\gamma_2^*\backslash(l_2) = \pi/2.$$

Proof. For η: $L_1(\mu)\to l_2^n$ quotient map and φ: $l_2^n \to C$ an isometric embedding,

$$/\gamma_2^*\backslash(l_2^n) = \gamma_2^*(\varphi\eta) \leqslant \pi_2(\eta)\pi_2(\varphi') = \big(\pi_1(l_2^n)\big)^2 n^{-1},$$

the inequality by [9]. For the reverse inequality, factor the identity of l_2^n as uv, where $v \in L(l_2^n, C)$, $u \in L(C, l_2^n)$ satisfy $\|u\|\|v\| = \gamma_\infty(l_2^n)$, the projection constant of l_2^n. But $\gamma_\infty(l_2^n)\pi_1(l_2^n) = n$, so

$$\begin{aligned} n = i_1(l_2^n) &= i_1(uvv'u') \leqslant \|u\|\gamma_{2|}^*(vv')\|u'\| \\ &\leqslant \|u\|\|u'\|\|v\|\|v'\|/\gamma_2^*\backslash(l_2^n). \end{aligned}$$

This establishes the first equality, and the second follows as $/\gamma_2^*\backslash(l_2) = \sup /\gamma_2^*\backslash(l_2^n)$.

Remarks. Using the equality $\alpha(l_2^n)\alpha^*(l_2^n) = n$ for any ideal norm α, we obtain

$$\backslash\gamma_2/(l_2^n) = n\big(/\gamma_2^*\backslash(l_2^n)\big)^{-1}$$

is the norm $\gamma_2(u)$ in the factorization 1: $l_2^n \xrightarrow{j} C \xrightarrow{u} L_1 \xrightarrow{\eta} l_2^n$.

Pełczyński [13] first proved that all of the p-absolutely summing norms coincide for operators between Hilbert spaces. The best constant relating the absolutely summing norm and Hilbert-Schmidt norm has been calculated by Grothendieck [8], and Garling [3] has done the same for the p-absolutely summing and Hilbert-Schmidt norms.

Corollary 2.5. *Let H_1 and H_2 be infinite-dimensional Hilbert spaces and $1 \leqslant q \leqslant p < \infty$. For u: $H_1 \to H_2$*

$$\pi_q(u) \leqslant /i_{q'p'}(l_2)\pi_p(u),$$

and the constant $/i_{q'p'}(l_2)$ is the best possible.

Proof. By Lemma 2.1, $\pi_q(u') \leqslant /i_{q'p'}(H_1)\pi_p(u)$. Also H_2' is a quotient of an $L_{q'}$-space, so by [6], Theorem 3.3, $\pi_q(u) \leqslant \pi_q(u')$. To see that the inequality is sharp, take u to be an operator which is the identity on l_2 and apply Theorem 2.2.

§ 3. Operator characterizations of l_1. We show here that a theorem of Lindenstrauss–Pełczyński is a consequence of the following result.

THEOREM 3.1. *Let $1 < p \leqslant 2$ and m be the positive integer with $m-1 < p' \leqslant m$. If E is an n-dimensional space and λ is the norm of the natural map from $L(l_\infty^n, E)$ into $I_1(l_p^n, E)$, then*

$$d(E, l_1^n) \leqslant \big(\lambda x(E)\big)^m.$$

Proof. There is no loss in generality in assuming $p' = m$. Let $B = (b_i)_{i\leqslant n}$ be a normalized basis for E with coefficient functionals $\{b_i\}_{i\leqslant n}$. We claim that for $k = 1, 2, \ldots, m$,

$$\|t\|_{p'/k} \leqslant \big(\lambda x(B)\big)^k \Big\|\sum_{i=1}^n t_i b_i\Big\|$$

for all n-tuples $t = (t_i)_{i\leqslant n}$ of scalars.

To see this for $k = 1$, choose $x_i' \in E'$, $1 \leqslant i \leqslant n$, so that $\|x_i'\| = 1$ and $\langle b_i, x_i'\rangle = 1$. For n-tuples of scalars, t and s, define $v \in L(l_\infty^n, E)$ and $u \in L(E, l_p^n)$ by

$$v(a) = \sum_{i=1}^n a_i t_i b_i \quad \text{and} \quad u(x) = (s_i\langle x, x_i'\rangle)_{i\leqslant n}.$$

Write w for the inclusion of l_p^n into l_∞^n. Then

$$|\langle s, t\rangle| = |\operatorname{trace}(uvw)| \leqslant i_1(uvw) \leqslant \lambda\|u\|\,\|v\|$$
$$\leqslant \|s\|_p \lambda x(B) \Big\|\sum_{i=1}^n t_i b_i\Big\|.$$

Maximizing for $\|s\|_p = 1$ gives the claim for $k = 1$.

Now assume the claim is true for some $k < m$. For s and t n-tuples of scalars, define v and w as above, and let $u \in L(E, l_p^n)$ be given by $u(x) = (\langle x, x_i'\rangle s_i)_{i\leqslant n}$. As the claim is true for k, $\|u\| \leqslant \|s\|_r \big(\lambda x(B)\big)^k$, where $1/r = 1/p - k/p'$, so that proceeding as above gives

$$|\langle s, t\rangle| \leqslant \|s\|_r \big(\lambda x(B)\big)^k \Big\|\sum_{i=1}^n t_i b_i\Big\|.$$

Taking the supremum over $\|s\|_r = 1$ gives the claim for $k+1$. Finally, as $p' \leqslant m$ and $\|b_i\| = 1$ for each i, the estimates

$$\Big\|\sum_{i=1}^n t_i b_i\Big\| \leqslant \|t\|_1 \leqslant \big(\lambda x(B)\big)^m \Big\|\sum_{i=1}^n t_i b_i\Big\|$$

hold for all n-tuples t, which completes the proof.

The projection constant of a space E is $\gamma_\infty(E)$. Recall that $i_1(v) \leqslant \gamma_\infty(E)\pi_1(v)$ for each operator v defined on E. Thus

COROLLARY 3.2 [11]. *There is a constant k, $2 \leqslant k \leqslant e$, such that $d(E, l_\infty^n) \leqslant k\big(x(E)\gamma_\infty(E)\big)^2$ for each n-dimensional space E.*

Proof. Let w be the inclusion of l_2^n into l_∞^n. For $u \in L(l_\infty^n, E')$

$$i_1(uw) = i_1(w'u') \leqslant \gamma_\infty(E)\pi_1(w'u') \leqslant \pi_1(w')\gamma_\infty(E)\|u\|.$$

It is well known and easy to show from Khintchin's inequalities that $\pi_1(w')$ is uniformly bounded, $2^{1/2} \leqslant \pi_1(w') \leqslant e^{1/2}$, the last by [17].

One infinite-dimensional version of Theorem 3.1 may be phrased as follows:

COROLLARY 3.3. *Let $1 < p \leqslant 2$ and let E be a space with an unconditional basis. If every continuous operator from c_0 into E naturally induces an integral operator from l_p to E, then E is isomorphic to l_1.*

Proof. The closed graph theorem gives a constant λ such that $i_1(uw) \leqslant \lambda\|u\|$ for all $u \in L(c_0, E)$, where w is the inclusion of l_p into c_0. The corollary follows by normalizing the basis and applying the inequalities of Theorem 3.1 to the span of the first n basis elements, $n = 1, 2, \ldots$

Remarks. Every space E satisfies the hypothesis of Corollary 3.3 for $p = 1$, as the inclusion of l_1 into c_0 is integral. However, no infinite-dimensional space E has the above property for $p > 2$ for the following reason: If $i_1(uw) \leqslant \lambda\|u\|$ for all $u \in L(c_0, E)$, then taking an arbitrary finite sequence $\{x_i\}_{i\leqslant n} \subset E$, let $x_i' \in E'$ with $\|x_i\| = \langle x_i, x_i'\rangle$, and define $u \in L(c_0, E)$ and $v \in L(E, l_p)$ by:

$$u(a) = \sum_{i=1}^n a_i x_i \quad \text{and} \quad v(x) = (s_i\langle x, x_i'\rangle)_{i\geqslant 1}.$$

Then

$$\sum_{i=1}^n s_i\|x_i\| = \operatorname{trace}(vuw) \leqslant i_1(vuw) \leqslant \|v\|\lambda\|u\|$$
$$\leqslant \lambda\|s\|_p \max_{\pm} \Big\|\sum_{i=1}^n \pm x_i\Big\|,$$

so that

$$\Big(\sum_{i=1}^n \|x_i\|^{p'}\Big)^{1/p'} \leqslant \lambda \max_{\pm} \Big\|\sum_{i=1}^n \pm x_i\Big\|,$$

that is, the identity operator on E is $(p', 1)$ summing, and according to the Dvoretzky-Rogers lemma (cf. [1], Lemma 1, p. 61), this is impossible for $2 > p' \geqslant 1$ and E infinite-dimensional.

An elementary localization argument, using Khintchin's inequality as in Corollary 3.2, shows that each $\mathscr{L}_1$-space has the stated property for $1 < p \leqslant 2$, so that Corollary 3.3 is a partial converse of this fact. Actually, a formally stronger converse follows from the same proof: If E has sufficiently many Boolean algebras of projections (in the sense of [12]) and if every operator from c_0 into E induces an integral operator on l_p, $1 < p \leqslant 2$, then E is an $\mathscr{L}_1$-space.

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Received March 20, 1973 (804)

STUDIA MATHEMATICA, T. LIV. (1975)

A unified approach to Riesz type representation theorems

by

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Abstract. We establish abstract versions of the Riesz representation theorem. Necessary and sufficient conditions for the existence of regular finitely additive, σ-additive and τ-additive representing measures are found. A methodological simplification is obtained by constructing the measures directly, rather than *via* a preliminary extension of the linear functional. Thus our approach is in agreement with the viewpoints of Alexandroff rather than with those of Bourbaki. We are able to easily deduce the Daniell extension theorem as well as numerous topological representation theorems such as those developed by Radon, Markoff, Alexandroff, Hewitt, LeCam, Mařík and Varadarajan. Indeed, these results are sometimes strengthened. Our method is based on the theory developed by the second author; hopefully, our results demonstrate the usefulness of this theory.

1. A common problem in Functional Analysis is whether a given bounded linear functional defined on a vector lattice of real valued functions is representable as an integral with respect to some suitable regular measure. By well-known techniques this problem can be reduced to the following situation:

On a set X there is given a convex cone $\mathscr{C}$ of non-negative real functions, closed under the finite lattice operations and containing the zero function. A non-negative, monotone, linear functional T is defined on $\mathscr{C}$. That is, our basic assumptions are:

A1. *$\mathscr{C}$ is a $(0, \bigvee f, \bigwedge f)$ convex cone in $[0, \infty[^X$;*

A2. *$T\colon \mathscr{C} \to [0, \infty[$,*
$T(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 Th_1 + \alpha_2 Th_2$ for $\alpha_1, \alpha_2 \geqslant 0$ and $h_1, h_2 \in \mathscr{C}$, $h_1 \leqslant h_2$ and $h_1, h_2 \in \mathscr{C}$ implies that $Th_1 \leqslant Th_2$.

* Supported by an Australian National University Ph. D. scholarship and by the Danish Natural Science Research Council.
** Supported by the Danish Natural Science Research Council.