Numerical range preserving operators on a Banach algebra

by

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Abstract. Let $A$ be a unital Banach algebra and let $T_1$ and $T_2$ be bounded linear operators on $A$. We find necessary and sufficient conditions for the numerical range of $T_1a$ to be contained in the numerical range of $T_2a$ for each $a \in A$.

Let $A$ be a complex Banach algebra with unit $e$. If $M$ is a linear subspace of $A$ containing $e$ and $M^*$ is its dual space, we let $D(M, e) = \{\varphi \in M^*: 1 = \varphi(e) = ||\varphi||\}$. The set $D(A, e)$ is called the state space of $A$. The numerical range of an element $a \in A$ is defined by

$$V(A, e) = \{\varphi(a): \varphi \in D(A, e)\}.$$  

The basic facts about the numerical range may be found in [1] and [8].

In this paper we determine necessary and sufficient conditions for a pair of operators $T_1$ and $T_2$ on $A$ to satisfy

$$V(A, T_1a) \subseteq V(A, T_2a)$$

for each $a \in A$ (see Theorem 2.2).

In the case where $A = C(X)$ for some compact space $X$ we have

$$V(A, f) = \text{co } f(X)$$

for each $f \in C(X)$ [8]. Here co means closed convex hull. Theorem 2.3 then states that $T_1f(X) \subseteq \text{co } T_2f(X)$ for each $f \in C(X)$ if and only if $T_1 = ST_2$ where $S$ is a positive operator on $C(X)$ (by positive we mean $S(1) = 1$ and $Sf \geq 0$ whenever $f \geq 0$). In the case where $T_1$ is the identity operator the theorem was originally proven by R. R. Phelps [7].

We begin in Section 1 with some remarks on positive operators. Section 2 contains the main result. Finally, in Section 3, we present an application of our results.

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1. In this section we will consider a class of operators whose adjoints preserve states. More precisely,
Definition 1.1. Let $M$ and $N$ be subspaces of Banach algebras $A$ and $A$ with units $e$ and $\hat{e}$. A bounded linear operator $S : M \to N$ will be called state preserving if $S^* D(N, e) \subset D(M, e)$.

We note two properties of state preserving operators.

Theorem 1.2. Let $S : M \to N$ be a state preserving operator where $M$ and $N$ are as in 1.1. Then

(i) $S e = \hat{e}$.

(ii) If $a \in M$ and $V(a, e) \subset R^+$, then $V(Sa, e) \subset R^+$.

Proof. If $\phi \in D(N, \hat{e})$, then $\phi(Sa) = S^* \phi e = \phi e = 1$. Hence $V(a, e) \subset \hat{e}$ by [1], page 34. Thus $S e = \hat{e}$ by Lemma 1.1.1.1.

Since $V(a, Sa) \subset R^+$ and $\phi \in D(N, e)$, then $\phi(Sa) = S^* \phi e \subset R^+$. Thus $V(\hat{e}, Sa) \subset R^+$.

We remark that (i) and (ii) do not imply that $S$ preserves states.

To see this we expand on the construction of Pheaps [6]. Let $H(A) = \{a \in A : V(a, e) \subset R\}$ and $J(A) = H(A) + iH(A)$. $J(A)$ is a closed subspace of $A$. The Vidar-Palmer Theorem [1] implies $J = J(A)$ if and only if $A$ is isometrically isomorphic to a C* algebra. Now let $A_0$ be any Banach algebra for which $J(A_0) \neq A_0$. Pick an element $a_0 \in A_0 \setminus J(A_0)$ of norm 1 and a closed subspace $N$ of $A_0$ such that $J(A_0) \subset N \subset \overline{N} = N \subset \overline{N}$. Let $a_0$ be the one-dimensional subspace spanned by $a_0$. Now select $\phi \in D(A_0, e)$ such that $\phi(a_0) \neq 0$ and $\delta > 1$ such that $\delta |\phi(a_0)| > 1$. Define an operator $T : A_0 \to A_0$ by $(a + i \lambda a_0) \mapsto a + i \lambda a_0$. Then $T$ is a bounded linear operator satisfying (i) and (ii) but $T \phi \neq \phi(D, e)$.

We close this section by remarking that if $A$ is a C* algebra the usual notion of positivity for an element $a \in A$ is equivalent to $V(a, e) \subset R^+$ [1]. An operator satisfying (i) and (ii) of 1.2 is called a normalized positive operator. It is easy to show that for a C* algebra an operator $T$ on $A$ is state preserving if and only if it is a normalized positive operator.

2. We begin by quoting a result of R. T. Moore [4] and A. M. Sinclair [7] which we will need in the sequel. (See also [2], §3.1).

Lemma 2.1. Let $A$ be a unital Banach algebra. If $L \in A^*$, there exist $a_1, \ldots, a_n \in R^+$ and $L_1, \ldots, L_n \in D(A, e)$ such that

\[
L = (a_1 L_1 - a_2 L_2) + i (a_3 L_3 - a_4 L_4)
\]

and

\[
a_1 + a_2 + a_3 + a_4 \leq \|L\| \leq \|L\| + \|L_4\|.
\]

where $\|L\| = \sup(|\lambda| : \lambda \in \sigma(A))$.

Theorem 2.2. Let $A$ be a unital Banach algebra and let $T_1$ and $T_2$ be bounded linear operators on $A$. Then the following are equivalent:

(i) $V(\hat{e}, T_1 a) \subset V(\hat{e}, T_2 a)$ for each $a \in A$.

(ii) $T_1^* D(A, e) \subset T_2^* D(A, e)$.

If $\phi \in E(T_2)$ then (i) and (ii) are equivalent to

(iii) $T_1 = \lambda T_2$, where $\lambda 
\geq \mathcal{E}(T_2) \to A$ is state preserving.

Proof. (i) $\Rightarrow$ (ii). Suppose that (ii) does not hold. Then there exists a state $\phi \in D(T_2^*, \hat{e})$ such that $T_1^* \phi \neq T_2^* \phi$. Since $T_2^*$ is continuous when $A^*$ has the weak* topology ([3], p. 478), $T_2^* D(A, e)$ is weak* compact and convex. By [3], p. 417, and [3], p. 421, there exists $a \in A$ and constants $e$ and $\epsilon > 0$, such that

\[
\mathcal{E}(T_1^* \phi) \leq e - \epsilon < e \leq \mathcal{E}(T_2^* \phi).
\]

This implies $\mathcal{E}(T_1 \phi) < e \leq \mathcal{E}(T_2 \phi)$ and thus contradicts (i).

(ii) $\Rightarrow$ (i). The proof is immediate from the definitions involved.

(iii) $\Rightarrow$ (ii). Let $T_1$ be $T_2$ considered as a map from $\mathcal{E}(T_2) \to A$. Thus $T_2^* : A^* \to A^*$. By the Hahn–Banach theorem the restriction mapping $\phi \mapsto \phi|A$ maps $D(A, e)$ onto $D(A_1, e)$. We also have $T_2^* \phi|A = T_2^* \phi$. Thus we have

\[
\]

Since by assumption, $V^* D(A, e) \subset D(A_1, e)$, we have

\[
\]

(iii) $\Rightarrow$ (i). By (2) and (ii) we conclude that $T_1^* D(A, e) \subset T_2^* D(A, e)$.

Hence by Lemma 2.3, $T_1^* A^* \subset T_2^* A^*$. Thus, for each $a \in A^*$, there exists a unique $\phi^L \in A^*$ such that $T_1^* \phi^L = T_2^* \phi^L$ (the uniqueness of $\phi^L$ follows from the uniqueness of $F(L)$ one can easily show that $F^L : A^* \to A^*$ is a linear map. Since $T_1^* D(A, e) \subset T_2^* D(A, e)$, we see that $F D(A, e) \subset D(A_1, e)$.

Thus we have

\[
L = (a_1 L_1 - a_2 L_2) + i (a_3 L_3 - a_4 L_4)
\]

where $a_1, a_2, a_3, a_4 \in R^+$ and $L_1, L_2, L_3, L_4 \in D(A, e)$. Thus, again by 2.1 and [3], page 34, we have

\[
\|F(L)\| \leq a_1 \leq a_2 + a_3 + a_4 \leq V^2 \sup \{\|L(a)\| : a \in A, \|a\| \leq 1\}
\]

\[
\leq \|L_4\|.
\]

If $\phi \in A^*$, then $|\phi(T_2 a)| \leq \|F(\phi)\| \|T_2 a\|$. Hence

\[
\|T_2 a\| = \sup \{\|\phi(T_2 a)\| : \phi \in A^*, \|\phi\| = 1\}
\]

\[
\leq \sup \{\|F(\phi)\| : \|T_2 a\| : \|\phi\| = 1\}
\]

\[
\leq \|F\| \|T_2 a\|.
\]
Define $V: R(T_2) \to A$ by $VT_2a = T_2a$. By (3), $V$ is well defined and bounded. Hence it may be extended to all of $A_4$. If we also call the extension $V$, then we have $V^* T_5 = T_5 V = T_1$. Hence $T_1^* V^* = T_1^* T_2^* F$ and we conclude, since $T_2^* F = F$, that $V^* = F$. Therefore, $VS(D(A_2) = D(A_2) e$.

THEOREM 2.3. Let $A$ be an unital Banach algebra and $T_1$ and $T_2$ bounded linear operators on $A$. Let $a \in A$, where $A_4 = R(T_2)$, $i = 1, 2$. Then the following are equivalent:

(i) $V(A, T_i a) = V(A, T_2 a)$, $a \in A$.

(ii) $T_i = VT_i$ where $V: A_4 \to A_4$ is an invertible operator such that $V$ and $V^{-1}$ are state preserving.

Proof. (ii) ⇒ (i). By Theorem 2.2 we have $V(A_i, T_i a) = V(A_2, T_2 a)$ for each $a \in A$. But, by (1), p. 16, we have that $V(A_i, b) = V(A, b)$ for all $a, b$, $i = 1, 2$. Hence (ii) follows.

(iii) ⇒ (ii). Theorem 2.2 implies that $T_1 = VT_2$ and $T_2 = VT_1$ where $V: A_4 \to A$ and $V: A_4 \to A$ are state preserving. Let $V$ be $V$ as a map from $A_4 \to A$ and let $W$ be $V$ as a map from $A_4 \to A$. Then $V$ and $W$ are state preserving and $V = W^{-1}$.

3. In this section we present an application of Theorem 2.2. We recall that an invertible linear transformation $J$ on a $C^*$-algebra $A$ is a $C^*$-isomorphism (also called a Jordan isomorphism) if $J(A^*) = J(A^*)$ and $J(a^*) = J(a)^*$ for each self-adjoint element $a$ and each positive integer $n$. These maps were first defined and studied by Kadison. An elegant statement of Kadison's results that will be used in this section can be found in [9], Theorem 1.1.

THEOREM 3.1. Let $A$ be a $C^*$-algebra with identity $e$ and $T_1$ and $T_2$ bounded linear operators on $A$. Suppose that $T_1$ and $T_2$ have dense ranges. Then, $V(A, T_1 a) = V(A, T_2 a)$ for each $a \in A$ if and only if $T_1 = J T_2$ where $J$ is a $C^*$-isomorphism.

Proof. Suppose $T_1 = J T_2$ Then, by [9], Theorem 1.1, we have that $J$ is bipositive (i.e., $V(A, J e) \subseteq R^+$ if and only if $V(A, e) \subseteq R^+$) and $J e = e$. For a $C^*$-algebra $A$ this implies $J$ and $J^{-1}$ preserve states. Hence $V(A, T_1 a) = V(A, J T_2 a) = V(A, T_2 a)$ for each $a \in A$ by Theorem 2.3. Conversely, suppose $V(A, T_1 a) = V(A, T_2 a)$ for all $a \in A$. Then, by Theorem 2.3, $T_1 = J T_2$ where $J e = e$ and $J$ and $J^{-1}$ preserve states. Thus, by Theorem 2.3, $T_1 = J T_2$ where $J e = e$ and $J$ and $J^{-1}$ preserve states. Thus, by Theorem 1.1, $J$ is bipositive. Thus, by [9], Theorem 1.1, $J$ is a $C^*$-isomorphism.

References


