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Numerical range preserving operators on a Banach algebra

by

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Abstract. Let A be a unital Banach algebra and let T_1 and T_2 be bounded linear operators on A. We find necessary and sufficient conditions for the numerical range of $T_1 a$ to be contained in the numerical range of $T_2 a$ for each $a \in A$.

Let A be a complex Banach algebra with unit e. If M is a linear subspace of A containing e and M^* is its dual space, we let $D(M, e) = \{\varphi \in M^*: 1 = \varphi(e) = \|\varphi\|\}$. The set $D(A, e)$ is called the *state space* of A. The *numerical range* of an element $a \in A$ is defined by

$$V(A, a) = \{\varphi(a): \varphi \in D(A, e)\}.$$

The basic facts about the numerical range may be found in [1] and [8].

In this paper we determine necessary and sufficient conditions for a pair of operators T_1 and T_2 on A to satisfy

$$V(A, T_1 a) \subset V(A, T_2 a) \tag{1}$$

for each $a \in A$ (see Theorem 2.2).

In the case where $A = C(X)$ for some compact space X we have $V(A, f) = \operatorname{co} f(X)$ for each $f \in C(X)$ [8]. Here co means closed convex hull. Theorem 2.2 then states $(T_1 f)(X) \subset \operatorname{co}(T_2 f)(X)$ for each $f \in C(X)$ if and only if $T_1 = ST_2$ where S is a positive operator on $C(X)$ (by *positive* we mean $S(1) = 1$ and $Sf \geqslant 0$ whenever $f \geqslant 0$). In the case where T_2 is the identity operator the theorem was originally proven by R. R. Phelps [5].

We begin in Section 1 with some remarks on positive operators. Section 2 contains the main result. Finally, in Section 3, we present an application of our results.

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1. In this section we will consider a class of operators whose adjoints preserve states. More precisely, DEFINITION 1.1. Let M and N be subspaces of Banach algebras A and $\hat{A}$ with units e and $\hat{e}$. A bounded linear operator $S\colon M\to N$ will be called *state preserving* if $S^*D(N,e)\subset D(M,e)$.

We note two properties of state preserving operators.

THEOREM 1.2. *Let $S\colon M\to N$ be a state preserving operator where M and N are as in* 1.1. *Then*

(i) $Se=\hat{e}$.

(ii) *If $a\in M$ and $V(A,a)\subset R^+$, then $V(A,Sa)\subset R^+$.*

Proof. If $\psi\in D(N,\hat{e})$, then $\psi(Se)=S^*\psi(e)=1$. Hence $V(\hat{A},Se-\hat{e})=\{0\}$ and thus $Se=\hat{e}$ by [1], page 34.

If $V(A,a)\subset R^+$ and $\varphi\in D(N,\hat{e})$, then $\varphi(Sa)=S^*\varphi(a)\in R^+$. Thus $V(\hat{A},Sa)\subset R^+$.

We remark that (i) and (ii) do not imply that S preserves states To see this we expand on a construction of Phelps [6]. Let $H(A)=\{a\in A\colon V(A,a)\subset R\}$ and $J(A)=H(A)+iH(A)$. $J(A)$ is a closed subspace of A. The Vidav–Palmer Theorem [1] implies $A=J(A)$ if and only if A is isometrically* isomorphic to a C^*-algebra. Now let A_0 be any Banach algebra for which $J(A_0)\neq A_0$. Pick an element $x_0\in A_0\setminus J(A_0)$ of norm 1 and a closed subspace N of A_0 such that $J(A_0)\subset N$, $N\cap[X_0]=\{0\}$ and $A_0=N+[X_0]$ where x_0 is the one-dimensional subspace spanned by x_0. Now select $\varphi_0\in D(A_0,e)$ such that $\varphi_0(x_0)\neq 0$ and $\delta>1$ such that $\delta|\varphi_0(x_0)|>1$. Define an operator $T\colon A_0\to A_0$ by $T(n+\lambda x_0)=n+\lambda\delta x_0$. Then T is a bounded linear operator satisfying (i) and (ii) but $T^*\varphi_0\notin D(A,e_0)$.

We close this section by remarking that if A is a C^*-algebra the usual notion of positivity for an element $a\in A$ is equivalent to $V(A,a)\subset R^+$ [1]. An operator satisfying (i) and (ii) of 1.2 is called a *normalized positive operator*. It is easy to show that for a C^*-algebra an operator T on A is state preserving if and only if it is a normalized positive operator.

2. We begin by quoting a result of R. T. Moore [4] and A. M. Sinclair [7] which we will need in the sequel. (See also [2], § 31.)

LEMMA 2.1. *Let A be a unital Banach algebra. If $L\in A^*$, there exist $a_1,\dots,a_4\in R^+$ and $L_1,\dots,L_4\in D(A,e)$ such that*

$$L=(a_1L_1-a_2L_2)-i(a_3L_3-a_4L_4)$$

and

$$a_1+a_2+a_3+a_4\leqslant\sqrt{2}\sup\{|L(a)|\colon a\in A,\ v(a)\leqslant 1\}$$

where $v(a)=\sup\{|\lambda|\colon \lambda\in V(A,a)\}$.

THEOREM 2.2. *Let A be a unital Banach algebra and let T_1 and T_2 be bounded linear operators on A. Then the following are equivalent:*

(i) *$V(A,T_1a)\subset V(A,T_2a)$ for each $a\in A$.*

(ii) *$T_1^*D(A,e)\subset T_2^*D(A,e)$.*

If $e\in\overline{R(T_2)}$ then (i) *and* (ii) *are equivalent to*

(iii) *$T_1=VT_2$ where $V\colon\overline{R(T_2)}\to A$ is state preserving.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (ii) does not hold. Then there exists a state φ_0 such that $T_1^*\varphi_0\notin T_2^*D(A,e)$. Since T_2^* is continuous when A^* has the weak* topology ([3], p. 478), $T_2^*D(A,e)$ is weak* compact and convex. By [3], p. 417, and [3], p. 421, there exists $a\in A$ and constants c and ε, $\varepsilon>0$, such that

$$\operatorname{Re}\hat{a}(T_1^*\varphi_0)\leqslant c-\varepsilon<c\leqslant\operatorname{Re}\hat{a}T_2^*D(A,e).$$

This implies $\varphi_0(T_1a)\notin V(A,T_2a)$ and thus contradicts (i).

(ii) $\Rightarrow$ (i). The proof is immediate from the definitions involved.

(iii) $\Rightarrow$ (ii). Let $\tilde{T}_2$ be T_2 considered as a map from $A\to\overline{R(T_2)}=A_2$. Thus $\tilde{T}_2^*\colon A_2^*\to A^*$. By the Hahn–Banach theorem the restriction mapping $\psi\to\psi/A_2$ maps $D(A,e)$ onto $D(A_2,e)$. We also have $\tilde{T}_2^*\psi/A_2=T_2^*\psi$. Thus we have

$$\tilde{T}_2^*D(A_2,e)=T_2^*D(A,e). \tag{2}$$

Since by assumption, $V^*D(A,e)\subset D(A_2,e)$, we have

$$T_1^*D(A,e)=\tilde{T}_2^*V^*D(A,e)\subset\tilde{T}_2^*D(A_2,e)=T_2^*D(A,e).$$

(ii) $\Rightarrow$ (iii). By (2) and (ii) we conclude that $T_1^*D(A,e)\subset\tilde{T}_2^*D(A_2,e)$. Hence by Lemma 2.1, $T_1^*A^*\subset\tilde{T}_2^*A_2^*$. Thus, for each $L\in A^*$, there exists a unique $F(L)\in A_2^*$ such that $T_1^*L=\tilde{T}_2^*F(L)$ (the uniqueness of $F(L)$ follows from the fact that $\tilde{T}_2$ has a dense range and thus $\tilde{T}_2^*$ is 1-1). From the uniqueness of $F(L)$ one can easily show that $F\colon A^*\to A_2^*$ is a linear map. Since $T_1^*D(A,e)\subset\tilde{T}_2^*D(A_2,e)$, we see that $FD(A,e)\subset D(A_2,e)$.

We now show that F is bounded. By 2.1, we may write

$$L=(a_1L_1-a_2L_2)+i(a_3L_3-a_4L_4)$$

where $a_1,\dots,a_4\in R^+$ and $L_1,\dots,L_4\in D(A,e)$. Thus, again by 2.1 and [1], page 34, we have

$$\begin{aligned}\|F(L)\|&\leqslant a_1+a_2+a_3+a_4\leqslant\sqrt{2}\sup\{|L(a)|\colon a\in A,\ v(a)\leqslant 1\}\\&\leqslant\sqrt{2}\,e\|L\|.\end{aligned}$$

If $\varphi\in A^*$, then $|\varphi(T_1a)|\leqslant\|F(\varphi)\|\cdot\|T_2a\|$. Hence

$$\begin{aligned}\|T_1a\|&=\sup\{|\varphi(T_1a)|\colon \varphi\in A^*,\ \|\varphi\|=1\}\\&\leqslant\sup\{\|F(\varphi)\|\cdot\|T_2a\|\colon \varphi\in A^*,\ \|\varphi\|=1\}\\&\leqslant\|F\|\cdot\|T_2a\|.\end{aligned} \tag{3}$$

Define $V: R(T_2)\to A$ by $VT_2 a = T_1 a$. By (3), V is well defined and bounded. Hence it may be extended to all of A_2. If we also call the extension V, then we have $V\tilde{T}_2 = VT_2 = T_1$. Hence $\tilde{T}_2^* V^* = T_1^* = T_2^* F$ and we conclude, since $\tilde{T}_2^*$ is 1-1, that $V^* = F$. Therefore, $V^* D(A, e) \subset D(A_2, e)$.

THEOREM 2.3. *Let A be a unital Banach algebra and T_1 and T_2 bounded linear operators on A. Let $e \in A_i$ where $A_i = \overline{R(T_i)}$, $i = 1, 2$. Then the following are equivalent*:

(i)′ $V(A, T_1 a) = V(A, T_2 a)$, $a \in A$.

(ii)′ $T_1 = VT_2$ *where* $V: A_2 \to A_1$ *is an invertible operator such that* V *and* V^{-1} *are state preserving.*

Proof. (ii)′ $\Rightarrow$ (i)′. By Theorem 2.2 we have $V(A_1, T_1 a) = V(A_2, T_2 a)$ for each $a \in A$. But, by [1], p. 16, we have that $V(A_i, b) = V(A, b)$ for all $b \in A$ $(i = 1, 2)$ Hence (ii)′ follows.

(i)′ $\Rightarrow$ (ii)′. Theorem 2.2 implies that $T_1 = \tilde{V}T_2$ and $T_2 = \tilde{W}T_1$ where $\tilde{V}: A_2 \to A$ and $\tilde{W}: A_1 \to A$ are state preserving. Let V be $\tilde{V}$ as a map from $A_2 \to A_1$ and let W be $\tilde{W}$ as a map from $A_1 \to A_2$. V and W are state preserving and $V = W^{-1}$.

3. In this section we present an application of Theorem 2.2. We recall that an *invertible linear transformation* J on a C^*-algebra A is a C^*-isomorphism (also called a *Jordan isomorphism*) if $J(X^*) = J(X)^*$ and $J(a^n) = J(a)^n$ for each self-adjoint element a and each positive integer n. These maps were first defined and studied by Kadison. An elegant statement of Kadison's results that will be used in this section can be found in [9], Theorem 1.1.

THEOREM 3.1. *Let A be a C^*-algebra with identity e and T_1 and T_2 bounded linear operators on A. Suppose that T_1 and T_2 have dense ranges. Then, $V(A, T_1 a) = V(A, T_2 a)$ for each $a \in A$ if and only if $T_1 = JT_2$ where J is a C^*-isomorphism.*

Proof. Suppose $T_1 = JT_2$. Then, by [9], Theorem 1.1, we have that J is bipositive (i.e. $V(A, Ja) \subset R^+$ if and only if $V(A, a) \subset R^+$) and $Je = e$. For a C^*-algebra this implies J and J^{-1} preserve states. Hence $V(A, T_1 a) = V(A, JT_2 a) = V(A, T_2 a)$ for each $a \in A$ by Theorem 2.3.

Conversely, suppose $V(A, T_1 a) = V(A, T_2 a)$ for all $a \in A$. Then, by Theorem 2.3, $T_1 = JT_2$ where $Je = e$ and J and J^{-1} preserve states. Thus, by Theorem 1.1, J is bipositive. Thus, by [9], Theorem 1.1, J is a C^*-isomorphism.

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