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On the moduli of convexity and smoothness in Orlicz spaces

by

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Abstract. Estimates are given for the moduli of convexity and smoothness of some Orlicz spaces $L_M(S, \Sigma, \mu)$.

1. Introduction. Let (S, Σ, μ) be a measure space. Necessary and sufficient conditions for reflexivity of the Orlicz space $L_M(S, \Sigma, \mu)$ were obtained by Luxemburg ([5], p. 60) under some restrictions for the measure μ. More precisely, he proved that *if $M(t)$ and $M^*(t)$ are complementary Orlicz functions then:*

(i) *If* $0 < \mu(S) < \infty$ (*if* $S = \{\sigma_i\}_1^\infty$, $\mu(\sigma_{i+1}) \leqslant \mu(\sigma_i)$, *then* $\liminf_i \big(\mu(\sigma_{i+1})/\mu(\sigma_i)\big) > 0$ *is assumed*), *then $L_M(S, \Sigma, \mu)$ is reflexive iff $M(t)$ and $M^*(t)$ have the property Δ_2 at infinity.*

(ii) *If $\mu(S) = \infty$, and S contains a set of infinite measure free of atoms, then $L_M(S, \Sigma, \mu)$ is reflexive iff $M(t)$ and $M^*(t)$ have the property Δ_2 at zero and at infinity.*

(iii) *If* $\mu(S) = \infty$, $S = \{\sigma_a\}_{a\in A}$, *and* $0 < \inf_{a\in A} \mu(\sigma_a) \leqslant \sup_{a\in A} \mu(\sigma_a) < \infty$, *then $L_M(S, \Sigma, \mu)$ is reflexive iff $M(t)$ and $M^*(t)$ have the property Δ_2 at zero.*

Recently Akimovich [1] proved that *if the measure μ is the same as in* (i), (ii), (iii), *then the reflexive Orlicz space $L_M(S, \Sigma, \mu)$ is isomorphic to a uniformly convex and uniformly smooth Orlicz space $L_N(S, \Sigma, \mu)$.*

For many results formulated in terms of moduli of convexity and smoothness, it is essential that the Banach space is isomorphic to a uniformly convex (uniformly smooth) space whose modulus of convexity (modulus of smoothness) can be estimated.

In Section 2 estimates are obtained for the moduli of convexity and smoothness of an Orlicz space $L_N(S, \Sigma, \mu)$ ((S, Σ, μ) is an arbitrary measure space) isomorphic to the initial space $L_M(S, \Sigma, \mu)$ under the assumption that the complementary Orlicz functions $M(t)$ and $M^*(t)$ have the property Δ_2 at zero and infinity.

Let the measure μ be the same as in (i), (ii), (iii) and let $L_M(S, \Sigma, \mu)$ be reflexive. An Orlicz function $N(t)$ equivalent to $M(t)$ at infinity, at zero and at infinity, at zero, respectively, is constructed and estimates for the moduli of convexity and smoothness of the space $L_N(S, \Sigma, \mu)$ isomorphic to $L_M(S, \Sigma, \mu)$ are found in Section 3. Moreover, if $(S, \Sigma, \mu) = (R, \mathcal{L}, \lambda)$ (λ is the Lebesgue measure) it can be proved using our methods that these estimates are the best possible in the class of all Orlicz spaces $L_N(S, \Sigma, \mu)$ isomorphic to $L_M(S, \Sigma, \mu)$ with $N(t)$ equivalent to $M(t)$ at infinity, at zero and at infinity, at zero, respectively (1). We omit here the proof of this fact as the Editorial Board of Studia Mathematica has kindly informed us that from a result of Figiel and Pisier [2] and some more recently discovered properties of moduli of convexity and smoothness it follows that these estimates are the best possible in the class of all Banach spaces isomorphic to $L_M(S, \Sigma, \mu)$.

Notations. Let X be a Banach space. The *modulus of convexity* of X is defined by

$$\delta_X(\varepsilon) = \inf\left(1-\tfrac{1}{2}\|(x+y)\|;\ x, y\in X,\ \|x\| = \|y\| = 1,\ \|x-y\|\geqslant \varepsilon\right),\qquad \varepsilon\in[0,2].$$

X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

The *modulus of smoothness* of X is defined by

$$\varrho_X(\tau) = \tfrac{1}{2}\sup\left(\|x+\tau y\|+\|x-\tau y\|-2;\ x, y\in X,\ \|x\| = \|y\| = 1\right),\qquad \tau > 0.$$

X is called *uniformly smooth* if $\lim \varrho_X(\tau)/\tau = 0$.

The function $M(t)$ is called *Orlicz function* if it is continuous, strictly increasing, and convex in $[0, \infty)$ and if $M(0) = 0$.

If $M(t)$ is an Orlicz function, then the Orlicz function

$$M^*(t) = \sup\left(ut - M(u);\ u\geqslant 0\right)$$

is called *complementary* to $M(t)$.

The Orlicz function $M(t)$ is said to *have the property* Δ_2 *at zero* (*at infinity*) if there exist two positive constants b and t_0 such that $M(2t) \leqslant bM(t)$, $t\in[0, t_0]$ ($t\in[t_0, \infty)$).

Two Orlicz functions $M(t)$ and $N(t)$ are said to be *equivalent at zero* (*at infinity*) if there exist constants C, K, c and k such that $CM(ct)\leqslant N(t) \leqslant KM(kt)$ in a neighbourhood of zero (of infinity).

To the Orlicz function M and a measure space (S, Σ, μ) can be associated the Orlicz space $L_M(S, \Sigma, \mu)$ of all real functions $f(t)$, μ-measurable on S such that

$$\int_S M\left(|f(t)|/\tau\right)\mu(dt) < \infty$$

for some $\tau > 0$. The norm in $L_M(S, \Sigma, \mu)$ is introduced by

$$\|f\| = \inf\left\{\tau > 0;\ \int_S M\left(|f(t)|/\tau\right)\mu(dt)\leqslant 1\right\}.$$

We notice that if the measure μ is the same as in (i), (ii), (iii) and the Orlicz functions $M(t)$ and $N(t)$ are equivalent at infinity, at zero and infinity, at zero, respectively, then the Orlicz spaces $L_M(S, \Sigma, \mu)$ and $L_N(S, \Sigma, \mu)$ are isomorphic (see [5], p. 52). A converse is in general not true (see [3]).

2. To the Orlicz function $M(t)$ and the interval $I\subset(0, \infty)$ we shall associate the functions

$$F_{M,I}(\varepsilon) = \varepsilon^2\inf\{M(uv)/u^2M(v);\ u\in[\varepsilon, 1],\ v\in I\},\qquad 0<\varepsilon\leqslant 1,$$

$$G_{M,I}(\tau) = \tau^2\sup\{M(uv)/u^2M(v);\ u\in[\tau, 1],\ v\in I\},\qquad 0<\tau\leqslant 1,$$

$$\mathscr{F}_{M,I}(\varepsilon) = \varepsilon^2\inf\left\{\frac{u^2M(v)}{M(uv)};\ u\in\left[1, \frac{1}{\varepsilon}\right],\ v\in I\right\},\qquad 0<\varepsilon\leqslant 1,$$

$$\mathscr{G}_{M,I}(\varepsilon) = \tau^2\sup\left\{\frac{u^2M(v)}{M(uv)};\ u\in\left[1, \frac{1}{\tau}\right],\ v\in I\right\},\qquad 0<\tau\leqslant 1.$$

If $I = (0, \infty)$ we shall write

$$F_{M,I}(\varepsilon) = F_M(\varepsilon),\qquad G_{M,I}(\tau) = G_M(\tau).$$

THEOREM 1. *Let $M(t)$ be an Orlicz function which satisfies*

$$M(2t)\leqslant bM(t),\qquad M(lt)\leqslant \tfrac{1}{2}lM(t) \tag{1}$$

for all $t\in(0, \infty)$, where b, l are positive constants, $l<1$.

Then there exists an Orlicz function $N(t)$ equivalent to $M(t)$ at zero and at infinity such that for the moduli of convexity and smoothness of the space $X = L_N(S, \Sigma, \mu)$ ((S, Σ, μ) is an arbitrary measure space) the following estimates hold

$$\delta_X(\varepsilon)\geqslant CF_M(\varepsilon),\qquad \varrho_X(\tau)\leqslant KG_M(\tau),\qquad \varepsilon, \tau\in[0, 1]. \tag{2}$$

Remark. The second condition in (1) is equivalent to $M^*(2t) \leqslant 2M^*(t)/l$, $t\in(0, \infty)$, i.e. to the requirement that $M^*(t)$ has the property Δ_2 at zero and at infinity.

In order to prove this theorem we need some lemmas. In all the lemmas, $M(t)$ is assumed to be an Orlicz function,

$$M_1(t) = \int_0^t \frac{M(u)}{u}\,du,\qquad N(t) = \int_0^t \frac{M_1(u)}{u}\,du.$$

LEMMA 1. *Let $u\in[0, \infty)$, $v\in[0, 1]$ and $t\in(0, 1]$. Then*

$$\frac{v^2M(u)}{G_M(t)} - M(u)\leqslant M\left(\frac{uv}{t}\right)\leqslant \frac{v^2M(u)}{F_M(t)} + M(u). \tag{3}$$

(1) This result (for the Orlicz sequence space l_M) was communicated by the second author at the Conference "Geometry of Banach spaces" in Oberwolfach, Fed. Rep. Germany, in October 1973.

Proof. Obviously, $F_M(t) \leqslant t^2 \leqslant G_M(t)$, $t \in [0,1]$. If $v/t \leqslant 1$, then

$$\frac{v^2 M(u)}{G_M(t)} - M(u) \leqslant \left(\frac{v}{t}\right)^2 M(u) - M(u) \leqslant M\left(\frac{uv}{t}\right) \leqslant \frac{v^2 M(u)}{F_M(t)} + M(u).$$

Suppose now that $v/t > 1$. Then $t \leqslant w = t/v < 1$ and

$$F_M(t) \leqslant \frac{t^2 M\big((u/w)\cdot w\big)}{w^2 M(u/w)} \leqslant G_M(t). \tag{4}$$

Since $1/w = v/t$, (4) implies

$$v^2 M(u)/G_M(t) \leqslant M\left(\frac{uv}{t}\right) \leqslant v^2 M(u)/F_M(t).$$

Thus Lemma 1 is proved.

LEMMA 2. *Let* $M(t)$ *satisfy* (1). *Then*

$$b^{-1}M(t) \leqslant M_1(t) \leqslant a^{-1}M(t),$$
$$a \leqslant tM_1'(t)/M_1(t) \leqslant b$$

for $t > 0$, *where* $a = (1-l/2)^{-1} > 1$.

Proof. Since $M_1'(t) = M(t)/t$, it is enough to observe that

$$b^{-1}M(t) \leqslant M(t/2) \leqslant \int_{t/2}^{t} M(u)/u\,du \leqslant M_1(t) = \int_0^{lt} M(u)/u\,du + \int_{lt}^{t} M(u)/u\,du$$
$$\leqslant M(lt) + (1-l)M(t) \leqslant (1-l/2)M(t).$$

LEMMA 3. *Let* $M(t)$ *satisfy* (1). *Then*

$$a^2 \leqslant M(t)/N(t) \leqslant b^2, \qquad a \leqslant tN'(t)/N(t) \leqslant b, \tag{5}$$
$$\theta^b N(t) \leqslant N(\theta b) \leqslant \theta^a N(t), \qquad \theta \in (0,1), \tag{6}$$
$$\theta^a N(t) \leqslant N(\theta b) \leqslant \theta^b N(t), \qquad \theta > 1, \tag{7}$$
$$a(a-1) \leqslant t^2 N''(t)/N(t) \leqslant b(b-1) \tag{8}$$

for all $t > 0$.

Proof. The first two inequalities follow immediately from the representation

$$N(t) = \int_0^t \frac{M(u)}{u}\cdot\frac{M_1(u)}{M(u)}\,du.$$

It is easy to get (6) and (7) by integration of (5) with respect to t. To obtain (8) it is enough to mention that

$$t^2 N''(t)/N(t) = \big(M(t) - M_1(t)\big)/N(t)$$

and apply Lemma 2.

LEMMA 4. *Let* $M(t)$ *satisfy* (1). *Then*

$$\frac{a(a-1)}{3^{b+2}}\cdot\frac{N(\zeta)}{\zeta^2}\left(\frac{\xi-\eta}{2}\right)^2 \leqslant N(|\xi|) + N(|\eta|) - 2N\big(|\tfrac{1}{2}(\xi+\eta)|\big) \tag{9}$$
$$\leqslant 18b(b-1)\frac{N(\zeta)}{\zeta^2}\left(\frac{\xi-\eta}{2}\right)^2,$$

where $\zeta = \max(|\xi|, |\eta|)$.

Proof. Without loss of generality we may suppose that $|\xi| \geqslant |\eta|$. We shall consider four cases:

(a) $0 \leqslant \frac{1}{3}\xi \leqslant \eta \leqslant \xi$. Since $N(t)$ has a continuous second derivative, one has

$$N(\xi) + N(\eta) - 2N\big(\tfrac{1}{2}(\xi+\eta)\big) = \big(\tfrac{1}{2}(\xi-\eta)^2 N''\big(\tfrac{1}{2}(\xi+\eta) + \tfrac{1}{2}\theta(\xi-\eta)\big) \tag{10}$$

for some $\theta \in (-1, 1)$.

Let us write $\sigma = \frac{1}{2}(\xi+\eta) + \frac{1}{2}\theta(\xi-\eta)$. From (8) it follows that

$$a(a-1)\frac{N(\sigma)}{\sigma^2} \leqslant N''(\sigma) \leqslant b(b-1)\frac{N(\sigma)}{\sigma^2}. \tag{11}$$

Obviously, $\frac{1}{3}\xi \leqslant \eta \leqslant \sigma \leqslant \xi$. Hence from (11) and (6) we obtain

$$\frac{a(a-1)}{3^b}\cdot\frac{N(\xi)}{\xi^2} \leqslant N''(\sigma) \leqslant 9b(b-1)\frac{N(\xi)}{\xi^2}. \tag{12}$$

From (10) and (12) follows (9).

(b) $0 \leqslant \eta \leqslant \frac{1}{3}\xi$. Let us define the function

$$\varphi(t) = N(\xi) + N(t) - 2N\big(\tfrac{1}{2}(\xi+t)\big).$$

Since

$$\varphi'(t) = N'(t) - N'\big(\tfrac{1}{2}(\xi+t)\big) \leqslant 0, \qquad 0 \leqslant t \leqslant \xi,$$

$\varphi(t)$ is decreasing for $t \in (0, \xi)$. Then

$$N(\xi) + N(\eta) - 2N\big(\tfrac{1}{2}(\xi+\eta)\big) \geqslant N(\xi) + N(\tfrac{1}{3}\xi) - 2N(\tfrac{2}{3}\xi). \tag{13}$$

As in the previous case,

$$N(\xi) + N(\tfrac{1}{3}\xi) - 2N(\tfrac{2}{3}\xi) \geqslant \frac{a(a-1)}{3^{b+2}}N(\xi). \tag{14}$$

On the other hand

$$N(\xi) + N(\eta) - 2N\big(\tfrac{1}{2}(\xi+\eta)\big) \leqslant 2N(\xi). \tag{15}$$

From (13), (14), (15) and the inequalities

$$\big(\tfrac{1}{2}(\xi-\eta)\big)^2/\xi^2 < 1 - 9\big(\tfrac{1}{2}(\xi-\eta)\big)^2/\xi^2$$

immediately follows (9).

(c) $0 \leqslant -\eta \leqslant \xi$. Then

(16) $$N(\xi)+N(-\eta)-2N\big(\tfrac{1}{2}(\xi+\eta)\big) \leqslant 2N(\xi)$$

and from (6) we obtain

(17) $$N(\xi)+N(-\eta)-2N\big(\tfrac{1}{2}(\xi+\eta)\big) \geqslant N(\xi)-2N(\tfrac{1}{2}\xi) \geqslant (1-2^{1-a})N(\xi).$$

To obtain (9) from (16) and (17) it is enough to consider

$$\big(\tfrac{1}{2}(\xi-\eta)\big)^2/\xi^2 \leqslant 1 \leqslant 4\big(\tfrac{1}{2}(\xi-\eta)\big)^2/\xi^2.$$

(d) $0 \leqslant \eta \leqslant -\xi$. This case can be treated exactly as (c).

Thus Lemma 4 is proved.

LEMMA 5. *Let $f, g \in L_N(S, \Sigma, \mu)$ ((S, Σ, μ) is an arbitrary measuer space) be such that $\|f\| = \|g\| = 1$. Then the following inequalities hold*

(18) $$cF_N\left(\left\|\frac{f-g}{8}\right\|\right) \leqslant \int_S \left[N(|f(x)|)+N(|g(x)|)-N\left(\left|\frac{f(x)+g(x)}{2}\right|\right)\right]\mu(dx) \leqslant kG_N\left(\left\|\frac{f-g}{8}\right\|\right),$$

where c, k are positive constants.

Proof. Let us write $h(x) = \max(|f(x)|, |g(x)|)$. Obviously, $h \in L_N$ and $1 \leqslant \|h\| \leqslant 2$. It follows from (9) that

(19) $$\frac{a(a-1)}{3^{b+2}}\int_S \frac{N(h(x))}{h^2(x)}\left(\frac{f(x)-g(x)}{2}\right)^2\mu(dx) \leqslant \int_S \left[N(|f(x)|)+N(|g(x)|)-2N\left(\left|\frac{f(x)+g(x)}{2}\right|\right)\right]\mu(dx) \leqslant 18b^2\int_S \frac{N(h(x))}{h^2(x)}\left(\frac{f(x)-g(x)}{2}\right)^2\mu(dx).$$

Applying (3) to the function $N(t)$ for $u = h(x)/2\|h\|$, $v = |f(x)-g(x)|\,\|h\|/4h(x)$, $t = \|f-g\|/8$, we easily obtain

$$\frac{\|h\|^2(f(x)-g(x))^2N(h(x)/2\|h\|)}{16h^2(x)G_N(\|f-g\|/8)} - N(h(x)/2\|h\|) \leqslant N(|f(x)-g(x)|/\|f-g\|) \leqslant \frac{\|h\|^2(f(x)-g(x))^2N(h(x)/2\|h\|)}{16h^2F_N(\|f-g\|/8)} + N(h(x)/2\|h\|).$$

Hence by using Lemma 3 we get

$$\frac{\|h\|^2(f(x)-g(x))^2N(h(x))}{16(2\|h\|)^b h^2(x)G_N(\|f-g\|/8)} - 2^{-a}N(h(x)/\|h\|) \leqslant N(|f(x)-g(x)|/\|f-g\|) \leqslant \frac{\|h\|^2(f(x)-g(x))^2N(h(x))}{16(2\|h\|)^a h^2(x)F_N(\|f-g\|/8)} + 2^{-a}N(h(x)/\|h\|).$$

After integration on S, using the estimates for $\|h\|$, we obtain

(20) $$\frac{1}{2^{b+2}}\cdot\frac{1}{G_N(\|f-g\|/8)}\int_S \frac{N(h(x))}{h^2(x)}\left(\frac{f(x)-g(x)}{2}\right)^2\mu(dx) - 2^{-a} \leqslant 1 \leqslant \frac{1}{2^{a+2}}\cdot\frac{1}{F_N(\|f-g\|/8)}\int_S \frac{N(h(x))}{h^2(x)}\left(\frac{f(x)-g(x)}{2}\right)^2\mu(dx) + 2^{-a}.$$

From (19) and (20) follows (18).

LEMMA 6. *If $f, g \in L_N(S, \Sigma, \mu)$, $\|f\| = \|g\| = 1$, then*

$$1-\|\tfrac{1}{2}(f+g)\| \geqslant (2b)^{-1}\int_S \left[N(|f(x)|)+N(|g(x)|)-2N\left(\tfrac{1}{2}|f(x)+g(x)|\right)\right]\mu(dx).$$

Proof. From (7) it follows that

$$\int_S N\left(\tfrac{1}{2}|f(x)+g(x)|\right)\mu(dx) \geqslant \|\tfrac{1}{2}(f+g)\|^b.$$

Then

$$\int_S \left[N(|f(x)|)+N(|g(x)|)-2N\left(\tfrac{1}{2}|f(x)+g(x)|\right)\right]\mu(dx) \leqslant 2\left(1-\|\tfrac{1}{2}(f+g)\|^b\right) \leqslant 2b\left(1-\|\tfrac{1}{2}(f+g)\|\right).$$

LEMMA 7. *There exist two positive constants γ, $\varkappa$ such that for every $f, g \in L_N(S, \Sigma, \mu)$, $\|f\| = \|g\| = 1$, $\tau \in [0, \gamma]$, the inequality*

(21) $$\|f+\tau g\|+\|f-\tau g\|-2 \leqslant \varkappa\left[1-\int_S N\left(\frac{1}{2}\left|\frac{f(x)+\tau g(x)}{\|f+\tau g\|}+\frac{f(x)-\tau g(x)}{\|f-\tau g\|}\right|\right)\mu(dx)+\tau^2\right]$$

holds.

Proof. It is easy to verify that

(22) $$(1+d)^{-1} \leqslant 1-d+2d^2, \qquad |d| \leqslant \tfrac{1}{2}.$$

Let $\gamma \in (0, \frac{1}{2})$ be such that

(23) $$(1+d)^{b-1} \leqslant 1+bd, \qquad d \in [0, \gamma],$$

(24) $$(1-d)^a \leqslant 1-\tfrac{1}{2}ad, \qquad d \in [0, \gamma].$$

Suppose now that $\tau \in [0, \gamma]$. Then

$$\frac{1}{2}\left|\frac{1}{\|f+\tau g\|}-\frac{1}{\|f-\tau g\|}\right| \leqslant 2\tau. \tag{25}$$

From (22) it follows that

$$\frac{1}{2}\left(\frac{1}{\|f+\tau g\|}+\frac{1}{\|f-\tau g\|}\right) \leqslant 2-\tfrac{1}{2}(\|f+\tau g\|+\|f-\tau g\|)+2\tau^2. \tag{26}$$

Since $N(t)$ is monotone and convex, from (25), (26) (by (6) and (7)) we have

$$N\left(\frac{1}{2}\left|\frac{f(x)+\tau g(x)}{\|f+\tau g\|}+\frac{f(x)-\tau g(x)}{\|f-\tau g\|}\right|\right)$$
$$\leqslant \frac{1}{1+\tau^2}\Big[N\big((1+\tau^2)\big(2-\tfrac{1}{2}(\|f+\tau g\|+\|f-\tau g\|)\big)|f(x)|\big)+$$
$$+\tfrac{1}{2}\tau^2\big(N\big(4(1+\tau^2)|f(x)|\big)+N\big(4(1+\tau^2)|g(x)|\big)\big)\Big]$$
$$\leqslant (1+\tau^2)^{b-1}\Big[\big(2-\tfrac{1}{2}(\|f+\tau g\|+\|f-\tau g\|)\big)^a N\big(|f(x)|\big)+$$
$$+4^b\tau^2\cdot\tfrac{1}{2}\big(N\big(|f(x)|\big)+N\big(|g(x)|\big)\big)\Big].$$

Integrating on S we obtain

$$1-\int_S N\left(\frac{1}{2}\left|\frac{f(x)+\tau g(x)}{\|f+\tau g\|}+\frac{f(x)-\tau g(x)}{\|f-\tau g\|}\right|\right)\mu(dx) \tag{27}$$
$$\geqslant 1-(1+\tau^2)^{b-1}\big(\big(2-\tfrac{1}{2}(\|f+\tau g\|+\|f-\tau g\|)\big)^a+4^b\tau^2\big).$$

To get (21) it is enough to make use of (23), (24) in (27).

Proof of Theorem 1. Let $X = L_N(S, \Sigma, \mu)$. From Lemmas 6 and 7 it follows that for every $f, g \in X$, $\|f\| = \|g\| = 1$, $\|f-g\| \geqslant \varepsilon$,

$$1-\|\tfrac{1}{2}(f+g)\| \geqslant (2b)^{-1}cF_N(\tfrac{1}{3}\varepsilon).$$

Since $M(t)$ is equivalent to $N(t)$, we have

$$\delta_X(\varepsilon) \geqslant CF_M(\varepsilon).$$

On the other hand, from Lemmas 6 and 8 it follows that for every $f, g \in X$, $\|f\| = \|g\| = 1$, $\tau \in [0, \gamma]$ the inequality

$$\|f+\tau g\|+\|f-\tau g\|-2 \leqslant \varkappa\left[kG_N\left(\frac{1}{8}\left\|\frac{f+\tau g}{\|f+\tau g\|}-\frac{f-\tau g}{\|f-\tau g\|}\right\|\right)+\tau^2\right] \tag{28}$$

holds.

Since

$$\left\|\frac{f+\tau g}{\|f+\tau g\|}-\frac{f-\tau g}{\|f-\tau g\|}\right\| \leqslant 8\tau \quad \text{and} \quad \tau^2 \leqslant G_N(\tau),$$

from (28) we obtain

$$\varrho_X(\tau) \leqslant \varkappa(k+1)G_N(\tau), \qquad \tau \in [0, \gamma].$$

Finally, from the equivalence of $M(t)$ and $N(t)$ it follows that

$$\varrho_X(\tau) \leqslant KG_M(\tau), \qquad \tau \in [0, 1].$$

Theorem 1 is proved.

3. COROLLARY 1. *Let* $0 < \mu(S) < \infty$ *(if* $S = \{\sigma_i\}_1^\infty$ *with* $\mu(\sigma_{i+1}) \leqslant \mu(\sigma_i)$, *then we assume* $\liminf_{i\to\infty} \mu(\sigma_{i+1})/\mu(\sigma_i) > 0$*) and let* $L_M(S, \Sigma, \mu)$ *be reflexive. Then there exists an Orlicz function* $N(t)$ *equivalent to* $M(t)$ *at infinity such that for the space* $X = L_N(S, \Sigma, \mu)$ *the estimates*

$$\delta_X(\varepsilon) \geqslant AF_{M,[1,\infty]}(\varepsilon), \quad \varepsilon \in [0, 1], \qquad \varrho_X(\tau) \leqslant BG_{M,[1,\infty]}(\tau) \tag{29}$$

hold.

Proof. Since $L_M(S, \Sigma, \mu)$ is reflexive, $M(t)$ and $M^*(t)$ have the property Δ_2 at infinity. Wihout loss of generality we may assume that $M(1) = 1$, $M'(1)$ exists, and $M'(t) > 1$.

Let us consider the function

$$N(t) = \begin{cases} t^2 & \text{for} \quad t \in [0, \alpha], \\ at+b & \text{for} \quad t \in [\alpha, 1], \\ M(t) & \text{for} \quad t \in [1, M), \end{cases}$$

where $\alpha = \tfrac{1}{2}\big(M'(1)-|M'(1)-2|\big) > 0$, $a = M'(1)$, $b = 1-M'(1)$.

Obviously, $N(t)$ and its complement $N^*(t)$ have the property Δ_2 at zero and at infinity, i.e. $N(t)$ satisfies (1). Then from Theorem 1 for $X = L_N(S, \Sigma', \mu)$ it follows that

$$\delta_X(\varepsilon) \geqslant A_1F_N(\varepsilon), \quad \varepsilon \in [0, 1], \qquad \varrho_X(\tau) \leqslant B_1G_N(\tau), \quad \tau \in [0, 1]. \tag{30}$$

On the other hand,

$$\alpha^{b+1} \leqslant F_N(t)/\mathscr{F}_{M,[1,\infty)}(t) \leqslant \frac{1}{\alpha^{b+1}}, \quad \alpha^{b+1} \leqslant G_N(t)/\mathscr{G}_{M,[1,\infty)}(t) \leqslant \frac{1}{\alpha^{b+1}}, \quad t \in (0,1]. \tag{31}$$

To obtain (31) it is enough to observe that for every $u \in [t, 1]$ and $v \in (0, \infty)$

$$\alpha^{b+1}\frac{u_1^2M(v_1)}{M(u_1v_1)} \leqslant \frac{N(uv)}{u^2N(v)} \leqslant \frac{1}{\alpha^{b+1}}\cdot\frac{u_2^2M(v_2)}{M(u_2v_2)}$$

for some $u_1, u_2 \in [t, 1]$, $v_1, v_2 \in [1, \infty)$.

By combination of (30) and (31) we obtain (29).

Corollary 2. *Let $\mu(S) = \infty$, let S contain a subset of infinite measure free of atoms, and let $L_M(S, \Sigma, \mu)$ be reflexive. Then for the moduli of convexity and smoothness of $X = L_M(S, \Sigma, \mu)$ the following estimates hold:*

$$(32) \qquad \delta_X(\varepsilon) \geqslant AF_M(\varepsilon), \ \varepsilon \in [0,1], \qquad \varrho_X(\tau) \leqslant BG_M(\tau), \ \tau \in [0,1].$$

Proof. $M(t)$ and $M^*(t)$ have the property Δ_2 at zero and at infinity. Hence (32) follows from Theorem 1.

Corollary 3. *Let $\mu(S) = \infty$, $S = \{\sigma_a\}_{a \in A}$, $0 < \inf_a \mu(\sigma_a) \leqslant \sup_a \mu(\sigma_a) < \infty$, and let $L_M(S, \Sigma, \mu)$ be reflexive. Then there exists an Orlicz function $N(t)$ equivalent to $M(t)$ at zero such that for the space $X = L_N(S, \Sigma, \mu)$ the following estimates hold*

$$(33) \quad \delta_X(\varepsilon) \geqslant AF_{M,[0,1]}(\varepsilon), \ \varepsilon \in [0,1], \qquad \varrho_X(\tau) \leqslant BG_{M,[0,1]}(\tau), \ \tau \in [0,1].$$

Proof. $M(t)$ and $N(t)$ have the property Δ_2 at zero. Without loss of generality we may assume that $M(1) = 1$ and $M'(1)$ exists.

Let us consider the function

$$N(t) = \begin{cases} M(t) & \text{for} \quad 0 \leqslant t \leqslant 1, \\ at+b & \text{for} \quad 1 \leqslant t \leqslant \alpha, \\ t^2 & \text{for} \quad \alpha \leqslant t, \end{cases}$$

where $\alpha = \frac{1}{2}\big(M'(1) + |M'(1) - 2|\big)$, $a = M'(1)$, $b = 1 - M'(1)$.

It is readily seen that $N(t)$ and its complement, $N^*(t)$, have the property Δ_2 at zero and at infinity. From Theorem 1 for $X = L_N(S, \Sigma, \mu)$ it follows that

$$(34) \qquad \delta_X(\varepsilon) \geqslant A_2 F_N(\varepsilon), \ \varepsilon \in [0,1], \qquad \varrho_X(\tau) \leqslant B_2 G_N(\tau), \ \tau \in [0,1].$$

On the other hand,

$$(35) \quad \alpha^{-2} \leqslant F_N(t)/F_{M,[0,1]}(t) \leqslant 1, \qquad 1 \leqslant G_N(t)/G_{M,[0,1]}(t) \leqslant \alpha^2, \ t \in (0,1].$$

From (34) and (35) follows (33).

Remarks. The function $N(t)$ we have constructed in Corollaries 1, 3 is equivalent to $M(t)$ at infinity, at zero, respectively, and therefore, the space $L_N(S, \Sigma, \mu)$ is isomorphic to $L_M(S, \Sigma, \mu)$. If $M(t) = t^p$, $1 < p < \infty$, and $(S, \Sigma, \mu) = (R, \mathscr{L}, \lambda)$ (λ is the Lebesgue measure) from Corollaries 1, 2, 3 follows the well-known estimates for the spaces L_p (see e.g. [4], p. 28).

$$\delta_{L_p}(\varepsilon) \geqslant c_p \varepsilon^r, \qquad \varrho_{L_p}(\tau) \leqslant k_p \tau^s,$$

where

$$r = \max(2, p), \qquad s = \min(2, p).$$

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