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On ideals of joint topological divisors of zero

by

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Abstract. Let A be a commutative Banach algebra with unit. Using an extension theorem of Słodkowski [3] we give a new proof for the main result of [4] which states that every maximal ideal of the Shilov boundary of A consists of joint topological divisors of zero.

Let A be a commutative Banach algebra with unit e and denote by $\mathscr{M}(A)$ the space of continuous complex-valued homomorphisms of A, by $\Gamma(A)$ the Shilov boundary, and by $\mathscr{L}(A)$ the closed set of all $\varphi \in \mathscr{M}(A)$ whose kernels consist of joint topological divisors of zero. (See [4] for the notion of joint topological divisors of zero.)

The set $\mathscr{L}(A)$ is nonempty. In fact, Żelazko [4] has proved the following

THEOREM 1. $\Gamma(A) \subset \mathscr{L}(A)$.

In [3] Słodkowski proved the following extension theorem, thereby solving a question posed in [4].

THEOREM 2. *Let I be an ideal consisting of joint topological divisors of zero. Then there exists a $\varphi \in \mathscr{L}(A)$ with $I \subset \operatorname{Ker}\varphi$.*

The main step in Słodkowski's proof ([3], Lemma 3) can also be found in [2] in a slightly different context. Independently, we proved a somewhat more general version of Theorem 2 in [1]. Our proof uses Theorem 1 whereas Słodkowski's proof does not use that theorem. Therefore it might be of some interest to note that Theorem 1 is a consequence of Theorem 2.

To see this, let $\varphi_0 \in \Gamma(A)$ and let U be an arbitrary neighbourhood of φ_0 in $\mathscr{M}(A)$. In order to prove that $\varphi_0 \in \mathscr{L}(A)$ it suffices to prove that $U \cap \mathscr{L}(A)$ is nonempty, because $\mathscr{L}(A)$ is closed. Choose $a \in A$ such that

$$\sup_{\varphi \in \mathscr{M}(A)} |\hat{a}(\varphi)| = 1 \quad \text{and} \quad \sup_{\varphi \in \mathscr{M}(A) \setminus U} |\hat{a}(\varphi)| < 1.$$

Next choose a complex number λ in the spectrum of a and with $|\lambda| = 1$. Then λ belongs to the topological boundary of the spectrum, so it follows that $a - \lambda e$ is a topological divisor of zero. Let I be the ideal generated by $a-\lambda e$. Then I consists of joint topological divisors of zero, so by Theorem 2 there is a $\varphi \in \mathscr{L}(A)$ with $I \subset \operatorname{Ker}\varphi$. In particular, $|\hat{a}(\varphi)| = |\lambda| = 1$ and it follows that $\varphi \in U$. This proves that $U \cap \mathscr{L}(A)$ is nonempty.

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A moment theory approach to the Riesz theorem on the conjugate function with general measures

by

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Abstract. A measure $\mu \geqslant 0$ belongs to the class $\mathfrak{R}_M$ if it satisfies the Riesz inequality

$$\int_0^{2\pi} |\check{f}(t)|^2 d\mu \leqslant M \int_0^{2\pi} |f(t)|^2 d\mu, \quad \forall f \in L^2(\mu),$$

where $\check{f}$ is (essentially) the conjugate functions of f, with fixed constant M.

Applying a moment theory approach we introduce (and give explicit formulae for) the canonical extremal (simple) elements for $\mathfrak{R}_M$, which prove to be given by $R(t)\,dt$, with $R(t)$ certain rational functions, making up a determinant set for $\mathfrak{R}_M$. These particular measures are for the class $\mathfrak{R}_M$ what the Dirac measures are for the class of all measures. Among the possible applications of this parallel construction is the analogue for the $\mathfrak{R}_M$-simple measures of Bochner's theorem of decomposition on Dirac measures.

1. Introduction. We say that a measure $\mu \geqslant 0$ satisfies the Riesz inequality in L^p if

$$\int |\tilde{F}(t)|^p d\mu \leqslant M \int |F(t)|^p d\mu, \quad \forall F \in L^p(\mu), \tag{1.1}$$

where $\tilde{F}$ is the conjugate function (or the Hilbert transform) of F, and μ is defined in $(0, 2\pi)$ (or in $\boldsymbol{R}^n$). Here we consider the simplest case when $p = 2$ and μ acts in the unit circle; the essential part of this exposition can be extended to $L^p(0, 2\pi)$ and $L^p(\boldsymbol{R})$ if p is even. The generalization for all p and $\boldsymbol{R}^n$ will be considered elsewhere.

Let $\mathscr{E}_N$ be the set of all complex trigonometric polynomials of the form

$$F(t) = \sum_{-N}^{N} a_n e_n(t), \quad e_n(t) = e^{int}, \tag{1.2}$$

$\mathscr{E} = \bigcup_{N=0}^{\infty} \mathscr{E}_N$, and for each F of the form (1.2) let us set

$$\check{F}(t) = \sum_{n=0}^{N} a_n e_n(t), \tag{1.2a}$$