On vector-valued analytic functions with constant norm
by
J. GLOBERNIK* (Ljubljana)

Abstract. Let $X$ be a complex Banach space and $B$ a domain in the complex plane. Let $f: B \to X$ be an analytic function such that $|f(z)|$ is constant as $z \in B$. If $X$ is the space of complex numbers then by the classical maximum modulus theorem $f(z)$ itself is constant on $B$. This is not the case in general. In the paper a characterization of the analytic functions with constant norm is given.

0. Introduction. If $f$ is a complex-valued analytic function, defined on a domain $B$ in the complex plane, then the classical maximum modulus theorem asserts that $|f(z)|$ has no maximum on $B$ or that $f(z)$ is constant on $B$. If $f$ has values in a complex $B$-space, the theorem holds, but its strong form, asserting $f(z)$ to be constant if $|f(z)|$ is constant on $B$, does not hold in general. E. Thorp and R. Whitley [1] characterized those complex $B$-spaces in which the strong form of the maximum modulus theorem holds — these are exactly the spaces in which every point on the unit sphere is a complex extreme point. Although many $B$-spaces have this property (e.g. strictly convex complex $B$-spaces), there remains a large class of the spaces which do not have this property (e.g. $C^*$-algebras of dimension greater than one [1]). Given such a space, we characterize those analytic functions with values in it, which have constant norm. We study the functions with values in $B$-spaces. The functions with values in $B$-algebras will be studied in a separate paper.

The idea is to linearize the problem in the sense that for every vector $a$ of the unit sphere we construct the subspace $E(a)$ of all vectors showing that $a$ is not a complex extreme point of the unit sphere, and then to obtain the characterization of the analytic functions with constant norm in terms of these subspaces.

1. Preliminaries. Throughout the paper $B$-space stands for Banach space. $X$ being a $B$-space, we denote by $S(X) = \{x \in X: \|x\| = 1\}$ the unit sphere of $X$ and by $X^*$ the dual space. The image of $\pi_X X$ under

* This work was supported by the Boris Kidrič Fund.
\( u \cdot X \) is denoted by \( \langle u, w \rangle \). If \( S \) is a subset of a \( B \)-space, we denote by \( \bar{S} \), \( S \) and \( \partial S \) the closure, the convex hull and the closed convex hull of the set \( S \), respectively. An open connected subset of the complex plane is called domain.

**Definition 1.0** (cf. [1]). Let \( X \) be a complex \( B \)-space. A point \( a \in \overline{S}(X) \) is called complex extreme point of \( S(X) \) if \( |a + z||z| \leq 1 \quad (|z| < 1) \) implies \( y = 0 \).

L. A. Harris [3] greatly simplified the original proof of Thorp--Whiteley's result, using the following lemma.

**Lemma 1.1** (L. A. Harris [2]). Let \( f \) be a complex valued function, analytic on the open unit disc in the complex plane, satisfying \( |f(z)| < 1 \quad (|z| < 1) \). Then

\[
|f(0)| + \frac{1-|z|}{2|z|} \left| f(z) - f(0) \right| \leq 1 \quad (|z| < 1, \ z \neq 0).
\]

This lemma is the main tool in our paper.

2. The subspace \( E(a) \).

**Definition 2.0.** Let \( X \) be a complex \( B \)-space and let \( a \in X \). The set \( E(a) = X \) is defined as follows. \( a \in E(a) \) if \( r > 0 \) exists such that \( |a + z||z| < |a| \quad (|z| < r) \).

**Proposition 2.1.** Let \( X \) be a complex \( B \)-space and let \( a \in \overline{S}(X) \). Then \( a \in E(a) \) if and only if a constant \( M > 0 < \infty \) exists such that

\[
|\langle a, u \rangle| < M(|\langle a \rangle - |\langle a, u \rangle|) \quad (u \in \overline{S}(X)).
\]

**Proof.** If

\[
|\langle a, u \rangle| < M(|\langle a \rangle - |\langle a, u \rangle|) \quad (u \in \overline{S}(X)),
\]

then

\[
|\langle a, u \rangle| < |\langle a \rangle| - |\langle a, u \rangle| \quad (u \in \overline{S}(X); |u| < 1/M),
\]

so

\[
|a + z||z| < |a| \quad (|z| < 1/M).
\]

Conversely, if \( r > 0 \) exists such that

\[
|a + z||z| < |a| \quad (|z| < r),
\]

then

\[
|\langle a, u \rangle + z| < |\langle a \rangle| - |\langle a, u \rangle| \quad (u \in \overline{S}(X))
\]

so

\[
|\langle a, u \rangle + z| < |\langle a \rangle| - |\langle a, u \rangle| \quad (u \in \overline{S}(X))
\]

and

\[
|\langle a, u \rangle| < (1/r)|\langle a \rangle| - |\langle a, u \rangle| \quad (u \in \overline{S}(X)). \quad \text{Q.E.D.}
\]

**Definition 2.2.** Let \( X \) be a complex \( B \)-space and let \( a \in \overline{S}(X) \). For \( a \in \overline{E}(a) \) we define

\[
|\langle a \rangle| = \inf \{ M : |\langle a, u \rangle| < M(|\langle a \rangle| - |\langle a, u \rangle|) \quad (u \in \overline{S}(X)) \}.
\]

**Proposition 2.3.** Let \( X \) be a complex \( B \)-space. Let \( a \in \overline{E}(a) \) and \( r \in \overline{E}(a) \), where

\[
|\langle a \rangle| = \inf \{ r : |\langle a + z \rangle| < |\langle a \rangle| \quad (|z| < r) \}.
\]

**Proof.** It is easy to see that the inequalities

\[
|\langle a + z \rangle| < |\langle a \rangle| \quad (|z| < r)
\]

and

\[
|\langle a, u \rangle| < (1/r)|\langle a \rangle| - |\langle a, u \rangle| \quad (u \in \overline{S}(X))
\]

are equivalent. Now the assertion follows immediately. Q.E.D.

**Proposition 2.4.** Let \( X \) be a complex \( B \)-space and let \( a \in \overline{S}(X) \). Then \( E(a) \) is a linear subspace of \( X \) and \( \|a\|_a = |a| \) is a norm in \( E(a) \).

**Proof.** Let \( a \in \overline{E}(a) \). By Proposition 2.1 and Definition 2.2 we have

\[
|\langle a, u \rangle| < \|a\|_a(|\langle a \rangle| - |\langle a, u \rangle|) \quad (u \in \overline{S}(X)).
\]

Clearly \( \|a\|_a > 0 \). If \( \|a\|_a = 0 \) then \( |\langle a, u \rangle| = 0 \quad (u \in \overline{S}(X)) \), so \( a = 0 \). If \( a \) is a complex number, we have

\[
|\langle a, u \rangle| = |a| \quad (|\langle a, u \rangle| < |a||\|a\|_a - |\langle a, u \rangle|) \quad (u \in \overline{S}(X))
\]

which shows that \( a \in \overline{E}(a) \) and that \( \|a\|_a = \|a\|_a \). Further, if also \( y \in \overline{E}(a) \), then

\[
|\langle y, u \rangle| < |\|a\|_a - |\langle a, u \rangle|\|y\|_a \quad (u \in \overline{S}(X))
\]

which shows that \( x + y \in \overline{E}(a) \) and that \( \|x + y\|_a < \|x\|_a + \|y\|_a \). Q.E.D.

**Proposition 2.5.** Let \( X \) be a complex \( B \)-space and let \( a \in \overline{S}(X) \). Then

\[
|a + z||z| \geq |a| \quad (a \in \overline{E}(a)).
\]

(Throughout, \( \overline{E}(a) \) is the closure of \( E(a) \) as a subset of \( X \).

**Proof.** Assume that \( |a + z||z| < |a| \) for an \( a \in \overline{E}(a) \). By the Hahn--Banach theorem an \( u \in \overline{S}(X) \) exists such that \( \langle a, u \rangle = |a| \). Since \( a + z \in \overline{S}(X) \), an \( r > 0 \) exists such that \( |a + z||z| < |a| \quad (|z| < r) \) which implies \( |\langle a, u \rangle + z| < |a| \quad (|z| < r) \). Now, by \( \langle a, u \rangle = |a| \) follows that \( |\langle a, u \rangle| = |a| \), contrary to the assumption \( |a + z||z| < |a| \). So \( a + z \in \overline{E}(a) \) which proves the assertion. Q.E.D.

**Lemma 2.6.** Let \( X \) be a complex \( B \)-space and let \( a \in \overline{S}(X) \). Let \( y \in \overline{E}(a) \) where \( y \in X \). Then \( E(a + y) = \overline{E}(a) \).

**Proof.** Since \( \|y\|_a < 1/2 \), by Proposition 2.3 an \( R > 2 \) exists such that \( |a + z||z| < |a| \quad (|z| < R) \). Consequently, an \( r > 1 \) exists such that
On vector-valued analytic functions

To prove the converse, let $a, E(a) \in \mathfrak{a}$ and let $\sum_{i=1}^{\infty} |a_i| \cdot \beta^i = M < \infty$ for an $r > 0$. By Definition 2.2 it follows that

$$\sum_{i=1}^{\infty} |a_i| \cdot \beta^i \leq M (|a_i| - |a_{i+1}|) \quad (a \in S(X')).$$

If $N = \max \{1, M\}$ it is easily seen that

$$|a_i| + \sum_{i=1}^{\infty} |a_i(r)|^2 \cdot \beta^i \leq |a_i| \quad (a \in S(X')).$$

so

$$\|a + \sum_{i=1}^{\infty} a_i \cdot \beta^i\| \leq |a| \quad (|\xi| \leq r/N)$$

and by the maximum modulus theorem

$$\|f(\xi)\| = |a| \quad (|\xi| \leq r/N). \quad \text{Q.E.D.}$$

Corollary 3.1. Let $X$ be a complex B-space and let $a, E(a) (i = 0, 1, \ldots, n)$ be a function with values in a complex B-space $X$, defined and analytic in a neighbourhood of the point 0 in the complex plane.

Then a neighbourhood of the point 0 in which $|f(\xi)|$ is constant exists if and only if

$$\sum_{i=1}^{\infty} |a_i| \cdot \beta^i \text{ converges for an } r > 0.$$

Proof. Let $|f(\xi)| = |a| \quad (|\xi| < R)$. This gives

$$|a_i| + \sum_{i=1}^{\infty} |a_i(r)|^2 \cdot \beta^i \leq |a| \quad (a \in S(X')).$$

By Proposition 2.1 it follows that $a, E(a) (i = 1, 2, \ldots)$ and $|a_i| \leq (3/R)^i (i = 1, 2, \ldots)$ clearly the series $\sum_{i=1}^{\infty} |a_i| \cdot \beta^i$ converges if $0 < r < R/\beta$. 

3 = Studia Mathematica LIII
34

J. Glo brnick

THEOREM 3.4. Let \( f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \) be a function with values in a complex B-space \( X \), defined and analytic in a neighbourhood of the point 0 in the complex plane. Let \( \dim E(a_0) < \infty \).

Then a neighbourhood of the point 0 in which \( |f(z)| < M \) is constant exists if and only if \( a_i E(a_0) \) (\( i = 1, 2, \ldots \)). Consequently the latter holds in the special case when \( X \) is finite-dimensional.

Proof. If \( |f(z)| \) is constant in a neighbourhood of the point 0, then by Theorem 3.4 \( a_i E(a_0) \) (\( i = 1, 2, \ldots \)).

To prove the converse, let \( \{z_1, z_2, \ldots, z_n\} \) be a basis of \( E(a_0) \). By Proposition 2.1 an \( M < \infty \) exists such that

\[
|\langle z_i, w \rangle| \leq M(|a_0| - |\langle a_0, w \rangle|) \quad (i = 1, 2, \ldots, n; w \in S(X')).
\]

Now, let \( a_i E(a_0) \) (\( i = 1, 2, \ldots \)) and let the series \( a_0 + a_1 z + \ldots \) converge for \( |z| < R \). Denote \( g(z) = a_0 + a_1 z + a_2 z^2 + \ldots \). The subspace \( E(a_0) \) being finite-dimensional, it is closed. It follows that \( g(z) E(a_0) \) (\( |z| < R \)). Now we may write \( g(z) = g_1(z) + g_2(z) + \ldots + g_n(z) \), \( |z| < R \) and an easy application of the Hahn–Banach theorem shows that the functions \( g_i (i = 1, 2, \ldots, n) \) are continuous for \( |z| < R \) with \( g_0(0) = 0 \) (\( i = 1, 2, \ldots, n \)). It follows that a positive \( r < R \) exists such that \( |g(z)| < 1/(nM) \) (\( i = 1, 2, \ldots, n; |z| < r \)). So

\[
|g(z), w)| = \left| \sum \gamma_i g_i(z), w) \right| 
\leq \sum \left| \gamma_i g_i(z) M(|a_0| - |\langle a_0, w \rangle|)
\leq |a_0| - |\langle a_0, w \rangle| \quad (|z| < r; w \in S(X'))
\]

what means that \( |f(z)| = |a_0 - g(z)| < |a_0| \) (\( |z| < r \)). By the maximum modulus theorem it follows that \( |f(z)| = |a_0| \) (\( |z| < r \)). Q.E.D.

4. The global characterization.

THEOREM 4.0. Let \( X \) be a complex B-space, \( \mathcal{B} \) a domain in the complex plane and \( f: \mathcal{B} \to X \) an analytic function. Let \( |f(z)| \) be constant on \( \mathcal{B} \). Then

(i) the subspace \( E[f(z)] \) does not depend on \( z \in \mathcal{B} \), i.e.

\[
E[f(z)] = \mathcal{B} \quad (z \in \mathcal{B}),
\]

(ii) \( f(z) = f(z) E[f(z)] \) (\( z \in \mathcal{B}, \mathcal{B} \in \mathcal{B} \)).

Conversely, let the following conditions be satisfied:

(iii) the closure \( E[f(z)] \) does not depend on \( z \in \mathcal{B} \), i.e.

\[
E[f(z)] = \mathcal{B} \quad (z \in \mathcal{B}),
\]

by Proposition 2.5 it follows that \( |f(z)| \geq |f(z)| \). Since \( z, \mathcal{B} \in \mathcal{B} \) were arbitrary, the last statement of the theorem follows. Q.E.D.

Remark 4.1. The result of Thorpe–Whiteley (14), Th. 3.1) follows immediately. One has only to notice that \( E(a) = \{0\} \) if \( x \) is a complex extreme point of \( S(X) \) and then to use Theorem 4.0 and Corollary 3.1. Later (Corollary 4.5) we shall generalize this result.
Proposition 4.2. Let $X$ be a complex $B$-space. Let $a, b \in X$ and $a \not\in S(X)$. Then the subspaces $E(a+1-a)b$ does not depend on $a \in (0,1)$, i.e.
\[ E(a+1-a)b = E \quad (0 < a < 1). \]
Further, $E(a) = E, E(b) = E$.

If in addition $b - a \in E(a) \cap E(b)$, then also $E(a) = E$ and $E(b) = E$.

Proof. Let $u, v \in X$, $\co{u, v} \subseteq S(X)$ and let $x = E(u)$. So an $r > 0$ exists such that $||u + zv|| < 1$ ($|z| < r$). Let $0 < a < 1$. Then
\[ ||u + (1-a)v|| < 1 \Rightarrow \co{u + (1-a)v} \subseteq S(X). \]
Since $\co{u, v} \subseteq S(X)$, we have $||u + (1-a)v|| = 1$. It follows that $ax \in E(u + (1-a)v)$. Now, by assumption $a \neq 0$ so Proposition 2.4 yields $x = E(u + (1-a)v)$, which proves that $E(u) = E(u + (1-a)v)$ ($0 < a < 1$). Interchanging the roles of $u$ and $v$ we get also
\[ E(v) = E(u + (1-a)v) \quad (0 < a < 1). \]

Now, let $0 < a_1 < a_2 < 1$. Put $a_1 + (1-a_2)b = v$ and $a = u$. By the assumption $a \not\in S(X)$ so that by the first part of the proof
\[ E(a_1 + (1-a_2)b) = E(a_1 + (1-a_2)b). \]
Putting $b = v$ and $a = u, b = u$, we have similarly
\[ E(a_1 + (1-a_2)b) \subseteq E(u_1 + (1-a_2)b). \]
So we proved that
\[ E(a+1-a)b = E \quad (0 < a < 1). \]

Similarly, for $a = u$ and $b = v$ we get $E(a) \subseteq E, E(b) \subseteq E$.

To prove the last statement of the proposition, let $b - a \in E(a) \cap E(b)$.

This means that $r > 0$ exists such that
\[ ||a + z(b-a)|| < ||a|| = 1 \quad \text{and} \quad ||b + z(b-a)|| < ||b|| = 1 \quad (|z| < r). \]

By the maximum modulus theorem we have
\[ ||a + z(b-a)|| = ||b + z(b-a)|| = 1 \quad (|z| < r). \]

By Theorem 4.0 it follows that the subspaces $E[a + z(b-a)]$ and $E[b + z(b-a)]$ do not depend on $z: |z| < 1$. Since $E[a + (1-a)b] = E$ ($0 < a < 1$) it follows that $E(a) = E(b)$. Q.E.D.

Lemma 4.3. Let $X$ be a complex $B$-space and let $\tilde{S}$ be a subset of $X$ satisfying $co S \subseteq S(X)$. Let $E(a) = E(a \cos \tilde{S})$ and let $x = y \in E(a \cos \tilde{S})$.

Then
\[ E(a) = E(a \cos \tilde{S}). \]

Proof. In view of Proposition 4.2 the proof is straightforward, so we omit it.

Theorem 4.4. Let $X$ be a complex $B$-space, $\mathcal{D}$ a domain in the complex plane and $f: \mathcal{D} \to X$ an analytic function, satisfying $\|f(z)\| = 1 \quad (z \in \mathcal{D})$.

Then
\[ E(a) = E(a \cos \tilde{S}). \]

Proof. By [4], Lemma 3.3, we have $\cos \tilde{S} \subseteq S(X)$, so that by Theorem 4.0 the set $f(S)$ satisfies the assumptions of Lemma 4.3 which proves the assertion.

Corollary 4.5. Let $X$ be a complex $B$-space, $\mathcal{D}$ a domain in the complex plane and $f: \mathcal{D} \to X$ an analytic function, satisfying $\|f(z)\| = 1 \quad (z \in \mathcal{D})$.

If $\cos \tilde{S}$ contains a complex extreme point of $S(X)$ then $f(z)$ is constant on $\mathcal{D}$.

Proof. Let a point $a \in \cos \tilde{S}$ be complex extreme of $S(X)$. This means that $E(a) = (0)$. By Theorem 4.4 it follows that $E(a) = E(a \cos \tilde{S})$. Since by Theorem 4.4 we have $f(z_1) - f(z_2) \in E(f(z_3))$ ($z_1, z_2 \in \tilde{S}$), it follows that $f(z_1) - f(z_2) = 0$ ($z_1, z_2 \in \tilde{S}$). Q.E.D.

Remark 4.6. In general, Theorem 4.4 and Corollary 4.5 do not hold for $\cos \tilde{S}$ instead of $\cos \tilde{S}$. To see this, let $X$ be the complex $B$-space of complex number pairs $z = [x, y]$, where $||x|| = \max{||x||, ||y||}$. Let $f(z) = (1, 1)$. Then $f$ is analytic, $||f(z)|| = 1 \quad (|z| < 1)$, but $f(1) = (1, 1)$ is a complex extreme point of $S(X)$. So $f(S)$ contains a complex extreme point of $S(X)$, but still $f(z)$ is not constant on $\mathcal{D}$ where $\mathcal{D} = \{|z| < 1\}$.

Acknowledgement. The author wishes to express his thanks to Professor J. Vidar who posed the problem and gave some helpful remarks, and to Professor L. A. Harris for his kind interest.

References

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS UNIVERSITY OF LJUBLJANA, YUGOSLAVIA