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On vector-valued analytic functions with constant norm

by

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Abstract. Let X be a complex Banach space and $\mathscr{D}$ a domain in the complex plane. Let $f\colon \mathscr{D}\to X$ be an analytic function such that $\|f(\zeta)\|$ is constant as $\zeta\in\mathscr{D}$. If X is the space of complex numbers then by the classical maximum modulus theorem $f(\zeta)$ itself is constant on $\mathscr{D}$. This is not the case in general. In the paper a characterization of the analytic functions with constant norm is given.

0. Introduction. If f is a complex-valued analytic function, defined on a domain $\mathscr{D}$ in the complex plane, then the classical maximum modulus theorem asserts that $|f(\zeta)|$ has no maximum on $\mathscr{D}$ or that $f(\zeta)$ is constant on $\mathscr{D}$. If f has values in a complex B-space, the theorem holds, but its strong form, asserting $f(\zeta)$ to be constant if $\|f(\zeta)\|$ is constant on $\mathscr{D}$, does not hold in general. E. Thorp and R. Whitley [4] characterized those complex B-spaces in which the strong form of the maximum modulus theorem holds – these are exactly the spaces in which every point on the unit sphere is a complex extreme point. Although many B-spaces have this property (e.g. strictly convex complex B-spaces), there remains a large class of the spaces which do not have this property (e.g. C^*-algebras of dimension greater than one [1]). Given such a space, we characterize those analytic functions with values in it, which have constant norm. We study the functions with values in B-spaces. The functions with values in B-algebras will be studied in a separate paper.

The idea is to linearize the problem in the sense that for every vector a of the unit sphere we construct the subspace $E(a)$ of all vectors showing that a is not a complex extreme point of the unit sphere, and then to obtain the characterization of the analytic functions with constant norm in terms of these subspaces.

1. Preliminaries. Throughout the paper B-space stands for Banach space. X being a B-space, we denote by $S(X) = \{x\in X\colon \|x\| = 1\}$ the unit sphere of X and by X' the dual space. The image of $x\in X$ under

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$u \in X'$ is denoted by $\langle x, u\rangle$. If S is a subset of a B-space, we denote by $\bar{S}$, $\mathrm{co}\, S$ and $\overline{\mathrm{co}}\, S$ the closure, the convex hull and the closed convex hull of the set S, respectively. An open connected subset of the complex plane is called domain.

Definition 1.0 (cf. [4]). Let X be a complex B-space. A point $a \in S(X)$ is called *complex extreme point of* $S(X)$ if $\|a+\zeta y\| \leqslant 1$ $(|\zeta| \leqslant 1)$ implies $y = 0$.

L. A. Harris [2] greatly simplified the original proof of Thorp–Whitley's result, using the following lemma.

Lemma 1.1 (L. A. Harris [2]). *Let f be a complex valued function, analytic on the open unit disc in the complex plane, satisfying* $|f(\zeta)| \leqslant 1$ $(|\zeta| < 1)$. *Then*

$$|f(0)| + \frac{1-|\zeta|}{2|\zeta|}\,|f(\zeta)-f(0)| \leqslant 1 \qquad (|\zeta|<1,\ \zeta \neq 0).$$

This lemma is the main tool in our paper.

2. The subspace $E(a)$.

Definition 2.0. Let X be a complex B-space and let $a \in X$. The set $E(a) \subset X$ is defined as follows. $x \in E(a)$ if $r > 0$ exists such that $\|a+\zeta x\| \leqslant \|a\|$ $(|\zeta| \leqslant r)$.

Proposition 2.1. *Let X be a complex B-space and let $a \in X$. Then $x \in E(a)$ if and only if a constant $M < \infty$ exists such that*

$$|\langle x, u\rangle| \leqslant M(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X')).$$

Proof. If

$$|\langle x, u\rangle| \leqslant M(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X'))$$

then

$$|\langle \zeta x, u\rangle| \leqslant \|a\| - |\langle a, u\rangle| \qquad (u \in S(X');\ |\zeta| \leqslant 1/M),$$

so

$$\|a+\zeta x\| \leqslant \|a\| \qquad (|\zeta| \leqslant 1/M).$$

Conversely, if $r > 0$ exists such that

$$\|a+\zeta x\| \leqslant \|a\| \qquad (|\zeta| \leqslant r)$$

then

$$|\langle a, u\rangle + \zeta\langle x, u\rangle| \leqslant \|a\| \qquad (|\zeta| \leqslant r;\ u \in S(X'))$$

so

$$|\langle a, u\rangle| + r|\langle x, u\rangle| \leqslant \|a\| \qquad (u \in S(X'))$$

and

$$|\langle x, u\rangle| \leqslant (1/r)(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X')).\ \text{Q.E.D.}$$

Definition 2.2. Let X be a complex B-space and let $a \in X$. For $x \in E(a)$ we define

$$\|x\|_a = \inf\{M:\ |\langle x, u\rangle| \leqslant M(\|a\| - |\langle a, u\rangle|)\ (u \in S(X'))\}.$$

Proposition 2.3. *Let X be a complex B-space. Let $a \in X$ and $x \in E(a)$. Then $\|x\|_a = 1/r(a)$, where*

$$r(a) = \sup\{r:\ \|a+\zeta x\| \leqslant \|a\|\ (|\zeta| \leqslant r)\}.$$

Proof. It is easy to see that the inequalities

$$\|a+\zeta x\| \leqslant \|a\| \qquad (|\zeta| \leqslant r)$$

and

$$|\langle x, u\rangle| \leqslant (1/r)(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X'))$$

are equivalent. Now the assertion follows immediately. Q.E.D.

Proposition 2.4. *Let X be a complex B-space and let $a \in X$. Then $E(a)$ is a linear subspace of X and $\|\ \|_a$ is a norm in $E(a)$.*

Proof. Let $x \in E(a)$. By Proposition 2.1 and Definition 2.2 we have

$$|\langle x, u\rangle| \leqslant \|x\|_a(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X')).$$

Clearly $\|x\|_a \geqslant 0$. If $\|x\|_a = 0$ then $\langle x, u\rangle = 0$ $(u \in S(X'))$, so $x = 0$. If α is a complex number, we have

$$|\langle \alpha x, u\rangle| = |\alpha|\,|\langle x, u\rangle| \leqslant |\alpha|\,\|x\|_a(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X'))$$

which shows that $\alpha x \in E(a)$ and that $\|\alpha x\|_a = |\alpha|\,\|x\|_a$. Further, if also $y \in E(a)$, then

$$|\langle y, u\rangle| \leqslant \|y\|_a(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X')),$$

so

$$\begin{aligned}|\langle x+y, u\rangle| &\leqslant |\langle x, u\rangle| + |\langle y, u\rangle| \\ &\leqslant (\|x\|_a + \|y\|_a)(\|a\| - |\langle a, u\rangle|) \qquad (u \in S(X'))\end{aligned}$$

which shows that $x+y \in E(a)$ and that $\|x+y\|_a \leqslant \|x\|_a + \|y\|_a$. Q.E.D.

Proposition 2.5. *Let X be a complex B-space and let $a \in X$. Then*

$$\|a+x\| \geqslant \|a\| \qquad (x \in \overline{E(a)}).$$

(Throughout, $\overline{E(a)}$ is the closure of $E(a)$ as a subset of X.)

Proof. Assume that $\|a+x\| < \|a\|$ for an $x \in \overline{E(a)}$. By the Hahn–Banach theorem an $u \in S(X')$ exists such that $\langle a, u\rangle = \|a\|$. Since $x \in \overline{E(a)}$, an $r > 0$ exists such that $\|a+\zeta x\| \leqslant \|a\|$ $(|\zeta| \leqslant r)$ what implies $|\langle a, u\rangle + \zeta\langle x, u\rangle| \leqslant \|a\|$ $(|\zeta| \leqslant r)$. Now, by $\langle a, u\rangle = \|a\|$ it follows that $\langle x, u\rangle = 0$, so that $\langle a+x, u\rangle = \|a\|$, contrarily to the assumption $\|a+x\| < \|a\|$. So $\|a+x\| \geqslant \|a\|$ $(x \in E(a))$ which proves the assertion. Q.E.D.

Lemma 2.6. *Let X be a complex B-space and let $a \in X$. Let $y \in E(a)$ where $\|y\|_a < 1/2$. Then $E(a+y) = E(a)$.*

Proof. Since $\|y\|_a < 1/2$, by Proposition 2.3 an $R > 2$ exists such that $\|a+\zeta y\| \leqslant \|a\|$ $(|\zeta| \leqslant R)$. Consequently, an $r > 1$ exists such that

$\|a+y+\zeta y\| \leqslant \|a\|$ $(|\zeta| \leqslant r)$. Since by Proposition 2.5 $\|a+y\| \geqslant \|a\|$, it follows that $y \in E(a+y)$ and by Proposition 2.3 we have $\|y\|_{a+y} < 1$.

Now, let $z \in E(a)$ and $\|z\|_a < 1$. We prove that $E(a+z) \supset E(a)$. By Proposition 2.3 we have $\|a+\zeta z\| \leqslant \|a\|$ $(|\zeta| \leqslant 1/\|z\|_a)$ and by Proposition 2.5 it follows that $\|a+z\| = \|a\|$. Let $x \in E(a)$, $x \neq 0$. Choose r such that $0 < r < (1-\|z\|_a)/\|z\|_a$. By Proposition 2.4, $\| \ \|_a$ is a norm so that we have

$$\|z+\zeta x\|_a < \|z\|_a + [(1-\|z\|_a)/\|x\|_a]\|x\|_a = 1 \qquad (|\zeta| \leqslant r).$$

By Proposition 2.3 it follows that

$$\|a+\xi(z+\zeta x)\| \leqslant \|a\| \qquad (|\zeta| \leqslant r;\ |\xi| \leqslant 1).$$

Taking $\xi = 1$ we get

$$\|(a+z)+\zeta x\| \leqslant \|a\| = \|a+z\| \qquad (|\zeta| \leqslant r)$$

which gives $x \in E(a+z)$. This proves that $E(a) \subset E(a+z)$. Consequently, $y \in E(a)$ and $\|y\|_a < 1$ imply $E(a+y) \supset E(a)$. Further, by the first part of the proof we have $y \in E(a+y)$ and $\|y\|_{a+y} < 1$, which implies $E(a) = E[(a+y)-y] \supset E(a+y)$. Q.E.D.

3. The local characterization.

THEOREM 3.0. *Let $\zeta \mapsto f(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \ldots$ be a function with values in a complex B-space X, defined and analytic in a neighbourhood of the point 0 in the complex plane.*

Then a neighbourhood of the point 0 in which $\|f(\zeta)\|$ is constant exists if and only if

(1) $$a_i \in E(a_0) \qquad (i = 1, 2, \ldots)$$

and

(ii) *the series* $\sum_{i=1}^{\infty} \|a_i\|_{a_0} \cdot r^i$ *converges for an* $r > 0$.

Proof. Let $\|f(\zeta)\| \equiv \|a_0\|$ $(|\zeta| < R)$. This gives

$$|\langle a_0, u\rangle + \langle a_1, u\rangle\zeta + \ldots| \leqslant \|a_0\| \qquad (|\zeta| < R;\ u \in S(X')).$$

Applying Lemma 1.1 to the function $\zeta \mapsto \langle f(R\zeta), u\rangle/\|a_0\|$, we obtain

$$|\langle a_1, u\rangle\zeta + \langle a_2, u\rangle\zeta^2 + \ldots| \leqslant \|a_0\| - |\langle a_0, u\rangle| \qquad (|\zeta| \leqslant R/3;\ u \in S(X')).$$

Cauchy's estimates (if γ is a complex valued analytic function and $|\gamma(\zeta)| \leqslant M$ $(|\zeta| < r)$, then $|\gamma^{(n)}(0)| \leqslant M \cdot r^{-n} \cdot n!$ $(n = 1, 2, \ldots)$ give

$$|\langle a_i, u\rangle| \leqslant (3/R)^i(\|a_0\| - |\langle a_0, u\rangle|) \qquad (i = 1, 2, \ldots;\ u \in S(X')).$$

By Proposition 2.1 it follows that $a_i \in E(a_0)$ $(i = 1, 2, \ldots)$ and $\|a_i\|_{a_0} \leqslant (3/R)^i$ $(i = 1, 2, \ldots)$. Clearly the series $\sum_{i=1}^{\infty} \|a_i\|_{a_0} \cdot r^i$ converges if $0 \leqslant r < R/3$.

To prove the converse, let $a_i \in E(a_0)$ $(i = 1, 2, \ldots)$ and let $\sum_{i=1}^{\infty} \|a_i\|_{a_0} \cdot r^i = M < \infty$ for an $r > 0$. By Definition 2.2 it follows that

$$\sum_{i=1}^{\infty} |\langle a_i r^i, u\rangle| \leqslant M(\|a_0\| - |\langle a_0, u\rangle|) \qquad (u \in S(X')).$$

If $N = \max\{1, M\}$ it is easily seen that

$$|\langle a_0, u\rangle| + \sum_{i=1}^{\infty} |\langle a_i (r/N)^i, u\rangle| \leqslant \|a_0\| \qquad (u \in S(X')).$$

so

$$\Big\|a_0 + \sum_{i=1}^{\infty} a_i\zeta^i\Big\| \leqslant \|a_0\| \qquad (|\zeta| \leqslant r/N)$$

and by the maximum modulus theorem

$$\|f(\zeta)\| \equiv \|a_0\| \qquad (|\zeta| \leqslant r/N). \qquad \text{Q.E.D.}$$

COROLLARY 3.1. *Let X be a complex B-space and let $a_i \in X$ $(i = 0, 1, \ldots, n)$. Then $\|a_0 + a_1\zeta + \ldots + a_n\zeta^n\|$ is constant in a neighbourhood of the point 0 if and only if $a_i \in E(a_0)$ $(i = 1, 2, \ldots, n)$.*

Proof. Trivial. Q.E.D.

COROLLARY 3.2. *Let X be a complex B-space and let $a_i \in X$ $(i = 0, 1, \ldots, m)$. Let $\|a_0 + a_1\zeta + \ldots + a_m\zeta^m\|$ be constant in a neighbourhood of the point 0. If $b_1, b_2, \ldots, b_n$ lie in the subspace spanned by $a_1, a_2, \ldots, a_m$ then a neighbourhood of the point 0 exists in which $\|a_0 + b_1\zeta + b_1\zeta^2 + \ldots + b_n\zeta^n\|$ is constant.*

Proof. By Theorem 3.0 we have $a_i \in E(a_0)$ $(i = 1, 2, \ldots, m)$. Let $b_1, b_2, \ldots, b_n$ be in the linear subspace, spanned by $a_1, a_2, \ldots, a_m$. By Proposition 2.4 $E(a_0)$ is linear subspace, so we have $b_i \in E(a_0)$ $(i = 1, 2, \ldots, n)$ whence the statement follows by Corollary 3.1. Q.E.D.

COROLLARY 3.3. *Let X be a complex B-space. Let $a_0 \in X$ and $a_i \in E(a_0)$ $(i = 1, 2, \ldots)$. Then the sequence $\{\alpha_i;\ i = 1, 2, \ldots\}$ of positive numbers exists with the following property: if $\{\gamma_i;\ i = 1, 2, \ldots\}$ is a sequence of (complex) numbers such that $|\gamma_i| \leqslant \alpha_i$ $(i = 1, 2, \ldots)$, then a neighbourhood of the point 0 exists in which $\|a_0 + (\gamma_1 a_1)\zeta + (\gamma_2 a_2)\zeta^2 + \ldots\|$ is constant.*

Proof. Choose $\{\alpha_i;\ i = 1, 2, \ldots\}$ so that the series $\sum_{i=1}^{\infty} \|\alpha_i a_i\|_{a_0} \cdot r^i$ converges for an $r > 0$ and apply Theorem 3.0. Q.E.D.

In Theorem 3.0 the assumption (ii) can be dropped if the function considered is a polynomial (Corollary 3.1). The same holds also for an arbitrary function if the subspace $E(a_0)$ is finite-dimensional, as the following theorem shows.

THEOREM 3.4. *Let $\zeta\mapsto f(\zeta)=a_0+a_1\zeta+a_2\zeta^2+\ldots$ be a function with values in a complex B-space X, defined and analytic in a neighbourhood of the point 0 in the complex plane. Let $\dim E(a_0)<\infty$.*

Then a neighbourhood of the point 0 in which $\|f(\zeta)\|$ is constant exists if and only if $a_i\in E(a_0)$ $(i=1,2,\ldots)$. Consequently the latter holds in the special case when X is finite-dimensional.

Proof. If $\|f(\zeta)\|$ is constant in a neighbourhood of the point 0, then by Theorem 3.0 $a_i\in E(a_0)$ $(i=1,2,\ldots)$.

To prove the converse, let $\{e_1,e_2,\ldots,e_n\}$ be a basis of $E(a_0)$. By Proposition 2.1 an $M<\infty$ exists such that

$$|\langle e_i,u\rangle|\leqslant M(\|a_0\|-|\langle a_0,u\rangle|)\quad (i=1,2,\ldots,n;\ u\in S(X')).$$

Now, let $a_i\in E(a_0)$ $(i=1,2,\ldots)$ and let the series $a_0+a_1\zeta+\ldots$ converge for $|\zeta|<R$. Denote $g(\zeta)=a_1\zeta+a_2\zeta^2+\ldots$ The subspace $E(a_0)$ being finite-dimensional, it is closed. It follows that $g(\zeta)\in E(a_0)$ $(|\zeta|<R)$. Now we may write $g(\zeta)=\gamma_1(\zeta)e_1+\gamma_2(\zeta)e_2+\ldots+\gamma_n(\zeta)e_n$ $(|\zeta|<R)$ and an easy application of the Hahn–Banach theorem shows that the functions γ_i $(i=1,2,\ldots,n)$ are continuous for $|\zeta|<R$ with $\gamma_i(0)=0$ $(i=1,2,\ldots,n)$. It follows that a positive $r<R$ exists such that $|\gamma_i(\zeta)|\leqslant 1/(nM)$ $(i=1,2,\ldots,n;\ |\zeta|\leqslant r)$. So

$$|\langle g(\zeta),u\rangle|=\Big|\sum_{i=1}^{n}\gamma_i(\zeta)\langle e_i,u\rangle\Big|$$
$$\leqslant\sum_{i=1}^{n}|\gamma_i(\zeta)|M(\|a_0\|-|\langle a_0,u\rangle|)$$
$$\leqslant\|a_0\|-|\langle a_0,u\rangle|\quad (|\zeta|<r;\ u\in S(X')),$$

what means that $\|f(\zeta)\|=\|a_0+g(\zeta)\|\leqslant\|a_0\|$ $(|\zeta|<r)$. By the maximum modulus theorem it follows that $\|f(\zeta)\|\equiv\|a_0\|$ $(|\zeta|<r)$. Q.E.D.

4. The global characterization.

THEOREM 4.0. *Let X be a complex B-space, $\mathscr{D}$ a domain in the complex plane and $f\colon\mathscr{D}\to X$ an analytic function.*

Let $\|f(\zeta)\|$ be constant on $\mathscr{D}$. Then

(i) *the subspace $E[f(\zeta)]$ does not depend on $\zeta\in\mathscr{D}$, i.e.*

$$E[f(\zeta)]\equiv E\quad (\zeta\in\mathscr{D}),$$

(ii) $f(\zeta_1)-f(\zeta_2)\in E$ $(\zeta_1\in\mathscr{D},\ \zeta_2\in\mathscr{D})$.

Conversely, let the following conditions be satisfied:

(i′) *the closure $\overline{E[f(\zeta)]}$ does not depend on $\zeta\in\mathscr{D}$, i.e.*

$$\overline{E[f(\zeta)]}\equiv F\quad (\zeta\in\mathscr{D}),$$

(ii′) $f(\zeta_1)-f(\zeta_2)\in F$ $(\zeta_1\in\mathscr{D},\ \zeta_2\in\mathscr{D})$.

Then $\|f(\zeta)\|$ is constant on $\mathscr{D}$.

Proof. Let $\|f(\zeta)\|\equiv M$ $(\zeta\in\mathscr{D})$ and let $\mathscr{D}$ contain a disc $|\zeta-\zeta_0|<r$. Then

$$|\langle f(\zeta),u\rangle|\leqslant M\quad (|\zeta-\zeta_0|<r;\ u\in S(X')),$$

and applying Lemma 1.1 to the function $\zeta\mapsto\langle f(\zeta_0+r\zeta),u\rangle/M$ we have

$$|\langle f(\zeta_0),u\rangle|+\frac{r-|\zeta-\zeta_0|}{2|\zeta-\zeta_0|}|\langle f(\zeta)-f(\zeta_0),u\rangle|\leqslant M$$
$$(0<|\zeta-\zeta_0|<r;\ u\in S(X')),$$

what gives

$$|\langle f(\zeta)-f(\zeta_0),u\rangle|\leqslant\frac{2|\zeta-\zeta_0|}{r-|\zeta-\zeta_0|}(\|f(\zeta_0)\|-|\langle f(\zeta_0),u\rangle|)$$
$$(|\zeta-\zeta_0|<r;\ u\in S(X')).$$

By Proposition 2.1 it follows that

$$f(\zeta)-f(\zeta_0)\in E[f(\zeta_0)]\quad (|\zeta-\zeta_0|<r) \tag{1}$$

where

$$\|f(\zeta)-f(\zeta_0)\|_{f(\zeta_0)}<1/2\quad (|\zeta-\zeta_0|<r/5).$$

Now Lemma 2.6 applies to show that

$$E[f(\zeta)]=E\big[f(\zeta_0)+\big(f(\zeta)-f(\zeta_0)\big)\big]=E[f(\zeta_0)]\quad (|\zeta-\zeta_0|<r/5). \tag{2}$$

Since the set $\mathscr{D}$ is open, for every $\zeta_0\in\mathscr{D}$ a disc with center at ζ_0 is contained in $\mathscr{D}$ so that for every $\zeta_0\in\mathscr{D}$ we can prove (1) and (2). Now, $\mathscr{D}$ being connected, any two points of $\mathscr{D}$ can be connected by a (compact) arc and by the compactness argument it follows that $E[f(\zeta)]\equiv E$ $(\zeta\in\mathscr{D})$. By (1) the same argument yields $f(\zeta_1)-f(\zeta_2)\in E$ $(\zeta_1,\zeta_2\in\mathscr{D})$.

Conversely, let (i′) and (ii′) hold. Let $\zeta_1,\zeta_2\in\mathscr{D}$. Writing

$$f(\zeta_1)=f(\zeta_2)+[f(\zeta_1)-f(\zeta_2)]$$

and noticing that by (ii′)

$$f(\zeta_1)-f(\zeta_2)\in F=\overline{E[f(\zeta_2)]},$$

by Proposition 2.5 it follows that $\|f(\zeta_1)\|\geqslant\|f(\zeta_2)\|$. Since $\zeta_1,\zeta_2\in\mathscr{D}$ were arbitrary, the last statement of the theorem follows. Q.E.D.

Remark 4.1. The result of Thorp–Whitley ([4], Th. 3.1) follows immediately. One has only to notice that $E(x)=\{0\}$ if x is a complex extreme point of $S(X)$ and then to use Theorem 4.0 and Corollary 3.1. Later (Corollary 4.5) we shall generalize this result.

PROPOSITION 4.2. *Let X be a complex B-space. Let $a, b \in X$ and $\mathrm{co}\{a, b\} \subset S(X)$. Then the subspace $E[\alpha a+(1-\alpha)b]$ does not depend on $\alpha \in (0,1)$, i.e.*

$$E[\alpha a+(1-\alpha)b] \equiv E \quad (0<\alpha<1).$$

Further, $E(a) \subset E$, $E(b) \subset E$.

If in addition $b-a \in E(a) \cap E(b)$, then also $E(a) = E$ and $E(b) = E$.

Proof. Let $u, v \in X$, $\mathrm{co}\{u, v\} \subset S(X)$ and let $x \in E(u)$. So an $r>0$ exists such that $\|u+\zeta x\| \leqslant 1$ $(|\zeta| \leqslant r)$. Let $0<\alpha<1$. Then

$$\|[\alpha u+(1-\alpha)v]+\zeta(\alpha x)\| = \|\alpha(u+\zeta x)+(1-\alpha)v\| \leqslant \alpha\|u+\zeta x\|+(1-\alpha)\|v\| \leqslant \alpha+(1-\alpha) = 1 \quad (|\zeta| \leqslant r).$$

Since $\mathrm{co}\{u, v\} \subset S(X)$, we have $\|\alpha u+(1-\alpha)v\| = 1$. It follows that $\alpha x \in E[\alpha u+(1-\alpha)v]$. Now, by supposition $\alpha \neq 0$ so Proposition 2.4 yields $x \in E[\alpha u+(1-\alpha)v]$, which proves that $E(u) \subset E[\alpha u+(1-\alpha)v]$ $(0<\alpha\leqslant 1)$. Interchanging the roles of u and v we get also

$$E(v) \subset E[\alpha u+(1-\alpha)v] \quad (0 \leqslant \alpha < 1).$$

Now, let $0<\alpha_1<\alpha_2<1$. Put $\alpha_1 a+(1-\alpha_1)b = v$ and $a = u$. By the assumption $\mathrm{co}\{a, b\} \subset S(X)$ so that by the first part of the proof

$$E[\alpha_1 a+(1-\alpha_1)b] \subset E[\alpha_2 a+(1-\alpha_2)b].$$

Putting $b = v$ and $\alpha_2 a+(1-\alpha_2)b = u$, we have similarly

$$E[\alpha_2 a+(1-\alpha_2)b] \subset E[\alpha_1 a+(1-\alpha_1)b].$$

So we proved that

$$E[\alpha a+(1-\alpha)b] \equiv E \quad (0<\alpha<1).$$

Similarly, for $a = u$ and $b = v$ we get $E(a) \subset E$, $E(b) \subset E$.

To prove the last statement of the proposition, let $b-a \in E(a) \cap E(b)$. This means that an $r>0$ exists such that

$$\|a+\zeta(b-a)\| \leqslant \|a\| = 1 \quad \text{and} \quad \|b+\zeta(b-a)\| < \|b\| = 1 \; (|\zeta| \leqslant r).$$

By the maximum modulus theorem we have

$$\|a+\zeta(b-a)\| \equiv \|b+\zeta(b-a)\| \equiv 1 \quad (|\zeta| \leqslant r).$$

By Theorem 4.0 it follows that the subspaces $E[a+\zeta(b-a)]$ and $E[b+\zeta(b-a)]$ do not depend on ζ: $|\zeta|<r$. Since $E[\alpha a+(1-\alpha)b] \equiv E$ $(0<\alpha<1)$ it follows that $E(a) = E(b) = E$. Q.E.D.

LEMMA 4.3. *Let X be a complex B-space and let S be a subset of X satisfying $\mathrm{co}\, S \subset S(X)$. Let $E(a) \equiv E$ $(a \in S)$ and let $x-y \in E$ $(x, y \in S)$. Then*

$$E(a) \equiv E \quad (a \in \mathrm{co}\, S).$$

Proof. In view of Proposition 4.2 the proof is straightforward, so we omit it.

THEOREM 4.4. *Let X be a complex B-space, $\mathscr{D}$ a domain in the complex plane and $f\colon \mathscr{D} \to X$ an analytic function, satisfying $\|f(\zeta)\| \equiv 1$ $(\zeta \in \mathscr{D})$. Then*

$$E(a) \equiv E \quad \big(a \in \mathrm{co} f(\mathscr{D})\big).$$

Proof. By [4], Lemma 3.3, we have $\mathrm{co} f(\mathscr{D}) \subset S(X)$, so that by Theorem 4.0 the set $f(\mathscr{D})$ satisfies the assumptions of Lemma 4.3 which proves the assertion.

COROLLARY 4.5. *Let X be a complex B-space, $\mathscr{D}$ a domain in the complex plane and $f\colon \mathscr{D} \to X$ an analytic function, satisfying $\|f(\zeta)\| \equiv 1$ $(\zeta \in \mathscr{D})$. If $\mathrm{co} f(\mathscr{D})$ contains a complex extreme point of $S(X)$ then $f(\zeta)$ is constant on $\mathscr{D}$.*

Proof. Let a point $a_0 \in \mathrm{co} f(\mathscr{D})$ be complex extreme of $S(X)$. This means that $E(a_0) = \{0\}$. By Theorem 4.4 it follows that $E(a) = \{0\}$ $\big(a \in \mathrm{co} f(\mathscr{D})\big)$. Since by Theorem 4.4 we have $f(\zeta_1)-f(\zeta_2) \in E[f(\zeta_2)]$ $(\zeta_1, \zeta_2 \in \mathscr{D})$, it follows that $f(\zeta_1)-f(\zeta_2) = 0$ $(\zeta_1, \zeta_2 \in \mathscr{D})$. Q.E.D.

Remark 4.6. In general, Theorem 4.4 and Corollary 4.5 do not hold for $\overline{\mathrm{co}} f(\mathscr{D})$ instead of $\mathrm{co} f(\mathscr{D})$. To see this, let X be the complex B-space of complex number pairs $z = \{z_1, z_2\}$, where $\|z\| = \max\{|z_1|, |z_2|\}$. Let $f(\zeta) = \{1, \zeta\}$. Then f is analytic, $\|f(\zeta)\| \equiv 1$ $(|\zeta|<1)$, but $f(1) = \{1, 1\}$ is a complex extreme point of $S(X)$. So $\overline{f(\mathscr{D})}$ contains a complex extreme point of $S(X)$, but still $f(\zeta)$ is not constant on $\mathscr{D}$ where $\mathscr{D} = \{\zeta\colon |\zeta|<1\}$.

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