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A short proof of the theorem of Crone and Robinson* on quasi-equivalence of regular bases

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Abstract. Recently L. Crone and W. Robinson* proved that in any nuclear Fréchet space with a regular basis all bases are quasi-equivalent. Here a simple proof of this fact is given.

Let $(e_n)_0^{\infty}$ and $(f_n)_0^{\infty}$ be bases in a linear topological space E. We say that (e_n) and (f_n) are equivalent if the operator $T \colon E \to E$ such that $Te_n = f_n$, ∇n , is an isomorphism; (e_n) and (f_n) are semi-equivalent if there exists a sequence (r_n) of non-zero scalars such that (e_n) and (r_nf_n) are equivalent; (e_n) and (f_n) are quasi-equivalent if there exists a permutation σ of natural numbers such that (e_n) and $(f_{\sigma(n)})$ are semi-equivalent.

A basis (e_n) in a Fréchet space E is regular (in the sense of Dragilev [2]) if there is a sequence of semi-norms $(p_t)_1^{\infty}$ generating the topology of E, such that

$$p_t(e_n)/p_{t+1}(e_n) \geqslant p_t(e_{n+1})/p_{t+1}(e_{n+1})$$
 for all t and n.

M. M. Dragilev, B. S. Mitiagin and V. P. Zaharjuta have proved, for wide classes of linear topological spaces, that all bases in the space are quasi-equivalent. For more information and historical remarks see Mitiagin [5].

Recently L. Crone and W. Robinson*, generalizing former Dragilev's results, proved in [1] the following

THEOREM. In every nuclear Fréchet space with a regular basis all bases are quasi-equivalent.

Since in any nuclear Fréchet space with a regular basis every basis admits a permutation which makes it regular ([2], Theorem 1), the proof of Theorem reduces to that of the following proposition.

Proposition 1. If E is a nuclear Fréchet space, then any two regular bases for E are semi-equivalent.

^{*} Remark added in proof. The author was informed that the same theorem was obtained independently by V. P. Kondakov (see [7]).

We shall present a simple proof of this proposition. Suppose that $(e_n)_0^{\infty}$ and $(f_n)_0^{\infty}$ are bases in E which are regular with respect to the sequences of semi-norms $(p_t)_1^{\infty}$ and $(q_t)_1^{\infty}$, respectively. Put

$$egin{aligned} p_t^*(x) &= \Big(\sum_n |\xi_n|^2 p_t^2(e_n)\Big)^{1/2} & ext{ for } & x &= \sum_n \xi_n e_n \ & ext{ and } & U_t &= \{x \, \epsilon \, E \colon \, p_t^*(x) < 1\}, \ & q_t^*(x) &= \Big(\sum_n |\eta_n|^2 q_t^2(f_n)\Big)^{1/2} & ext{ for } & x &= \sum_n \eta_n f_n \ & ext{ and } & V_t &= \{x \, \epsilon \, E \colon \, q_t^*(x) < 1\}. \end{aligned}$$

Since all bases in nuclear Fréchet spaces are absolute (Dynin-Mitiagin [3]; cf. [4] and [6]), each sequence of semi-norms (p_t^*) and (q_t^*) generates the topology of E. Without loss of generality one may assume that

$$U_1 \supset V_1 \supset U_2 \supset \ldots \supset U_t \supset V_t \supset U_{t+1} \supset \ldots$$

(if not, we pass to suitable subsequences of the seminorms, and replace the seminorms by their positive multiples).

Since the bases are regular, we have the following expressions for the diameters (cf. [4], § 2, or [6], 9.1):

$$d_n(U_s, U_t) = a_n^t/a_n^s, d_n(V_s, V_t) = b_n^t/b_n^s,$$
where $a_n^t = p_t(e_n), b_n^t = q_t(f_n).$

Therefore

$$\begin{split} &\text{(I) if } t\leqslant s \text{ then } U_t\supset V_t\supset V_s\supset U_{s+1} \text{ and we have} \\ &d_n(U_{s+1},\,U_t)\leqslant d_n(V_s,\,V_t), \quad a_n^t/a_n^{s+1}\leqslant b_n^t/b_n^s \quad \text{for } \quad n=0\,,1\,,\ldots; \end{split}$$

(II) if
$$t > s$$
 then $V_s \supset U_{s+1} \supset U_t \supset V_t$ and we have

$$d_n(V_t, V_s) \leqslant d_n(U_t, U_{s+1}), \quad b_n^s/b_n^t \leqslant a_n^{s+1}/a_n^t \quad \text{for} \quad n = 0, 1, \dots$$

By (I) and (II),
$$a_n^t/b_n^t \leqslant a_n^{s+1}/b_n^s$$
 for all t,s and n . Let

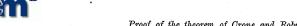
$$r_n = \sup_t a_n^t / b_n^t.$$

Then $a_n^t \leqslant r_n b_n^t \leqslant a_n^{t+1}$ for all t and n. Hence the operator $T \colon E \to E$ such that $Te_n = r_n f_n$ for every n is an isomorphism. This completes the proof of the proposition.

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