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A short proof of the theorem of Crone and Robinson* on quasi-equivalence of regular bases

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Abstract. Recently L. Crone and W. Robinson* proved that in any nuclear Fréchet space with a regular basis all bases are quasi-equivalent. Here a simple proof of this fact is given.

Let $(e_n)_0^\infty$ and $(f_n)_0^\infty$ be bases in a linear topological space E. We say that (e_n) and (f_n) are *equivalent* if the operator $T\colon E \to E$ such that $Te_n = f_n$, $\forall n$, is an isomorphism; (e_n) and (f_n) are *semi-equivalent* if there exists a sequence (r_n) of non-zero scalars such that (e_n) and $(r_n f_n)$ are equivalent; (e_n) and (f_n) are *quasi-equivalent* if there exists a permutation σ of natural numbers such that (e_n) and $(f_{\sigma(n)})$ are semi-equivalent.

A basis (e_n) in a Fréchet space E is *regular* (in the sense of Dragilev [2]) if there is a sequence of semi-norms $(p_t)_1^\infty$ generating the topology of E, such that

$$p_t(e_n)/p_{t+1}(e_n) \geqslant p_t(e_{n+1})/p_{t+1}(e_{n+1}) \quad \text{for all } t \text{ and } n.$$

M. M. Dragilev, B. S. Mitiagin and V. P. Zaharjuta have proved, for wide classes of linear topological spaces, that all bases in the space are quasi-equivalent. For more information and historical remarks see Mitiagin [5].

Recently L. Crone and W. Robinson*, generalizing former Dragilev's results, proved in [1] the following

THEOREM. *In every nuclear Fréchet space with a regular basis all bases are quasi-equivalent.*

Since in any nuclear Fréchet space with a regular basis every basis admits a permutation which makes it regular ([2], Theorem 1), the proof of Theorem reduces to that of the following proposition.

PROPOSITION 1. *If E is a nuclear Fréchet space, then any two regular bases for E are semi-equivalent.*

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* Remark added in proof. The author was informed that the same theorem was obtained independently by V. P. Kondakov (see [7]).

We shall present a simple proof of this proposition. Suppose that $(e_n)_0^\infty$ and $(f_n)_0^\infty$ are bases in E which are regular with respect to the sequences of semi-norms $(p_t)_1^\infty$ and $(q_t)_1^\infty$, respectively. Put

$$p_t^*(x) = \Big(\sum_n |\xi_n|^2 p_t^2(e_n)\Big)^{1/2} \quad \text{for} \quad x = \sum_n \xi_n e_n$$
$$\text{and} \quad U_t = \{x \in E\colon\ p_t^*(x) < 1\},$$
$$q_t^*(x) = \Big(\sum_n |\eta_n|^2 q_t^2(f_n)\Big)^{1/2} \quad \text{for} \quad x = \sum_n \eta_n f_n$$
$$\text{and} \quad V_t = \{x \in E\colon\ q_t^*(x) < 1\}.$$

Since all bases in nuclear Fréchet spaces are absolute (Dynin–Mitiagin [3]; cf. [4] and [6]), each sequence of semi-norms (p_t^*) and (q_t^*) generates the topology of E. Without loss of generality one may assume that

$$U_1 \supset V_1 \supset U_2 \supset \ldots \supset U_t \supset V_t \supset U_{t+1} \supset \ldots$$

(if not, we pass to suitable subsequences of the seminorms, and replace the seminorms by their positive multiples).

Since the bases are regular, we have the following expressions for the diameters (cf. [4], § 2, or [6], 9.1):

$$d_n(U_s, U_t) = a_n^t/a_n^s,\ d_n(V_s, V_t) = b_n^t/b_n^s,$$
$$\text{where} \quad a_n^t = p_t(e_n), \quad b_n^t = q_t(f_n).$$

Therefore

(I) if $t \leqslant s$ then $U_t \supset V_t \supset V_s \supset U_{s+1}$ and we have

$$d_n(U_{s+1}, U_t) \leqslant d_n(V_s, V_t), \quad a_n^t/a_n^{s+1} \leqslant b_n^t/b_n^s \quad \text{for} \quad n = 0, 1, \ldots;$$

(II) if $t > s$ then $V_s \supset U_{s+1} \supset U_t \supset V_t$ and we have

$$d_n(V_t, V_s) \leqslant d_n(U_t, U_{s+1}), \quad b_n^s/b_n^t \leqslant a_n^{s+1}/a_n^t \quad \text{for} \quad n = 0, 1, \ldots$$

By (I) and (II), $a_n^t/b_n^t \leqslant a_n^{s+1}/b_n^s$ for all t, s and n. Let

$$r_n = \sup_t a_n^t/b_n^t.$$

Then $a_n^t \leqslant r_n b_n^t \leqslant a_n^{t+1}$ for all t and n. Hence the operator $T\colon E \to E$ such that $Te_n = r_n f_n$ for every n is an isomorphism. This completes the proof of the proposition.

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