STUDIA MATHEMATICA, T. LIII. (1975)

On commutative approximate identities

by

T. PYTLIK (Wrocław)

Abstract. It is shown that every locally compact metric group G has a commutative approximate identity for $L_2(G)$.

This result was earlier obtained in [2] even for $L_1(G)$, but in a more complicated way. A simple construction of commutative approximate identity for a C^*-algebra was given in [1].

Let G be a locally compact group, and assume for simplicity that G is unimodular. We say that a bounded operator T, which acts on $L_2(G)$, is a *convolution operator* if there exists a function f in $L_2(G)$ such that

$$Tg(s) = \int_G f(su^{-1})g(u)\,du \tag{1}$$

for all $g \in L_2(G)$. We then write $T = T_f$. The conjugated operator T_f^* is of the form T_{f^*} where $f^*(s) = \overline{f(s^{-1})}$, $s \in G$. If $f \in L_1(G) \cap L_2(G)$ then (1) defines a bounded convolution operator with norm at most $\|f\|_1$.

LEMMA. *For a locally compact metric group there exists a continuous function $f = f^*$ with a compact support and such that* $\ker T_f = \{0\}$.

For the proof see [3], Theorem 1.

THEOREM. *Let f be in $L_1(G) \cap L_2(G)$, and suppose* $\ker T_f = \ker T_f^* = \{0\}$, $\|f\|_1 \leqslant 1$. *There is a family $\{p_t\}_{t>0}$ of functions in $L_2(G) \cap C_0(G)$ which has the following properties:*

(i) $p_t = p_t^*$;

(ii) $p_t * p_s = p_{t+s}$;

(iii) *if* $g \in L_2(G)$ *then* $p_t * g \in L_2(G)$, $t > 0$, *and* $g = \lim_{t \searrow 0} p_t * g$;

(iv) $\int_0^\infty e^{-t} p_t\,dt = f^* * f$, *the integral being convergent in $L_2(G)$ and in* $C_0(G)$.

icm©

Proof. Let $T = T_{f^* * f} = T_f^* T_f$. We have $\mathrm{Sp}\, T \subset [0,1]$ and $\ker T = \{0\}$. Let

$$T = \int_0^1 \lambda E(d\lambda)$$

be the spectral resolution of T and for $t \in (0, \infty)$ denote by P_t, Q_t the operators

$$P_t = \int_0^1 \exp t(1-1/\lambda) E(d\lambda),$$

$$Q_t = \int_0^1 \lambda^{-1} \exp t(1-1/\lambda) E(d\lambda).$$

Then $P_t = TQ_t$ and $P_t P_s = P_{t+s}$,

$$\|P_t\| \leqslant \sup_{\lambda \in [0,1]} \exp t(1-1/\lambda) = 1,$$

$$\|Q_t\| \leqslant \sup_{\lambda \in [0,1]} \lambda^{-1} \exp t(1-1/\lambda) \leqslant \max\{t^{-1}, 1\}.$$

It is clear that $\ker P_t = \ker T = \{0\}$ for all $t > 0$ and that P_t strongly converges to the identity operator when t tends to zero. Since $f \in L_2(G)$ and $P_t = TQ_t$, each of the P_t's is a continuous mapping from $L_2(G)$ into $C_0(G)$ with norm at most $\|f\|_2 \|Q_t\|$, and

$$L_2(G) \ni g \to (P_t g)(e) \in \boldsymbol{C}$$

is a continuous linear functional on $L_2(G)$. Consequently, there is a function $p_t \in L_2(G)$, $\|p_t\|_2 \leqslant \|f\|_2 \|Q_t\|$ such that

$$(P_t g)(e) = \langle g, p_t \rangle = \int_G g(s) \overline{p_t(s)}\, ds.$$

Since each P_t commutes with the right translations on G, we have

$$P_t g(u) = \int_G g(su) \overline{p_t(s)}\, ds = p_t^* * g(u).$$

We also have $p_t^* = p_t$ and $p_t * p_s = p_{t+s}$, thus $p_t \in L_2(G) \cap C_0(G)$.

For a fixed $\varepsilon > 0$, both $\|p_t\|_2$ and $\|p_t\|_\infty = \|p_{t/2}\|_2^2$ are continuous functions of t, bounded on the interval (ε, ∞), therefore the integral $\int_\varepsilon^\infty e^{-t} p_t\, dt$ is convergent to a function h_ε in $L_2(G) \cap C_0(G)$. Since

$$\int_0^\infty e^{-t} P_t\, dt = \int_0^1 \int_0^\infty e^{-t} e^{t(1-1/\lambda)}\, dt\, E(d\lambda) = \int_0^1 \lambda E(d\lambda) = T,$$

we have

$$e^{-\varepsilon} h_\varepsilon = e^{-\varepsilon} \int_\varepsilon^\infty e^{-t} p_t\, dt = \int_0^\infty e^{-t} p_{t+\varepsilon}\, dt = \int_0^\infty e^{-t} P_t p_\varepsilon\, dt = f^* * f * p_\varepsilon.$$

Hence, since both functions h_ε and $f^* * f * p_\varepsilon$ are continuous, we have

$$e^{-\varepsilon} h_\varepsilon - f^* * f = f^* * f * p_\varepsilon - f^* * f$$

and also

$$\|e^{-\varepsilon} h_\varepsilon - f^* * f\|_\infty \leqslant \|f^*\|_2 \|f * p_\varepsilon - f\|_2$$

which by (iii) shows that

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty e^{-t} p_t\, dt = f^* * f,$$

whence the convergence is both in L_2 and C_0.

References

[1] J. A. Aarnes and R. V. Kadison, *Pure states and approximate identities*, Proc. Amer. Math. Soc. 21 (1969), pp. 749–752.

[2] A. Hulanicki and T. Pytlik, *On commutative approximate identities and cyclic vectors of induced representations*, Studia Math. 48 (1973), pp. 189–199.

[3] T. Pytlik, *A nuclear space of functions on a locally compact group*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. et Phys. 17 (1969), pp. 161–166.

Received December 13, 1973 (778)