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On the analytic transform of bounded linear functionals on certain Banach algebras

by

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Abstract. A generalization of the Fourier–Carleman transform is introduced for bounded linear functionals on certain commutative Banach algebras. The investigation concerns the relations between the set of singularities of the transform and the structure of the largest annihilating ideal of the functional. Generalizations are obtained of earlier results which have been proved by different and more limited methods.

1. Introduction. In many investigations of special commutative Banach algebras with a subspace M of C as regular maximal ideal space, one has found it useful to map the dual Banach spaces of bounded linear functionals into the space of functions, holomorphic on $C\setminus M$, using transformations of Borel type. Following ideas from Carleman [3], this approach has been used for instance in the papers [1], [2], [8], [11]–[17], [20], [21], [23], [24].

Of particular concern in these studies are bounded linear functionals which annihilate some non-trivial ideal in the algebra. It was shown that every such functional has a transform which can be continued across some part of the boundary of M to a meromorphic function in the interior of M. Close relations were found between the singularities of the continued transform and the spectrum of the functional, i.e. the co-spectrum of its largest annihilating ideal. By means of estimates from above of the modulus of the transform and its continuation, it was possible to obtain information on the nature of the singularities of the transforms of certain functionals, and this in its turn could be used to reveal interesting properties of the ideal structure of the studied Banach algebras. A typical question which was settled in many Banach algebras by this method is the problem of determining all closed ideals which are primary, i.e. have one-point co-spectrum.

The methods which were employed in the mentioned papers to establish basic properties of the analytic transform are essentially function-theoretic and depend very much on particular properties of the individual

Banach algebras. Hence these methods cannot be used to build a unified and more general theory. It is, however, possible to attain this by means of the following simple observation: For every commutative Banach algebra B with unit, a bounded linear functional F with spectrum $\Lambda(F)$ and largest annihilating ideal $I(F)$ can be interpreted as a bounded linear functional on $B/I(F)$, a Banach algebra with maximal ideal space $\Lambda(F)$. This makes it possible to define the analytic transform of F directly as a function, holomorphic in $\boldsymbol{C}\setminus\Lambda(F)$. It turns out that many properties of the analytic transform, earlier proved only for very special algebras, can be proved very generally using Banach algebra theory applied to $B/I(F)$.

It is necessary to remark here that the analytic transform was used in the papers mentioned only as a means of investigating the ideal structure in special algebras, it was not studied for its own interest. The main result in these papers concern the ideal structure and they do not generalize as extensively as the transform theory, since they depend very strongly on the special nature of the Banach algebra and on function-theoretic results of limited validity, for instance the lemma of Sjöberg and Levinson ([5], [18], [22]). It is, however, probable that our general theory of the analytic transform can be of value in the continued detailed study of the ideal structure, even in more complicated situations, for instance for the quasi-analytic algebras studied by Geisberg and Konjuhovskii [9] and Vretblad [24].

The contents of this paper are as follows. The analytic transform is defined in Section 2 and the examples in Section 3 connect our definition with those used earlier. Section 4 contains a presentation of some basic properties of the transform. Especially important is Theorem 4.1, which extends the above-mentioned special results on the meromorphicity of the analytic continuation. Section 5 deals with the subspace of bounded linear functionals with rational transform. In Section 6 relations are established between the spectrum of a functional and the set of singularities of its analytic transform. A general result (Theorem 6.3) gives a corollary (Theorem 6.10) which says that if a functional has totally disconnected spectrum, then the spectrum is the natural boundary of the analytic transform of the functional. In Section 7, a representation formula is given, which is convenient to use in order to estimate the modulus of the analytic transform of F in the set $M\setminus\Lambda(F)$. Sections 8 and 9 contain results for more special types of algebras, in Section 8 regular algebras and in Section 9 algebras satisfying the analytic Ditkin condition, introduced by Bennett and Gilbert [1].

Especially in Sections 5 and 6 it would have been possible to make a more thorough use of the existing theory of analytic operator-valued functions, but we have preferred to make the exposition as self-contained as possible. In this context it should be mentioned that the paper [7] contains comparisons between the spectral notion used here and other notions, more closely related to operator theory.

2. Preliminaries. Definition of the analytic transform. B is always assumed to be a commutative Banach algebra with identity e and containing an element a such that rational functions of a with poles outside the spectrum $\mathrm{Sp}(a)$ of a form a dense subspace of B. We present some simple and essentially well-known properties of B.

Let us first observe that B need not be semisimple. This is shown by choosing as B an arbitrary radical algebra with unit, of dimension $\geqslant 2$ and generated by one of its elements. Secondly, the assumptions do not imply that B is singly generated. An example of this is $C(\boldsymbol{T})$. For if we consider $\boldsymbol{T}$ as the unit circle in $\boldsymbol{C}$, we find that the element $z\to z$, $z\in\boldsymbol{T}$, has spectrum $\boldsymbol{T}$, and rational functions of it, with poles outside $\boldsymbol{T}$, give a dense subspace of $C(\boldsymbol{T})$. Hence $B = C(\boldsymbol{T})$ fulfils our conditions. But it is well known and easily proved that two elements are needed in order to generate $C(\boldsymbol{T})$.

Let M denote the maximal ideal space of B. The Gelfand transform of an element $b\in B$ is denoted by $x\to b(x)$, $x\in M$. The range of $x\to b(x)$, $x\in M$, coincides with $\mathrm{Sp}(b)$. If $x, y\in M$ give $a(x) = a(y)$, then $Q(a)(x) = Q(a)(y)$ for all rational Q with poles outside $\mathrm{Sp}(a)$, and thus $b(x) = b(y)$ for every $b\in B$ since the elements $Q(a)$ form a dense subspace of B. Hence $x = y$, which shows that $x\to a(x)$, $x\in M$, is injective. Since $x\to a(x)$ is a continuous bijection from the compact space M to the Hausdorff space $\mathrm{Sp}(a)\subset\boldsymbol{C}$, it is a homeomorphism. Thus we can in the following put $M = \mathrm{Sp}(a)$, $a(z) = z$ for every $z\in M$. For rational functions Q on $\boldsymbol{C}$ with poles outside M, $Q(a)(z) = Q(z)$, for every $z\in M$. The Gelfand transform of an arbitrary element in B is continuous on M. Since it is the uniform limit of rational functions $z\to Q(z)$, $z\in M$, it is holomorphic in the interior M° of M.

For every closed ideal $I\subset B$ its *co-spectrum* $\mathrm{Cosp}(I)$ is the set of all maximal ideals containing it, or, otherwise expressed, the set of common zeros for the Gelfand transforms of its elements. B/I is a Banach algebra with maximal ideal space $\mathrm{Cosp}(I)$ in the sense that the element $b+I$, $b\in B$, has the Gelfand transform

$$z\to b(z),\quad z\in\mathrm{Cosp}(I).$$

B^* is the Banach space of bounded linear functionals on B. To every $F\in B^*$ which annihilates a closed ideal $I\subset B$, it corresponds a bounded linear functional F_I on B/I with

$$\langle b+I,\, F_I\rangle = \langle b, F\rangle,$$

for every $b \in B$. To every given $F \in B^*$ there exists a largest (closed) ideal $I(F)$, annihilated by F. $\Lambda(F) = \mathrm{Cosp}\, I(F)$ is called the *spectrum* of F. Obviously,

(2.1) $$\Lambda(F) = \emptyset \Leftrightarrow F = 0.$$

Definition 2.2. The *analytic transform* $z \to F(z)$ *of a functional* $F \in B^*$ is defined by the relation

(2.3) $$F(z) = \big\langle \big(a - ze + I(F)\big)^{-1}, F_{I(F)} \big\rangle, \quad z \in C \setminus \Lambda_F.$$

Theorem 2.4. (2.3) *is well defined and represents a holomorphic function in* $C \setminus \Lambda(F)$.

Proof. $B/I(F)$ has maximal ideal space $\Lambda(F)$, and if $z_0 \in C \setminus \Lambda(F)$ the Gelfand transform $z \to z - z_0$ of $a - z_0 e + I(F)$ in $B/I(F)$ does not vanish on $\Lambda(F)$. Hence $\big(a - z_0 e + I(F)\big)^{-1}$ is a well-defined element in $B/I(F)$, and $F(z_0)$ is defined. Now $\big(a - ze + I(F)\big)^{-1}$ is moreover a holomorphic function of z, $z \in C \setminus \Lambda(F)$, with values in $B/I(F)$, and hence $z \to F(z)$, $z \in C \setminus \Lambda(F)$ is holomorphic.

Remark. It is important to observe that if I is any closed ideal, contained in $I(F)$, then

(2.5) $$F(z) = \langle (a - ze + I)^{-1}, F_I \rangle,$$

if $z \in C \setminus \mathrm{Cosp}(I)$. In particular, choosing $I = \{0\}$, we obtain

(2.6) $$F(z) = \langle (a - ze)^{-1}, F \rangle,$$

if $z \in C \setminus M$.

3. Examples. The following examples are chosen in order to show the connection between our definition of the analytic transform and those used earlier.

Example 3.1. $M = T$. Let $F \in B^*$. The normal situation is that $\Lambda(F) = T$. Then $z \to F(z)$ is defined and holomorphic in the inside and in the outside of $T \subset C$. But if $\Lambda(F) \neq T$, i.e. if $I(F)$ contains an element b with $b(z_0) \neq 0$ for some $z_0 \in T$, then, by the continuity of $z \to b(z)$, $C \setminus \Lambda(F)$ contains some arc of T. The function in the outside is the continuation of the function in the inside over every such arc.

It follows from Definition 2.2 (or from (2.6)), that $F(z) \to 0$ as $z \to \infty$. Thus we have representations of the form

$$F(z) = \begin{cases} \sum_{1}^{\infty} F_n z^{n-1}, & \text{for} \quad |z| < 1, \\ \sum_{-\infty}^{0} F_n z^{n-1}, & \text{for} \quad |z| > 1. \end{cases}$$

It is easy to show from (2.6) that

$$F_n = \langle a^{-n}, F \rangle,$$

for every n. Hence the analytic transform in this case agrees essentially with the Fourier–Carleman transform, used for instance in [1], [8], [15], [16], [21], [23].

We specialize still further. Let p_n, $n \in Z$, form a positive submultiplicative sequence with $p_n \geqslant 1$, $n \in Z$, and satisfying $\log p_n = o(n)$, as $|n| \to \infty$. We define B as the Banach space of functions b defined by

$$z \to \sum_{-\infty}^{\infty} b_n z^n, \quad z \in T \subset C,$$

where b_n are complex numbers and

$$\|b\|_B = \sum_{-\infty}^{\infty} p_n |b_n| < \infty.$$

B is a Banach algebra under pointwise multiplication, and it is well known from the early Gelfand theory that its maximal ideal space is T in the sense that the Gelfand transform of every $b \in B$ is b itself. Taking a as the element $z \to z$, $z \in T$, we have $\mathrm{Sp}(a) = T$, and rational functions of a with poles only in 0 and ∞ form a dense subspace of B. Hence B satisfies all our conditions. It is interesting to observe that we obtain the analytic continuation between inside and outside as $\Lambda(F) \neq T$, without needing to assume

$$\sum_{-\infty}^{\infty} (1 + n^2)^{-1} \log p_n < \infty,$$

a condition which cannot be avoided if the continuation is proved by function-theoretic methods of the type used in the earlier mentioned investigations.

Example 3.2. The Banach algebra $A(R)$ of absolutely convergent Fourier integrals does not directly fit into our theory since it has no unit. We shall show how this difficulty is overcome, by simply adjoining a unit to the algebra.

We consider R as the real axis of a complex plane, where the points are denoted by w. Rational functions, regular on the extended R and vanishing at infinity, belong to $A(R)$ and form a dense subspace of it. By adding a unit e to the algebra, and mapping the w-plane into a z-plane by the formula $z = (w - i)(w + i)^{-1}$, we obtain a function algebra on $T \subset C$ with maximal ideal space T, and such that rational functions of a: $z \to z$, with poles outside T, are dense in it. Hence the enlarged algebra is of type B.

Let F be a bounded linear functional on $A(\boldsymbol{R})$. It can be extended to a bounded linear functional on the enlarged algebra by defining $\langle e, F\rangle = 0$. The analytic transform of F is for $z \notin \boldsymbol{T}$ defined by the formula

$$F(z) = \langle (a-ze)^{-1}, F\rangle .$$

Mapping back to the w-plane, we obtain a function $w \to G(w)$, defined by

$$G(w_0) = F\big((w_0-i)(w_0+i)^{-1}\big) = \langle (w_0+i)\big((w_0+i)a-(w_0-i)e\big)^{-1}, F\rangle ,$$

$w_0 \notin \boldsymbol{R}$. Here a is the element $w \to (w-i)(w+i)^{-1}$ in the enlarged algebra. Thus $G(w_0) = \langle b_{w_0}, F\rangle$, where b_{w_0} is the element

$$w \to (w_0+i)\left(\frac{1}{2i}+\frac{w_0+i}{2i(w-w_0)}\right),$$

$w \in \boldsymbol{R}$, in the enlarged algebra. But $\langle e, F\rangle = 0$, which gives

$$G(w_0) = \frac{(w_0+i)^2}{2i}\langle c_{w_0}, F\rangle , \tag{3.3}$$

$w_0 \notin \boldsymbol{R}$, where c_{w_0} is the element $w \to (w-w_0)^{-1}$ in $A(\boldsymbol{R})$.

For every $b \in A(\boldsymbol{R})$, we write

$$b(w) = \int_{-\infty}^{\infty} \hat{b}(t)e^{-itw}dt, \qquad w \in \boldsymbol{R},$$

where $\hat{b} \in L^1(\boldsymbol{R})$, and observe that a bounded linear functional F on $A(\boldsymbol{R})$ can be represented by a function $\varphi \in L^\infty(\boldsymbol{R})$ in the sense that

$$\langle b, F\rangle = \int_{-\infty}^{\infty} \varphi(t)\hat{b}(-t)\,dt.$$

Since, for $w \in \boldsymbol{R}$,

$$c_{w_0}(w) = \begin{cases} i\displaystyle\int_0^\infty e^{itw_0-itw}dt, & \text{if} \quad \operatorname{Im}(w_0) > 0,\\ -i\displaystyle\int_{-\infty}^0 e^{itw_0-itw}dt, & \text{if} \quad \operatorname{Im}(w_0) < 0,\end{cases}$$

we see that

$$\langle c_{w_0}, F\rangle = \begin{cases} i\displaystyle\int_{-\infty}^0 \varphi(t)e^{-itw_0}dt, & \text{if} \quad \operatorname{Im}(w_0) > 0,\\ -i\displaystyle\int_0^\infty \varphi(t)e^{-itw_0}dt, & \text{if} \quad \operatorname{Im}(w_0) < 0.\end{cases} \tag{3.4}$$

(3.3) and (3.4) show that $w_0 \to G(w_0)$ apart from an unessential factor coincides with the Carleman transform of φ, as defined in [3].

Similar results hold for more general function algebras on $\boldsymbol{R}$, for instance for the algebra of Fourier transforms b of functions $\hat{b}$ with $\hat{b}p \in L^1(\boldsymbol{R})$ and

$$\|b\| = \int_{-\infty}^{\infty} |\hat{b}(t)|p(t)\,dt,$$

where the weight function p satisfies $p(x) \geqslant 1$, $x \in \boldsymbol{R}$, is continuous, sub-multiplicative and satisfies

$$\log p(t) = o(t), \quad \text{as} \quad |t| \to \infty .$$

Hence the analytic continuation of the Carleman transform for such algebras follows from our general definition, and there is no need to assume that

$$\int_{-\infty}^{\infty} \frac{\log p(t)}{1+t^2}\,dt < \infty .$$

This is of interest for the type of algebras studied in [2], [3], [9], [20], [21], [24].

Example 3.5. We start by considering the case where M is the closed unit disc D. Normally, $\Lambda(F) = D$, and in that case $z \to F(z)$ is only defined outside D. But if $\Lambda(F) \neq D$, $\Lambda(F)$ is contained in the set of zeros of some not identically vanishing function $z \to b(z)$, $b \in B$. This function is continuous in D, holomorphic in D°. Thus, $\Lambda(F)$ is a totally disconnected set, and all points in $\Lambda(F) \cap D^\circ$ are isolated. Similar results hold if M is an annulus or a more general subset of $\boldsymbol{C}$ such that M° is connected and the boundary of M consists of sufficiently well-behaved curves.

By Möbius transformations we see, as in Example 3.2, that the discussion applies as well to certain Banach algebras without unit and with M as, for instance, a half-plane or a parallel strip. All these cases cover the variants of the Carleman transform discussed in [1], [8], [11]–[17], [21], [23], and our definition gives directly the analytic continuation, which was proved in these papers by more limited methods.

4. The mapping $(F, b) \to Fb$ **and its properties.** For every $F \in B^*$ and $b \in B$, Fb denotes the functional defined by the relation

$$\langle c, Fb\rangle = \langle bc, F\rangle ,$$

$c \in B$. Obviously, $Fb \in B^*$ and

$$\|Fb\|_{B^*} \leqslant \|F\|_{B^*}\|b\|_B ,$$

assuming that the norm in B is defined so that it is sub-multiplicative. The mapping $(F, b) \to Fb$ determines B^* as a module over B. Evidently,

$$I(F) \subset I(Fb), \quad \Lambda(Fb) \subset \Lambda(F), \tag{4.1}$$

for every choice of $F\in B^*$, $b\in B$. Another evident relation is

$$Fb = 0 \Leftrightarrow b\in I(F), \tag{4.2}$$

$F\in B^*$, $b\in B$.

In the remainder of this section we assume that F is a fixed element of B^*. We shall first prove four lemmas which describe some simple properties of $\Lambda(F)$ and the mappings $(F, b)\to Fb$ for various types of $b\in B$. We need the following definition.

DEFINITION 4.3. $U(F)$ is the union of all those components of M° which are not included in $\Lambda(F)$.

LEMMA 4.4. *All points in* $U(F)\cap\Lambda(F)$ *are isolated.*

Proof. For every component V of $U(F)$, there exists a $b\in I(F)$ such that $z\to b(z)$ does not vanish identically in V. $V\cap\Lambda(F)$ is included in the set of zeros of the holomorphic function $z\to b(z)$, $z\in V$. Hence all points in $V\cap\Lambda(F)$ are isolated, and from this the assertion follows.

LEMMA 4.5. *For every choice of* $c\in C\setminus M$ *and* $z\in C\setminus\Lambda(F)$,

$$(z-c)F(a-ce)^{-1}(z) = F(z)-F(c). \tag{4.6}$$

Proof. $(a-ce)^{-1}$ exists as an element in B, and by (4.1) the spectrum of $F(a-ce)^{-1}$ is included in $\Lambda(F)$. Hence the left-hand member is defined. Using (2.5), we obtain with $b_0\in B$ chosen as an arbitrary element in $\big(a-ze+I(F)\big)^{-1}$,

$$\begin{aligned}(z-c)F(a-ce)^{-1}(z) &= (z-c)\,\big\langle\big(a-ze+I(F)\big)^{-1},\ \big(F(a-ce)^{-1}\big)_{I(F)}\big\rangle\\ &= (z-c)\,\langle b_0,\ F(a-ce)^{-1}\rangle = (z-c)\,\langle(a-ce)^{-1}b_0,\ F\rangle\\ &= (z-c)\,\big\langle\big(a-ce+I(F)\big)^{-1}\big(a-ze+I(F)\big)^{-1},\ F_{I(F)}\big\rangle\\ &= \big\langle\big(a-ze+I(F)\big)^{-1},\ F_{I(F)}\big\rangle-\big\langle\big(a-ce+I(F)\big)^{-1}\ F_{I(F)}\big\rangle\\ &= F(z)-F(c).\end{aligned}$$

LEMMA 4.7. *For every rational function Q with poles outside M we form the rational function R, which is completely determined by the conditions that it is holomorphic on M and such that $F(z)Q(z)+R(z)$ has only removable singularities on $C\setminus M$ and tends to 0 as $z\to\infty$. Then*

$$FQ(a)(z) = F(z)Q(z)+R(z) \tag{4.8}$$

holds when $z\in C\setminus\Lambda(F)$, *except at the poles of* Q.

Proof. Dividing (4.6) by $(z-c)$, we obtain (4.8) in the case where Q has only one pole, of order 1, and not placed at ∞. Taking linear combinations of such relations (4.8), we obtain (4.8) in the case where Q has arbitrarily many poles, all of them of order 1 and not placed at ∞. Observing the inequality

$$\begin{aligned}|Fb(z)| &= \big|\big\langle\big(a-ze+I(F)\big)^{-1},\ Fb\big\rangle\big|\\ &\leqslant \|Fb\|_{B^*}\big\|\big(a-ze+I(F)\big)^{-1}\big\|_{B/I(F)}\\ &\leqslant \|F\|_{B^*}\|b\|_B\big\|\big(a-ze+I(F)\big)^{-1}\big\|_{B/I(F)},\end{aligned} \tag{4.9}$$

for $z\in C\setminus\Lambda(F)$, $b\in B$, we obtain the general case of (4.8) by a simple passage to the limit.

LEMMA 4.10. *For every $b\in B$ there exists a function $z\to f(z)$, defined and continuous in $\big(M\setminus\Lambda(F)\big)\cup U(F)$, holomorphic in $U(F)$ and such that*

$$Fb(z)-F(z)b(z) = f(z), \tag{4.11}$$

for $z\in M\setminus\Lambda(F)$.

Proof. Here we use the existence of a sequence Q_n, $n\in \boldsymbol{Z}_+$, of rational functions with poles outside M and such that $Q_n(a)\to b$ as $n\to\infty$. By Lemma 4.7 there exist rational functions R_n, $n\in\boldsymbol{Z}_+$, with poles outside M, and such that

$$FQ_n(a)(z)-F(z)Q_n(z) = R_n(z),$$

for $n\in\boldsymbol{Z}_+$, $z\in M\setminus\Lambda_F$. By (4.9), $FQ_n(a)(z)$ converges to $Fb(z)$, uniformly on compact subsets of $C\setminus\Lambda_F$, as $n\to\infty$, since

$$\big\|\big(a-ze+I(F)\big)^{-1}\big\|_{B/I(F)}$$

is bounded on such sets. Furthermore, $Q_n(z)\to b(z)$ uniformly on M, as $n\to\infty$. Hence $R_n(z)$ converges, as $n\to\infty$, uniformly on compact subsets of $M\setminus\Lambda(F)$, and if f denotes its limit function, (4.11) holds in $M\setminus\Lambda(F)$. By Lemma 4.4, all points in

$$\Big(\big(M\setminus\Lambda(F)\big)\cup U(F)\Big)\setminus\big(M\setminus\Lambda(F)\big) = U(F)\cap\Lambda(F)$$

are isolated. Let z_0 be an arbitrary point in this set. f is defined and holomorphic in a deleted open neighborhood of z_0, and it remains to prove that z_0 is a removable singularity for f. Take any circle around z_0 situated in this neighborhood, and such that the whole circle disc is contained in M°. On the circumference, the functions R_n, $n\in\boldsymbol{Z}_+$, are uniformly bounded and they are holomorphic in the disc since all singularities of the functions R_n lie outside M. By the maximum principle the functions R_n are uniformly bounded in the circle disc. Hence f is bounded in the deleted disc, and z_0 is a removable singularity for f.

We end this section with the following theorem, previously known only in special cases.

THEOREM 4.12. *For every $F\in B^*$, $z\to F(z)$ in $U(F)\setminus\Lambda(F)$ is the restriction of a function, meromorphic in $U(F)$.*

Proof. Take any component V of $U(F)$. By Definition 4.3 there exists a $b \in I(F)$ such that $z \to b(z)$ does not vanish identically in V. By (4.2) we have $Fb = 0$, and hence by Lemma 4.10

$$F(z)b(z) = -f(z), \tag{4.13}$$

for $z \in V \setminus \Lambda(F)$. $z \to f(z)$ is holomorphic in V by Lemma 4.10, and since $z \to b(z)$ is holomorphic and not identically vanishing in V, the theorem is proved.

5. Elements in B^* with rational analytic transform. We shall investigate the properties of elements $F \in B^*$ for which the analytic transform is the restriction to $C \setminus \Lambda(F)$ of a rational function. We start with the following special theorem.

THEOREM 5.1. *Let $F \in B^*$. $F = 0$ implies that $F(z) = 0$ on C. $F(z) = 0$ on $C \setminus M$ implies that $F = 0$.*

Proof. The first part is trivial. Using a simple passage to the limit, $F(z) = 0$ on $C \setminus M$ implies that $\langle Q(a), F\rangle = 0$ for every rational function Q with poles outside M. Since the corresponding elements $Q(a)$ form a dense subspace of B, $F = 0$.

For every $F \in B^*$ we introduce the notation

$$L(F) = \{Fb \mid b \in B\}. \tag{5.2}$$

LEMMA 5.3. *$I(F)$ is the annihilator of $L(F)$.*

Proof. $c \in I(F)$ is equivalent to $\langle bc, F\rangle = 0$ for all $b \in B$. c annihilates $L(F)$ is equivalent to $\langle c, Fb\rangle = 0$ for all $b \in B$. Since $\langle bc, F\rangle = \langle c, Fb\rangle$, our lemma follows.

The following theorem is basic for the continued investigations in this section.

THEOREM 5.4. *For every $F \in B^*$ the following statements are equivalent:*

(1) *$L(F)$ is finite-dimensional.*

(2) *$I(F)$ has finite co-dimension.*

(3) *There exists a not identically vanishing polynomial P such that $I(F)$ is the smallest closed ideal containing $P(a)$.*

(4) *There exist points $z_i \in M$, $i = 1, 2, \ldots, m$, non-negative integers n_i, $i = 1, 2, \ldots, m$, and complex constants C_{ij}, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots \ldots, n_i$, such that*

$$\langle Q(a), F\rangle = \sum_{i=1}^{m} \sum_{j=0}^{n_i} C_{ij} Q^{(j)}(z_i), \tag{5.5}$$

for all rational functions Q with poles outside M.

(5) *$z \to F(z)$ coincides on $C \setminus M$ with a rational function.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 5.3.

(2) $\Rightarrow$ (3). If (2) holds and n is the co-dimension of $I(F)$ then

$$e + I(F),\quad a + I(F),\quad a^2 + I(F),\ \ldots,\quad a^n + I(F)$$

are linearly dependent in $B/I(F)$. Thus there exists a polynomial P, not identically 0 and of degree $\leqslant n$, such that $P(a) \in I(F)$. Let B_P denote the closure in B of the subspace of all $P(a)Q(a)$, where Q is rational with poles outside M. B_P together with $e, a, a^2, \ldots, a^{n-1}$ spans B, and hence B_P has co-dimension $\leqslant n$. But B_P is contained in $I(F)$, and hence $B_P = I(F)$. Obviously, B_P is contained in every closed ideal containing $P(a)$, and hence (3) is proved.

(3) $\Rightarrow$ (4). We start from (3), where we assume that P is chosen with minimal degree. Put

$$P(z) = \prod_{i=1}^{m} (z - z_i)^{n_i+1},$$

where the points z_i are all different and $n_i \geqslant 0$. Then every $z_i \in M$, for otherwise $P(z)(z - z_i)^{-1}$ would be a polynomial of lower degree, still satisfying (3). The elements of type $P(a)Q(a)$, Q rational with poles outside M, are contained in $I(F)$ and from this it is easily proved that $\langle Q(a), F\rangle$, Q rational with poles outside M, does only depend (linearly) of $Q^{(j)}(z_i)$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n_i$. Hence we have a formula of type (5.5)

(4) $\Rightarrow$ (5). (5.5) gives with $Q(a) = (a - ze)^{-1}$, $z \in C \setminus M$,

$$F(z) = \langle (a - ze)^{-1}, F\rangle = \sum_{i=1}^{m} \sum_{j=0}^{n_i} C_{i,j} \frac{j!}{(z_i - z)^{j+1}}.$$

(5) $\Rightarrow$ (1). We start from (5), and denote by Q_0 the rational function on C which coincides with $F(z)$ on $C \setminus M$. For every rational Q with poles outside M, it follows from Lemma 4.7 that we can find a rational R with poles outside M such that

$$FQ(a)(z) = Q_0(z)Q(z) + R(z),$$

for $z \in C \setminus M$. The right-hand member is a rational function, and it follows from the properties of R, stated in Lemma 4.7, that all poles of the right-hand member are at the same time poles for Q_0 of not lower degree. This implies that the functions $z \to FQ(a)(z)$, $z \in C \setminus M$ form a finite-dimensional linear space. Hence, by Theorem 5.1, the elements $F(Qa)$ form a finite-dimensional subspace of B^*. But this subspace is dense in $L(F)$, and hence L_F is finite-dimensional.

By Theorem 5.1 there exists at most one $F \in B^*$, satisfying (5.5) for given z_i and C_{ij}. Hence the following definition determines a unique functional.

DEFINITION 5.6. For $z \in M$ and $j \in \boldsymbol{N}$, $D_{z,j}$ denotes the functional in B^* determined by

$$\langle Q(a), D_{z,j}\rangle = Q^{(j)}(z),$$

for rational Q with poles outside M if such a functional exists *in* B^*.

By taking $\langle b, D_{z,0}\rangle = b(z)$, for every $b \in B$, we see that $D_{z,0}$ exists in B^* for every $z \in M$. If $j > 0$, we can assure the existence of $D_{z,j}$ if $z \in M^\circ$, namely by defining

$$\langle b, D_{z,j}\rangle = b^{(j)}(z).$$

The analyticity of $z \to b(z)$ in M° and the inequality $|b(z)| \leqslant \|b\|_B$ show that this is a bounded linear functional. It should be observed that $D_{z_0,j}$ may exist for special algebras B even at points z_0 where derivatives of $z \to b(z)$ are not defined, for instance at isolated points of M.

In Theorem 5.9 we shall show that the functionals $D_{z,j}$ in B^* can be used to represent functionals, satisfying the equivalent conditions in Theorem 5.4. To prove Theorem 5.9 we need the following lemma.

LEMMA 5.7. *For every $z_0 \in M$, $j \in \boldsymbol{N}$, every $D_{z_0,j} \in B^*$ has $\{z_0\}$ as spectrum, and*

$$D_{z_0,j}(z) = \frac{j!}{(z_0 - z)^{j+1}}, \tag{5.8}$$

if $z \in \boldsymbol{C} \setminus \{z_0\}$.

Proof. $D_{z_0,j}$ annihilates the closed ideal generated by $(a - z_0)^{j+1}$, and z_0 is the only common zero of the Gelfand transforms of the elements in this ideal. Hence the spectrum is included in $\{z_0\}$, By (2.1) it coincides with $\{z_0\}$.

Hence $D_{z_0,j}(z)$ is holomorphic outside $\{z_0\}$. (5.8) holds for $z \in \boldsymbol{C} \setminus M$ by (2.6) and Definition 5.6. By analytic continuation, (5.8) holds for every $z \in \boldsymbol{C} \setminus \{z_0\}$.

THEOREM 5.9. *Let $F \in B^*$ satisfy Theorem 5.4, (4) with $C_{i,n_i} \neq 0$, $i = 1, 2, \ldots, m$. Then $D_{z_i,j}$ exists in B^* and belongs to $L(F)$ for every $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n_i$. We have the representation*

$$F = \sum_{i=1}^{m} \sum_{j=0}^{n_i} C_{ij} D_{z_i,j}. \tag{5.10}$$

Furthermore,

$$\Lambda(F) = \{z_1, z_2, \ldots, z_m\}, \tag{5.11}$$

and

$$F(z) = \sum_{i=1}^{m} \sum_{j=0}^{n_i} C_{ij} \frac{j!}{(z_i - z)^{j+1}}, \tag{5.12}$$

if $z \in \boldsymbol{C} \setminus \Lambda(F)$. $L(F)$ coincides with the span of $D_{z_i,j}$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n_i$. The dimension of L_F is

$$\sum_{i=1}^{m} (n_i + 1).$$

Proof. Forming $FP(a)$ for convenient polynomials P, we find that $D_{z_i,j} \in L_F \subset B^*$ to the extent stated in the theorem, and that all functionals $FQ(a)$, Q rational with poles outside M, are contained in their span. Taking limits we find that $L(F)$ is the linear span of these functionals $D_{z_i,j}$. The representation (5.10) is obvious. By Lemma 5.7 we find that $\Lambda(F)$ is included in $\{z_1, z_2, \ldots, z_n\}$. Conversely, $D_{z_i,0} \in L(F)$ implies that $z_i \in \Lambda(F)$ due to (4.1), hence (5.11) holds. (5.12) is a direct consequence of (5.10) and Lemma 5.7. The assertion on the dimension of $L(F)$ follows from the linear independence of all $D_{z_i,j}$, which in its turn follows from the linear independence of their analytic transforms.

Theorems 5.4 and 5.9 give various characterizations of the functionals $F \in B^*$ such that $L(F)$ is finite-dimensional. Turning to the dual space B, we can easily use these results to obtain the structure of the closed ideals in B of finite co-dimension. It should, however, be observed that it is not always trivial to find out whether a given closed ideal has finite co-dimension or not. We state here a theorem which is proved and applied to classical Banach algebras in [7].

THEOREM 5.13. *Let $\{z_1, \ldots, z_m\} \subset M$, and suppose that $D_{z_i,n_i} \in B^*$, where $n_i \geqslant 1$, $i = 1, 2, \ldots, m$, and that*

$$\|P(a)bc\|_B \leqslant C\|P(a)b\|_B\|P(a)c\|_B,$$

for some $C > 0$ and every $b, c \in B$, where

$$P(a) = \prod_{i=1}^{m} (a - z_i e)^{n_i}.$$

Let I be a closed ideal of B, only contained in the maximal ideals corresponding to $z_1, z_2, \ldots, z_m$. If, for every i, I contains an element b_i with

$$\langle b_i, D_{z_i,n_i}\rangle \neq 0,$$

I has finite co-dimension.

6. Relations between $\Lambda(F)$ and the set of singularities of the analytic transform. In this section we assume that $F \in B^*$ and that $E \subset \Lambda(F)$ with $E \neq \emptyset$ and both E and $\Lambda(F) \setminus E$ compact. Our main result is Theorem 6.3. In order to formulate this theorem we need two definitions.

DEFINITION 6.1. $z \to F(z)$, $z \in \boldsymbol{C} \setminus \Lambda(F)$, is said to be *meromorphic* (*holomorphic*) *at* E if there exists a meromorphic (holomorphic) function $z \to G(z)$ in a neighborhood N of E such that $F(z) = G(z)$ for $z \in (\boldsymbol{C} \setminus \Lambda(F)) \cap N$.

The subspace $L(F)$ of B^* was defined in (5.2). $\overline{L(F)}$ denotes its weak* closure.

DEFINITION 6.2. $L(F, E)$ is the subspace in B^* of all $G \in \overline{L(F)}$ with $\Lambda(G) \subset E$.

It should be remarked that the dimension of $L(F, E)$ is positive, since E is non-empty and $L(F, E)$ contains the functionals $D_{z,0}$, $z \in E$.

THEOREM 6.3. *$L(F, E)$ is finite-dimensional if and only if $z \to F(z)$ is meromorphic at E. If $L(F, E)$ is finite-dimensional, then E is finite, $z \to F(z)$ has a pole at every point of E, and the dimension of $L(F, E)$ equals the sum of the pole orders at E. In particular, $z \to F(z)$ is never holomorphic at E.*

Proof. By a theorem of Shilov ([10], § 14), applied to $B/I(F)$, or, even simpler, by the elementary theory of Banach space valued analytic functions, there exists an element $b_0 \in B$ such that

$$b_0(z) = \begin{cases} 1 & \text{on } E, \\ 0 & \text{on } \Lambda(F) \setminus E, \end{cases}$$

while $b_0(e - b_0) \in I(F)$. We shall show that

$$L(F, E) = \overline{L(Fb_0)}. \tag{6.4}$$

By (4.2), $Fb_0(e - b_0) = 0$, which shows that Fb_0 annihilates the closed ideal I in B, generated by $e - b_0$. Hence the whole subspace $\overline{L(Fb_0)}$ annihilates I. Since $(e - b_0)(z) = 1 \neq 0$, for $z \in \Lambda(F) \setminus E$, all elements $G \in \overline{L(Fb_0)}$ have $\Lambda(G) \subset E$, and hence

$$\overline{L(Fb_0)} \subset L(F, E).$$

To prove the opposite inclusion we take an arbitrary $G \in L(F, E)$. $\Lambda(G - Gb_0)$ is contained in E, and since G annihilates $I(F)$, we have

$$(G - Gb_0)b_0 = G(e - b_0)b_0 = 0.$$

Since $b_0(z) \neq 0$ for $z \in E$, no point in E belongs to the spectrum of $G - Gb_0$, and thus, by (2.1), $G = Gb_0$. Let N be any neighborhood of G in the weak* topology of B^*. The mapping $H \to Hb_0$, $H \in B^*$, is continuous in this topology, and hence we can find a neighborhood N_0 of $G = Gb_0$ which is mapped into N. Since $G \in \overline{L(F)}$, N_0 contains an element Fb, $b \in B$, and thus N contains the element $Fbb_0 = Fb_0 b$, $b \in B$. This proves $G \in \overline{L(Fb_0)}$, and hence (6.4) holds.

(6.4) implies that Fb_0 itself is an element with its spectrum included in E. Changing the rôle of E and $\Lambda(F) \setminus E$ in the discussion above, we find similarly that $F(e - b_0)$ has spectrum included in $\Lambda(F) \setminus E$. But

$$F = Fb_0 + F(e - b_0),$$

and this yields

$$\Lambda(F) \subset \Lambda(Fb_0) + \Lambda\big(F(e - b_0)\big),$$

and hence

$$\Lambda(Fb_0) = E, \tag{6.5}$$

$$\Lambda\big(F(e - b_0)\big) = \Lambda(F) \setminus E. \tag{6.6}$$

We shall now prove the theorem, using the identity

$$F(z) = Fb_0(z) + F(e - b_0)(z), \quad z \in C \setminus \Lambda(F). \tag{6.7}$$

Let us first assume that $z \to F(z)$, $z \in C \setminus \Lambda(F)$, is meromorphic at E. We keep the notation F for the continuation of the analytic transform into a neighborhood N of E. We can assume that N is disjoint from $\Lambda(F) \setminus E$. By (6.6),

$$z \to F(e - b_0)(z), \quad z \in N,$$

is holomorphic, and hence

$$z \to F(z) - F(e - b_0)(z), \quad z \in N, \tag{6.8}$$

is meromorphic. By (6.7) and the inclusion $N \cap (C \setminus E) \subset C \setminus \Lambda_F$, (6.8) and the holomorphic function

$$z \to Fb_0(z), \quad z \in C \setminus E, \tag{6.9}$$

coincide in their common set of definition. Hence they can be combined to a meromorphic function in $N \cup (C \setminus E) = C$. This function is therefore rational, thus (6.9) is rational. By Theorem 5.4, $L(Fb_0)$ is finite-dimensional, hence by (6.4) $L(F, E)$ is finite-dimentional. Applying Theorems 5.4 and 5.9 to $L(Fb_0)$ one finds that E is finite and that the dimension of $L(F, E)$ equals the sum of the pole orders of $z \to F(z)$ at E.

It remains to prove that finite-dimensional $L(F, E)$ implies that $z \to F(z)$ is meromorphic at E. By (6.4), $L(F, E)$ finite-dimensional implies that $L(Fb_0)$ is finite-dimensional, thus by Theorem 5.4 and Theorem 5.9 the function (6.9) is rational. Hence (6.7) proves that $z \to F(z)$ is meromorphic at E.

A corollary of Theorem 6.3 is the following theorem.

THEOREM 6.10. *If $\Lambda(F)$ is totally disconnected, then $\Lambda(F)$ is the natural boundary of $z \to F(z)$.*

Proof. Since $\Lambda(F)$ is compact, the assumption implies that every $z_0 \in \Lambda_F$ has an arbitrarily small open neighborhood N such that $\Lambda_F \cap N$ is closed. By Theorem 6.3, $z \to F(z)$ cannot be continued to a holomorphic function in N, and the theorem follows.

Theorem 6.10 should be compared to the corollary of Theorem 3 in Lindahl [19]. Our results in this section are related to his and the proof of our Theorem 6.3 is similar to the proof of his Theorem 3.

Theorem 6.3 can be applied for instance to the type of algebras, discussed in Example 3.5. Theorems 6.3 and 6.8 extend results which have been proved before only in very special cases and using special methods. One such earlier result is given by Theorem 2 in Nyman [21]. His theorem and its proof are reproduced in Gurarii [14].

7. Estimates for $|F(z)|$ in $M\setminus\Lambda(F)$. As mentioned in Section 1, estimates of $|F(z)|$ in $C\setminus\Lambda(F)$ play an important rôle in the investigations of the ideal structure of B. If $z\in C\setminus M$ we have by (2.6)

$$|F(z)|\leqslant\|F\|_{B^*}\|(a-ze)^{-1}\|_B,$$

and it is often easy to determine the behavior of the right-hand member of this inequality, as z approaches the boundary of M. If $z\in M\setminus\Lambda(F)$, (2.3) gives

$$|F(z)|\leqslant\|F\|_{B^*}\left\|\left(a-ze+I(F)\right)^{-1}\right\|_{B/I(F)},$$

but this is in general more difficult to use, due to the less explicit definition of the norm in $B/I(F)$. A more suitable way to obtain the desired estimates when $z\in M\setminus\Lambda(F)$ is to look for estimates of the norm of the function f in Lemma 4.10. Choosing b in that lemma so that $Fb=0$, we obtain then estimates for $|F(x)|$ when $z\in M\setminus\Lambda(F)$. First we prove a representation theorem for f in (4.11).

THEOREM 7.1. *Let $F\in B^*$, $z_0\in M\setminus\Lambda(F)$. Then B^* contains an element G_{z_0} with*

$$\Lambda(F)\subset\Lambda(G_{z_0})\subset\Lambda(F)\cup\{z_0\},\tag{7.2}$$

and

$$G_{z_0}(z)=F(z)(z-z_0)^{-1},\tag{7.3}$$

$z\in C\setminus\left(\Lambda(F)\cup\{z_0\}\right)$. For every $b\in B$, the function f in Lemma 4.10 satisfies

$$f(z_0)=\langle b,G_{z_0}\rangle.\tag{7.4}$$

Proof. $a-z_0e+I(F)$ has a Gelfand transform $z\to z-z_0$, which does not vanish on the maximal ideal space $\Lambda(F)$ of $B/I(F)$. Hence $\left(a-z_0e+I(F)\right)^{-1}$ exists and can be represented by $d+I(F)$, where $d\in B$. Define $H=Fd$. Obviously, $\Lambda(H)\subset\Lambda(F)$. Since $d(a-z_0e)-e\in I(F)$, we have

$$H(a-z_0e)=Fd(a-z_0e)=F,$$

and Lemma 4.7 gives

$$F(z)=H(z)(z-z_0)+C,$$

where C is a constant. Thus $H+CD_{z_0,0}=F_{z_0}$ is a functional, satisfying (7.3). The relation (7.2) is easy to see, its proof is omitted.

In order to prove (7.4), let us first observe that it follows from (4.11) that $f(z_0)$ is a linear functional of b, and an application of (4.9) shows that it is continuous. Hence it suffices to prove (7.4) for $b=(a-ce)^{-1}$, where $c\notin M$. We have then

$$\langle b,G_{z_0}\rangle=\langle(a-ce)^{-1},G_{z_0}\rangle=G_{z_0}(c)=F(c)(c-z_0)^{-1}.$$

By (4.11) and Lemma 4.5

$$\begin{aligned}f(z_0)&=Fb(z_0)-F(z_0)b(z_0)\\&=\left(F(z_0)-F(c)\right)(z_0-c)^{-1}-F(z_0)(z_0-c)^{-1}=F(c)(c-z_0)^{-1},\end{aligned}$$

and (7.4) is proved.

Remark. It is easy to prove that G_{z_0} with the properties (7.2) and (7.3) exists when z_0 is isolated in $\Lambda(F)$ with $F(z)$ having a pole of order p at z_0, assuming that $D_{z_0,p}\in B^*$.

A direct consequence of Theorem 7.1 is the following

THEOREM 7.5. *Let $F\in B^*$, $z_0\in M\setminus\Lambda(F)$, and suppose that $b\in I(F)$ is chosen with $b(z_0)\neq 0$. Then*

$$|F(z_0)|\leqslant\frac{\|G_{z_0}\|_{B^*}\|b\|_B}{|b(z_0)|}.$$

Proof. Since $Fb=0$, Lemma 4.10 gives

$$|F(z_0)|\,|b(z_0)|=|f(z_0)|=|\langle b,G_{z_0}\rangle|\leqslant\|G_{z_0}\|_{B^*}\|b\|_B.$$

EXAMPLE 7.6. Let p_n, $n\in N$, be a submultiplicative positive sequence satisfying $p_n\geqslant 1$, $n\in N$, and $n^{-1}\log p_n\to 0$, as $n\to\infty$. B is the Banach algebra of all functions $b\colon z\to\sum_0^\infty b_nz^n$ on the unit disc D, with

$$\|b\|_B=\sum_0^\infty|b_n|p_n<\infty.$$

It is well known that this algebra is of the type discussed in Example 3.5 with $M=D$. If p_n, $n\in N$, is monotonically increasing, elementary calculations show that

$$\|G_{z_0}\|_{B^*}\leqslant(1-|z_0|)^{-1}\|F\|_{B^*}\tag{7.7}$$

for every $z_0\in D^\circ\setminus\Lambda(F)$. In cases like this, the representation in Theorem 7.1 has been used earlier, both to define the analytic transform in $M\setminus\Lambda(F)$ and to obtain the estimate of $|F(z)|$ in this set. The reason why this has worked without using $B/I(F)$, is that only such cases have been studied, where the existence of $G_{z_0}\in B^*$ and the analytic continuation between $C\setminus M$ and $M\setminus\Lambda(F)$ have been provable by special methods.

One interesting consequence of the estimate (7.7) in Example 7.6 is the following. It is easy to find an element $F\in B^*$ such that $F(z)$ can be continued to a function, regular except at $z=1$, while F is not of bounded characteristic in D. Then there exists no bounded holomorphic function $b(z)\not\equiv 0$ in D, such that $F(z)b(z)$ is of bounded characteristic, and hence no relation

$$|F(z_0)|\leqslant\frac{\|F\|_{B^*}\|b\|_B}{(1-|z_0|)\,|b(z_0)|},$$

$b\in I(F)$, $z_0\in D^0\setminus\Lambda(F)$, can hold unless $b\equiv 0$. By Theorem 7.5 and (7.7) this means that $\Lambda(F)=D$. Thus $\Lambda(F)$ is the whole circle disc, while $F(z)$ has just one singularity, a striking contrast to the result in Theorem 6.10 for the case of totally disconnected $\Lambda(F)$.

8. Regular algebras B. The Banach algebra B is said to be *regular* if it is semisimple and if there exists, for every $z_0\in M$ and every neighborhood N of z_0, an element $b\in B$ such that $z\to b(z)$ vanishes outside N, while $b(z_0)\neq 0$. Obviously, regularity implies that M° is empty. It is not known if regularity implies that the spectrum of every $F\in B^*$ is the natural boundary of the analytic transform, but the following shows that this is true under a certain condition on B, slightly stronger than the condition of regularity.

THEOREM 8.1. *Suppose B is semi-simple and that there exists, for every $z_0\in M$ and every neighborhood N of z_0, a $b\in B$ with $b(z)=0$ outside N, $b(z_0)\neq 0$, and such that b can be approximated arbitrarily closely in B by elements*

$$c=\int_C (a-ze)^{-1}\,d\mu(z),$$

formed by uniformly bounded complex Borel measures μ, with their support included in $N\setminus M$.

Then, for every $F\in B^*$, $\Lambda(F)$ is the natural boundary of $z\to F(z)$.

Proof. Taking an indirect approach, we assume that $F\in B^*$, $z_0\in\Lambda(F)$ and that $z\to F(z)$ can be continued to a function, holomorphic in some neighborhood N of z_0. Let b be an element of B, of the kind described in the formulation of the theorem. We shall prove that $Fb=0$. Let $(c_n)_1^\infty$, with

$$c_n=\int_C (a-ze)^{-1}\,d\mu_n(z),$$

be a sequence of elements of the kind described in the lemma, and convergent to b.

A consequence of the regularity is that $\Lambda(Fb)\subset N$. Hence $z\to Fb(z)$ is holomorphic outside N. We shall prove that it is holomorphic inside N. For $w\in N\setminus M$ we have by Lemma 4.5

$$Fc_n(w)=\Big\langle (a-we)^{-1},\ F\int_C (a-ze)^{-1}d\mu_n(z)\Big\rangle$$

$$=\int_C d\mu_n(z)\langle (a-we)^{-1},\ F(a-ze)^{-1}\rangle=\int_C d\mu_n(z)\,\frac{F(w)-F(z)}{w-z}.$$

By the assumptions of F and $\int_C |d\mu_n(z)|$, $n\in \boldsymbol{Z}_+$, we see that $Fc_n(z)$, $n\in\boldsymbol{Z}_+$, is a sequence of uniformly bounded holomorphic functions in $N\setminus M$, analytically continuable to N. Its limit is $Fb(z)$, in $N\setminus M$, and since M° is empty, $z\to Fb(z)$ is continuable to N, too.

Thus, $Fb(z)$ can be extended to a bounded holomorphic function in $\boldsymbol{C}$. This means that $Fb(z)$ is constant, and since it vanishes at infinity, it vanishes identically. Thus $Fb=0$, which means that $b\in I(F)$. But $b(z_0)\neq 0$, which gives $z_0\notin\Lambda(F)$, a contradiction, which proves the theorem.

We shall apply Theorem 8.1 to the case where B is a weighted l_1-algebra of the kind described in Example 3.1, with p_n, $n\in\boldsymbol{Z}$, satisfying

$$\sum_{-\infty}^{\infty}(1+n^2)^{-1}\log p_n<\infty. \tag{8.2}$$

It is well known that B is regular (a more general result is Theorem 2.11 in [4]). Let b be any element in B. Then, if $0<r<1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(a-re^{i\theta})^{-1}b(e^{i\theta})e^{-i\theta}d\theta-\frac{1}{2\pi}\int_{-\pi}^{\pi}(a-r^{-1}e^{i\theta})^{-1}b(e^{i\theta})e^{i\theta}d\theta$$

$$=\sum_{-\infty}^{-1}a^n r^{|n|-1}b_n+\sum_{0}^{\infty}a^n r^{n+1}b_n,$$

which converges to b, when $r\to 1-0$. It follows from this that the assumptions of Theorem 8.1 hold for these algebras. It is interesting to observe that if (8.2) does not hold, then it can be deduced from Theorem 22 in Levinson [18] that $b=0$ is the only element of B such that $b(z)$ vanishes on a non-empty open arc of $\boldsymbol{T}$. Hence $\Lambda(F)\neq\boldsymbol{T}$ implies in this case that $\Lambda(F)$ is totally disconnected, i.e. $\Lambda(F)$ is the natural boundary of $z\to F(z)$.

For weighted L^1-algebras on $\boldsymbol{R}$ of the type described in Example 3.2 we have, at least for weight functions p which are monotonic on $]-\infty, 0]$ and $[0,\infty[$ a similar classification into two types, regular algebras where Theorem 8.1 can be applied, and algebras where $\Lambda(F) \neq \boldsymbol{R}$ implies that $\Lambda(F)$ is totally disconnected, hence $\Lambda(F)$ is the natural boundary of $z \to F(z)$.

9. On the analytic Ditkin condition. Let $b \in B$, $z_0 \in M$. Following Bennett and Gilbert [1] we say that B fulfils the *analytic Ditkin condition* for (b, z_0), if there exists a sequence c_n, $n \in \boldsymbol{Z}_+$, of elements in B such that $c_n(z_0) = 1$, for every n, while $bc_n \to 0$, as $n \to \infty$.

It is easy to see that a necessary condition in order to have the analytic Ditkin condition fulfilled for (b, z_0) is that $\langle b, D_{z_0,j}\rangle = 0$ whenever $D_{z_0,j}$ exists in B. In [1] a large family of Banach algebras has been shown to fulfil the analytic Ditkin condition for every (b, z_0) with $b(z_0) = 0$.

THEOREM 9.1. *Let $b \in B$, $z_0 \in M$, and let B fulfil the analytic Ditkin condition for (b, z_0). If $F \in B^*$, and if z_0 is an isolated point of $\Lambda(Fb)$, then z_0 is an essential singularity of $z \to Fb(z)$.*

Proof. We choose c_n, $n \in \boldsymbol{Z}_+$, in accordance with the formulation of the analytic Ditkin condition for (b, z_0), and choose $\varepsilon > 0$ such that $\{z \mid 0 < |z - z_0| \leqslant \varepsilon\}$ and $\Lambda(Fb)$ are disjoint. Since $\Lambda(Fbc_n) \subset \Lambda(Fb)$, for every n, an application of (4.9) shows that there exists, for every $m \in \boldsymbol{Z}$, a constant C_m, such that

$$\Big|\int_{|z-z_0|=\varepsilon} z^m Fbc_n(z)\, dz\Big| \leqslant C_m \|Fbc_n\|_{B^*}, \tag{9.2}$$

for every $n \in \boldsymbol{Z}_+$.

Let us assume that $Fb(z)$ has not an essential singularity at z_0. By Theorem 6.3 and the assumption $z_0 \in \Lambda(Fb)$, the singularity is a pole of order $p \geqslant 1$. We have Laurent expansions

$$\begin{aligned} Fb(z) &= \sum_{m=-p}^{\infty} F_m (z-z_0)^m, \\ Fbc_n(z) &= \sum_{m=-p}^{\infty} F_{m,n}(z-z_0)^m, \end{aligned} \tag{9.3}$$

in $\{z \mid 0 < |z - z_0| \leqslant \varepsilon\}$. The particular form of (9.3) follows from Lemma 4.10, which moreover shows that

$$F_{-p,n} = F_{-p} \neq 0, \tag{9.4}$$

for every $n \in \boldsymbol{Z}_+$. Now (9.2) and the assumption $\|bc_n\|_B \to 0$ show, since

$$\|Fbc_n\|_{B^*} \leqslant \|F\|_{B^*} \|bc_n\|_B,$$

that $F_{-p,n} \to 0$, as $n \to \infty$. This contradicts (9.4), and the singularity has to be essential.

Remark. Theorem 9.1 was proved for a special class of algebras by Bennett and Gilbert [1], but their proof differs from the one presented here and is difficult to generalize. They applied the theorem in the following situation.

Suppose that we know that $\Lambda(F)$ is denumerable. Then, by a classical theorem, everyone of its non-void compact subsets contains at least one isolated point. Let $b \in B$ have the property that $b(z) = 0$ on $\Lambda(F)$. We want to conclude that under certain additional conditions on b, $Fb = 0$. It suffices to have conditions which guarantee that the analytic Ditkin condition holds for (b, z), for every $z \in \Lambda(F)$, and that $z \to Fb(z)$ is meromorphic at all isolated points of $\Lambda(Fb)$, for $\Lambda(Fb)$ must then be empty, hence $Fb = 0$.

For the subalgebra $A_+(\boldsymbol{R}) \subset A(\boldsymbol{R})$ of Fourier transforms of L^1-functions on $\boldsymbol{R}$, vanishing outside $\boldsymbol{R}_+$, Gurarii [14] proved the same result as Bennett and Gilbert in [1] by a method, slightly differing from theirs.

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Systeme einiger singulärer Gleichungen vom nicht normalen Typ und Projektionsverfahren zu ihrer Lösung

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Zusammenfassung. In dieser Arbeit werden einige von I. Z. Gochberg und I. A. Feldman [3] studierte Funktionalgleichungen in dem Fall betrachtet, wenn das Symbol in einzelnen Punkten entartet (sogenannte Gleichungen vom nicht normalen Typ). Es werden exakte Paare von Banachräumen angegeben, in denen solche Gleichungen mit Matrixkoeffizienten aus einer bestimmten Algebra von stetigen Funktionen auf dem Einheitskreis Noethersche (Fredholmsche) Operatoren erzeugen. Anschließend wird die Konvergenz von Projektionsverfahren für solche Gleichungen in einem allgemeinen Schema untersucht. Diese Ergebnisse werden auf Systeme von singulären Integralgleichungen über dem Einheitskreis, auf Systeme von Wiener–Hopfschen Integralgleichungen sowie deren diskretes Analogon angewandt. Auf diesem Wege lassen sich selbst für Gleichungen mit nicht entartetem Symbol einige neue Konvergenzaussagen über Projektionsverfahren (z. B. für singuläre Integralgleichungen in Räumen hölderstetiger Funktionen) gewinnen.

Als einen wichtigen Vertreter der genannten Gleichungen stelle man sich die Wiener–Hopfsche Integralgleichung der Gestalt

$$A\varphi \equiv \varphi(x) - \int_0^\infty k(x-y)\varphi(y)\,dy = f(x) \qquad (0 < x < \infty) \tag{0.1}$$

vor; hierbei sei $k(x)$ eine Matrixfunktion der Ordnung n mit Elementen aus $L^1(-\infty, \infty)$, $f(x)$ eine gegebene n-dimensionale Vektorfunktion mit Komponenten aus dem Raum $L^p_+ = L^p(0, \infty)$, $1 \leqslant p < \infty$ $\big(: f(x) \in (L^p_+)_n\big)$ und $\varphi(x) \in (L^p_+)_n$ die gesuchte Vektorfunktion.

Bekanntlich gilt folgender Satz, der auf I. Z. Gochberg und M. G. Krein zurückgeht (siehe [3], S. 284):

Dafür, daß der Operator A im Raum $(L^p_+)_n$ ein Φ-Operator (1) ist, ist notwendig und hinreichend, daß

$$\det \mathscr{A}(\lambda) \neq 0 \qquad (-\infty < \lambda < \infty) \tag{0.2}$$

(1) Bezüglich der Definition eines Φ-Operators siehe § 2, Punkt 1.