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Error estimates for approximation of translation invariant operators

by

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Abstract. We consider A a translation invariant operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. We construct an approximating operator A_h, $h > 0$, and establish estimates for $\|Au - A_h u\|_p$ when $u \in W_p^m$ or $B_p^{s,q}$.

1. Introduction. In this paper we give error estimates for approximation of translation invariant operators on $L^p(R^n)$. These estimates improve and extend the results given in [5]. The main results are Theorems 4.1 and 4.4 for approximations with functions in the Sobolev space W_p^m, and Theorem 5.1 with functions in the Besov space $B_p^{s,q}$.

Section 2 contains notation and standard results on Fourier multipliers. In Section 3 we define the approximation A_h for A a translation invariant operator on L^p, $1 < p < \infty$, and summarize the principle theorems from [5]. We define in Section 4 the approximation for $p = 1, \infty$, and prove estimates for $\|Au - A_h u\|_p$ when $u \in W_p^m$, $1 \leqslant p \leqslant \infty$. For $1 < p < \infty$ the error bounds need involve only pure derivatives of u. Section 5 contains error estimates for $u \in B_p^{s,q}$.

In Section 6 we give results similar to those of Sections 4 and 5 for approximation of singular integrals with variable kernels. In a subsequent paper we shall establish a discrete smoothing property for commutators of approximations of singular integrals.

2. Preliminaries. R^n denotes n-dimensional Euclidean space, $\mathbf{Z}^n$ the points in R^n with integer coordinates, and T^n the dual group of $\mathbf{Z}^n$. For $h > 0$ we define $Q_h = \{\xi \in R^n\colon -\pi < h\xi_j \leqslant \pi,\ j = 1, \ldots, n\}$ and we identify T^n with $Q \equiv Q_1$. For $1 \leqslant p \leqslant \infty$, $L^p(G)$ denotes the usual L^p space of functions on G, where G is R^n, $\mathbf{Z}^n$, or T^n. $\|u\|_p$ denotes the norm of u in $L^p(G)$, and G will be clear from the context. χ_h denotes the characteristic function of Q_h. We write $D_j = -i\partial/\partial x_j$.

$S(R^n)$ denotes the family of C^∞ functions u on R^n such that $\sup\{|x^\beta D^\alpha u(x)|\colon x \in R^n\} < \infty$. $S(\mathbf{Z}^n)$ is the family of functions u on $\mathbf{Z}^n$ such that $\sup\{|\mu^\beta u(\mu)|\colon \mu \in \mathbf{Z}^n\} < \infty$. The Fourier transform $\hat{u}$ of a function

$u \in S(R^n)$ is defined by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{R^n} u(x) e^{-i\langle x,\xi\rangle} dx$$

where $\langle x, \xi\rangle = \sum_{j=1}^{n} x_j \xi_j$. The Fourier transform $\hat{u}$ of $u \in S(Z^n)$ is defined by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \sum_{\mu \in Z^n} u(\mu) e^{-i\langle \mu,\xi\rangle}, \quad \xi \in Q.$$

$\tilde{u}$ denotes the inverse Fourier transform of u.

A bounded linear operator is said to be *translation invariant* if it commutes with translations. The basic properties of translation invariant operators and Fourier multipliers are collected in [1], [2], and [3]. L_p^p denotes the space of distributions $T \in S'(R^n)$ such that

(2.1) $$\|T * u\|_p \leqslant C\|u\|_p, \quad u \in S(R^n).$$

The smallest constant C for which (2.1) holds is $L_p^p(T)$. The space of distributions $T \in S'(Z^n)$ such that

(2.2) $$\|T * u\|_p \leqslant C\|u\|_p, \quad u \in S(Z^n),$$

is called l_p^p and $l_p^p(T)$ is the smallest constant C for which (2.2) holds.

The space of Fourier transforms $\hat{T}$ of distributions $T \in L_p^p$ or l_p^p is denoted by M_p or m_p, respectively. We write $M_p(\hat{T}) = L_p^p(T)$ and $m_p(\hat{T}) = l_p^p(T)$. $\hat{T}$ is called a *Fourier multiplier of type* (p, p). It is well known that $M_2 = L^\infty(R^n)$, $m_2 = L^\infty(Q)$, $M_p \subset M_2$, $m_p \subset m_2$, and $M_1 = M_\infty = M$ the space of Fourier transforms of bounded measures on R^n.

We shall use the following theorems, which are proved in [3].

THEOREM 2.1. *There is a constant C such that if $f \in M_p$ and $f = 0$ outside Q, then $f \in m_p$ and $m_p(f) \leqslant CM_p(f)$.*

THEOREM 2.2. *Suppose that g is periodic with period 2π and in $L^\infty(R^n)$, and $f \in L^\infty(Q)$ with $f(\xi) = g(\xi)$ for $\xi \in Q$. Then $g \in M_p$ if and only if $f \in m_p$, and if $g \in M_p$, then $M_p(g) = m_p(f)$.*

3. Approximation in Bessel potential spaces. In [5] we constructed a family of approximations A_h, $h > 0$, of a translation invariant operator A on L^p, $1 < p < \infty$. We recall here the definition of A_h and state the principle theorems. It is known that there is a unique $T \in S'(R^n)$ such that $Au = T * u$ for all $u \in S(R^n)$, and thus $T \in L_p^p$ (see [2]). For $h > 0$ define $\hat{T}_h$ to be the periodic function with period $2\pi/h$ such that $\hat{T}_h(\xi) = \hat{T}(\xi)$, $\xi \in Q_h$. Then we have the following theorem.

THEOREM 3.1. *For $h > 0$ we have $T_h \in L_p^p$, and there is a constant C independent of h and T such that for all $h > 0$ we have*

$$L_p^p(T_h) \leqslant CL_p^p(T).$$

Define A_h as the closure of the mapping

$$L^p \supset S(R^n) \ni u \to (2\pi)^{n/2} (\hat{T}_h \hat{u})^{\sim} \in L^p.$$

We proved in [5] estimates for $\|Au - A_h u\|_p$ when u is in the Bessel potential space $L^{p,s}$ defined as follows. For $s > 0$ the Bessel potential $J_s u$ of $u \in L^p$ is defined by

$$(J_s u)^{\hat{}} = (1 + |\xi|^2)^{-s/2} \hat{u}.$$

$L^{p,s}$ is the range of L^p under J_s and is a subspace of L^p. Define the operator Λ^s by $(\Lambda^s u)^{\hat{}} = |\xi|^s \hat{u}$. It is well known that if $u \in L^{p,s}$ then $\Lambda^s u \in L^p$.

THEOREM 3.2. *Let $s > 0$. There is a constant C independent of h such that for $h > 0$ and for all $u \in L^{p,s}$ we have*

$$\|Au - A_h u\|_p \leqslant CL_p^p(T) h^s \|\Lambda^s u\|_p.$$

Since $C_0^\infty \subset L^{p,s}$ and C_0^∞ is dense in L^p, we obtained the following corollary.

COROLLARY 3.3. *If $u \in L^p$ then $\|Au - A_h u\|_p \to 0$ as $h \to 0$.*

We note that T_h is given by

$$T_h = h^{n/2} \sum_{\beta \in Z^n} a_\beta(\hat{T}_h) \delta_{-h\beta}$$

where δ_x is the Dirac measure supported at x and

$$a_\beta(\hat{T}_h) = (h/2\pi)^{n/2} \int_{Q_h} \hat{T}_h(\xi) e^{-ih\langle\beta,\xi\rangle} d\xi.$$

4. Approximation in Sobolev spaces. For $1 \leqslant p \leqslant \infty$ and m a positive integer, let W_p^m denote the space of functions $u \in L^p(R^n)$ all of whose distribution derivatives of order at most m are in $L^p(R^n)$. Define

$$\|u\|_{p,m} = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_p.$$

It is well known that for $1 < p < \infty$ and m a positive integer, $L^{p,m} = W_p^m$. Thus it is possible to apply Theorem 3.2 to estimate $\|Au - A_h u\|_p$ for $u \in W_p^m$. We shall present error estimates for $u \in W_p^m$ without using $L^{p,m}$ which are sharper than those obtained using $L^{p,m}$ directly. Also if A is bounded on L^1 (or equivalently, on L^∞) we shall construct A_h, show that A_h is bounded on L^1 (or on L^∞), and prove error estimates for $u \in W_1^m$ (or in W_∞^m). These cases were not considered in [5].

First we consider the case $1 < p < \infty$. Using the Hörmander–Mihlin multiplier theorem (see [2]) it is possible to show that there is a constant $C_{p,m}$ such that for all $u \in W_p^m$

$$C_{p,m}\|u\|_{p,m} \leqslant \|u\|_p + |u|_{p,m} \leqslant \|u\|_{p,m} \tag{4.1}$$

where we define

$$|u|_{p,m} = \sum_{j=1}^{n} \|D_j^m u\|_p . \tag{4.2}$$

Let $\|A\|$ denote the norm of A as an operator from L^p to L^p. Then we have the following estimate.

THEOREM 4.1. *Let $1 < p < \infty$ and let m be a positive integer. There exists a constant C independent of h such that for $h > 0$ and for all $u \in W_p^m$ we have*

$$\|Au - A_h u\|_p \leqslant C\|A\| h^m |u|_{p,m}. \tag{4.3}$$

Proof. Let $Au = T*u$ and $A_h u = T_h * u$ for $u \in S(R^n)$. Then it follows from the definition of $\hat{T}_h$ that

$$(Au - A_h u)\hat{} = (2\pi)^{n/2}(\hat{T} - \hat{T}_h)[1 - \chi_h(\xi)]\hat{u}. \tag{4.4}$$

The following identity is easily established.

$$\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j = \sum_{J,L}\Big(\prod_{j \in J} b_j\Big)\Big[\prod_{l \in L}(a_l - b_l)\Big], \tag{4.5}$$

where the sum is over all sets J and L of integers such that $J \cap L = \emptyset$, $L \neq \emptyset$, and $J \cup L = \{1, \ldots, n\}$. Since $L \neq \emptyset$, we may write $L = K \cup \{l\}$ with $l \notin K$. With $\chi_{h,j}$ the characteristic function of $\{\xi\colon -\pi < h\xi_j \leqslant \pi\}$ it follows from (4.5) that

$$[1 - \chi_h(\xi)]\hat{u} = h^m \sum_{J,K,l}\Big[\prod_{j \in J}\chi_{h,j}(\xi)\Big]\Big\{\prod_{k \in K}[1 - \chi_{h,k}(\xi)]\Big\}\{[1 - \chi_{h,l}(\xi_l)]/(h\xi)^m\}(D_l^m u)\hat{}. \tag{4.6}$$

Since $\chi_{h,j}$ is a multiplier with norm independent of h, (4.3) follows from (4.4), (4.6), and Theorem 3.1.

COROLLARY 4.2. *Let $1 < p < \infty$ and let m be a positive integer. There exists a constant C independent of h such that for $0 < h \leqslant 1$, $k = 0, 1, \ldots, m$, and for all $u \in W_p^m$ we have*

$$\|Au - A_h u\|_{p,k} \leqslant C\|A\| h^{m-k} |u|_{p,m}. \tag{4.7}$$

Proof. The case $k = 0$ is Theorem 4.1. For $k > 0$ it follows from (4.1) that it suffices to prove

$$|Au - A_h u|_{p,k} \leqslant C\|A\| h^{m-k} |u|_{p,m}. \tag{4.8}$$

Now

$$[D_j^k(Au - A_h u)]\hat{} = (2\pi)^{n/2}(\hat{T} - \hat{T}_h)[1 - \chi_h(\xi)]\xi_j^k \hat{u}. \tag{4.9}$$

We apply the expansion (4.5) again to $[1 - \chi_h(\xi)]\xi_j^k \hat{u}$ and obtain in the sum terms of each of the forms $\chi_{h,j}(\xi)\xi_j^k \hat{u}$ and $[1 - \chi_{h,j}(\xi)]\xi_j^k \hat{u}$. In the first case write

$$\chi_{h,j}(\xi)\xi_j^k = h^{-k}\chi_{h,j}(\xi)(h\xi_j)^k \varphi(h\xi_j) \tag{4.10}$$

where $\varphi \in C_0^\infty(R^1)$ and $\varphi = 1$ on $[-\pi, \pi]$.

In the second case write

$$[1 - \chi_{h,j}(\xi)]\xi_j^k \hat{u} = h^{m-k}\{[1 - \chi_{h,j}(\xi)]/(h\xi_j)^{m-k}\}(D_j^m u)\hat{}. \tag{4.11}$$

Since $\chi_{h,j}(\xi_j)(h\xi_j)^k \varphi(h\xi_j)$ and $[1 - \chi_{h,j}(\xi_j)]/(h\xi_j)^{m-k}$ are multipliers on R^1 with norms independent of h, (4.8) now follows easily from (4.9), (4.10), and (4.11).

The cases $p = 1$ and $p = \infty$ are equivalent since $L_1^1 = L_\infty^\infty = B$, the space of bounded measures on R^n. In general, A_h cannot be constructed using χ_h since $\chi_1 \notin M_1^1$. Choose $\varphi \in C_0^\infty(Q)$ such that $\varphi = 1$ on Q_2. Let $Au = \mu * u$ where $\mu \in B$. Define $\hat{\mu}_h$ as the function with period $2\pi/h$ such that $\hat{\mu}_h(\xi) = \varphi(h\xi)\hat{\mu}(\xi)$ for $\xi \in Q_h$. Then we have the following result.

THEOREM 4.3. *For $h > 0$ we have $\mu_h \in L_1^1$, and there is a constant C independent of h and μ such that*

$$L_1^1(\mu_h) \leqslant C L_1^1(\mu).$$

Proof. The result follows immediately from Theorems 2.1 and 2.2 and the fact that dilation of R^n preserves multipliers and their norms in M_p.

Define $A_h u = \mu_h * u$. Then we have the following error estimate.

THEOREM 4.4 *Let m be a positive integer and $p = 1$ or ∞. There is a constant C independent of h such that for $h > 0$ and for all $u \in W_p^m$ we have*

$$\|Au - A_h u\|_p \leqslant C\|A\| h^m \sum_{j=1}^{n} \|D_j^m u\|_p. \tag{4.12}$$

Proof. Choose $f \in C_0^\infty(-\pi/2, \pi/2)$ such that $f = 1$ on $[-\pi/4, \pi/4]$. Define η by

$$\eta(\xi) = \prod_{j=1}^{n} f(\xi_j).$$

Then it follows from the definition of $\hat{\mu}_h$ that

$$(Au - A_h u)\hat{} = (2\pi)^{n/2}(\hat{\mu} - \hat{\mu}_h)[1 - \eta(h\xi)]\hat{u}.$$

Now we apply (4.5) to $1-\eta(h\xi)$ and proceed as in the proof of Theorem 4.1. Using the fact that f and $1-f$ are Fourier transforms of bounded measures, we obtain (4.12).

COROLLARY 4.5. *Let m be a positive integer and $p = 1$ or ∞. There is a constant C independent of h such that for $0 < h \leqslant 1$, $k = 0, 1, \ldots, m$, and for all $u \in W_p^m$ we have*

$$\|Au - A_h u\|_{p,k} \leqslant C\|A\| h^{m-k} \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{4.13}$$

Proof. The proof is similar to that of Corollary 4.2 except that here we must include all derivatives of order at most k in $\|Au - A_h u\|_{p,k}$, so the right-hand side of (4.13) must contain mixed derivatives of order m.

5. Approximation in Besov spaces. In this section we shall prove estimates for $\|Au - A_h u\|_p$ when u is in the Besov space $B_p^{s,q}$. The estimates involve the modified Besov spaces $\dot{B}_p^{s,q}$ in a natural way. For the definitions and properties of these spaces, see [4] and the references given there. We summarize the results we need here.

Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $0 < s < \infty$. For $u \in L^p(R^n)$ and $0 < t < \infty$, define

$$\omega_p^1(t; u) = \sup_{|y|<t} \|u(\cdot + y) - u\|_p,$$

$$\omega_p^2(t; u) = \sup_{|y|<t} \|u(\cdot + y) - 2u + u(\cdot - y)\|_p.$$

For $s = J + \sigma$, $0 < \sigma \leqslant 1$ and J a non-negative integer, the Besov space $B_p^{s,q}$ is defined by the norm

$$\sum_{|\alpha|\leqslant J} \|D^\alpha u\|_p + \sum_{|\alpha|=J} \Big\{\int_0^\infty [t^{-\sigma}\omega_p^1(t; D^\alpha u)]^q \frac{dt}{t}\Big\}^{1/q}, \quad 0 < \sigma < 1, \tag{5.1}$$

and for $\sigma = 1$, ω_p^1 is replaced by ω_p^2. We make the usual change in (5.1) if $q = \infty$.

We shall use an equivalent definition of $B_p^{s,q}$ given in [4]. Let $\Phi \in C_0^\infty$ be positive on $E = \{\xi : 2^{-1} < |\xi| < 2\}$ and zero outside E, and

$$\sum_{k=-\infty}^{\infty} \Phi(2^{-k}\xi) = 1, \quad \xi \neq 0.$$

Write

$$\Phi_k(\xi) = \Phi(2^{-k}\xi), \quad k = 0, \pm 1, \pm 2, \ldots,$$

$$\Psi(\xi) = 1 - \sum_{k=1}^{\infty} \Phi_k(\xi). \tag{5.2}$$

Then $B_p^{s,q}$ may be defined as the Banach space corresponding to the norm (which is equivalent to the expression in (5.1))

$$\|u\|_{B_p^{s,q}} = \Big\{\sum_{k=0}^{\infty} [2^{sk}\|u_k\|_p]^q\Big\}^{1/q}, \tag{5.3}$$

with $\hat{u}_0 = (2\pi)^{n/2}\Psi\hat{u}$; $\hat{u}_k = (2\pi)^{n/2}\Phi_k\hat{u}$, $k = 1, 2, \ldots$ It is apparent from (5.3) that $B_p^{s,q} \subset B_p^{r,q}$ if $0 < r < s$.

We shall also work with the modified Besov spaces $\dot{B}_p^{s,q}$ which are defined by the seminorm

$$\|u\|_{\dot{B}_p^{s,q}} = \Big\{\sum_{k=-\infty}^{\infty} [2^{sk}\|u_k\|_p]^q\Big\}^{1/q}, \tag{5.4}$$

with $\hat{u}_k = (2\pi)^{n/2}\Phi_k\hat{u}$, $k = 0, \pm 1 \pm 2, \ldots$ It is known that $\|u\|_{\dot{B}_p^{s,q}}$ is equivalent to the second sum in (5.1) and that $\|u\|_{B_p^{s,q}}$ is equivalent to $\|u\|_p + \|u\|_{\dot{B}_p^{s,q}}$, that is, there are constants C_1 and C_2 such that

$$C_1\|u\|_{B_p^{s,q}} \leqslant \|u\|_p + \|u\|_{\dot{B}_p^{s,q}} \leqslant C_2\|u\|_{B_p^{s,q}}. \tag{5.5}$$

Let A be a translation invariant operator from L^p to L^p with norm $\|A\|$. For $1 < p < \infty$ let A_h be the approximation constructed in Section 3, and for $p = 1, \infty$, let A_h be the approximation constructed in Section 4. It is clear from (5.3) that for $s > 0$ and $1 \leqslant q \leqslant \infty$, A and A_h are bounded operators from $B_p^{s,q}$ to $B_p^{s,q}$ with norms no larger than their norms as operators from L^p to L^p. We have the following error estimates.

THEOREM 5.1. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $s > 0$. Then there is a constant C independent of h that for $h > 0$ and for all $u \in B_p^{s,q}$ we have*

$$\|Au - A_h u\|_p \leqslant C\|A\| h^s \|u\|_{\dot{B}_p^{s,q}}. \tag{5.6}$$

Proof. We give the proof for $1 < q < \infty$. The cases $q = 1, \infty$ are similar. Write $Au = T * u$ and $A_h u = T_h * u$. Then we have

$$(Au - A_h u)^\wedge = (2\pi)^{n/2}(\hat{T} - \hat{T}_h)\hat{u}.$$

It follows from the definition of Φ_j that there is a constant $\varkappa > 0$ such that

$$\sum_{j=j_0}^{\infty} \Phi_j(\xi) = 1, \quad \xi \notin Q_h,$$

where j_0 is the integer part of $\log_2(\varkappa/h)$. Hence

$$\|Au - A_h u\|_p \leqslant C\|A\| \sum_{j=j_0}^{\infty} \|(\Phi_j\hat{u})^\sim\|_p.$$

Applying Hölder's inequality we obtain

$$\|Au-A_h u\|_p \leqslant C\|A\|\Big\{\sum_{j=j_0}^{\infty} 2^{-sjq'}\Big\}^{1/q'}\|u\|_{\dot{B}_p^{s,q}},$$

where $1/q+1/q' = 1$. An elementary estimate now yields the result

$$\|Au-A_h u\|_p \leqslant C\|A\| 4^s(2^{sq'}-1)^{-1/q'}(h/\varkappa)^s\|u\|_{\dot{B}_p^{s,q}}.$$

and the proof is complete.

THEOREM 5.2. *Let $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$, and $0<r<s$. There is a constant C independent of h that for $h>0$ and for all $u\in B_p^{s,q}$ we have*

(5.7) $$\|Au-A_h u\|_{\dot{B}_p^{r,q}} \leqslant C\|A\|h^{s-r}\|u\|_{\dot{B}_p^{s,q}}.$$

Proof. Let j_0 be as in the proof of Theorem 5.1. Then

$$\|Au-A_h u\|_{\dot{B}_p^{r,q}} \leqslant C\|A\|\Big\{\sum_{j=j_0}^{\infty}[2^{rj}\|(\Phi_j\hat{u})^{\sim}\|_p]^q\Big\}^{1/q} \leqslant C\|A\|2^{-(s-r)j_0}\|u\|_{\dot{B}_p^{s,q}}.$$

(5.7) now follows easily.

COROLLARY 5.3. *Let $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$, and $0<r<s$. There is a constant C independent of h such that for $0<h\leqslant 1$ and for all $u\in B_p^{s,q}$ we have*

$$\|Au-A_h u\|_{B_p^{r,q}} \leqslant C\|A\|h^{s-r}\|u\|_{\dot{B}_p^{s,q}}.$$

Proof. This follows immediately from (5.5) and Theorems 5.1 and 5.2.

6. Approximation of singular integrals with variable kernels. We consider now $1<p<\infty$. Let $\beta>0$ and $r=[\beta]$. We shall say that $f\in B_\beta$ provided

$$\sup\{|D^\alpha f(x)|:\ x\in R^n,\ |\alpha|\leqslant r\}<\infty$$

and

$$\sup\{|y|^{-(\beta-r)}|D^\alpha f(x+y)-D^\alpha f(x)|:\ x\in R^n,\ y\neq 0,\ |\alpha|=r\}<\infty.$$

We denote by $\|f\|_{B_\beta}$ the maximum of these quantities.

We consider an operator A defined by

$$Au(x) = a(x)u(x)+\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} k(x, x-y)u(y)\,dy$$

where $u\in S(R^n)$ and k has the following properties. For each $x\in R^n$, $k(x,z)$ is C^∞ in z for $z\neq 0$, $k(x,z)$ is positively homogeneous of degree $-n$ in z, and $k(x,z)$ has mean value zero on $\{z:\ |z|=1\}$. Let $\hat{T}(x,\xi)$ denote the Fourier transform in the second variable of principal value $k(x,z)$. Then for each $x\in R^n$, $\hat{T}(x,\xi)$ is C^∞ in ξ for $\xi\neq 0$, $\hat{T}(x,\xi)$ is positively homogeneous of degree 0 in ξ, and $\hat{T}(x,\xi)$ has mean value 0 on $\{\xi:\ |\xi|=1\}$. We write

$$\sigma(A)(x,\xi) = a(x)+\hat{T}(x,\xi)$$

and call $\sigma(A)$ the symbol of A. We review the assumptions on $\sigma(A)$ and their consequences which may be found in [0]. We suppose that for each index α with $0\leqslant|\alpha|\leqslant 2n$ and for each ξ with $|\xi|=1$, the functions $D_\xi^\alpha\sigma(A)(x,\xi)$ are in B_β in x, and we say that A is an *operator of type* β. We define

$$\|A\|_\beta = \sup\{\|D_\xi^\alpha\sigma(A)(x,\xi)\|_{B_\beta}:\ |\xi|=1,\ 0\leqslant|\alpha|\leqslant 2n\}.$$

Then A may be extended to a bounded operator on W_p^M for $0\leqslant M\leqslant r$ and there is a constant C depending on p and β such that if $u\in W_p^M$, then

$$\|Au\|_{p,M}\leqslant C\|A\|_\beta\|u\|_{p,M}.$$

This result was proved by establishing series representations for $k(x,z)$ and $\sigma(A)(x,\xi)$ using spherical harmonics. Let $\{Y_{lm}\}$ be a complete orthonormal system of spherical harmonics in $L^2(\Sigma)$, where $\Sigma=\{\xi:\ |\xi|=1\}$. The positive integer m is the degree of Y_{lm} and the number of harmonics of degree m is $O(m^{n-2})$. The expansions are

(6.1) $$k(x,z) = \sum_{l;m\geqslant 1} a_{lm}(x)Y_{lm}(z)|z|^{-n}$$

and

(6.2) $$\sigma(A)(x,\xi) = a(x)+\sum_{l;m\geqslant 1} b_{lm}(x)Y_{lm}(\xi).$$

The coefficients in the expansions satisfy

(6.3) $$\|a\|_\beta\leqslant C\|A\|_\beta,\quad \|b_{lm}\|_\beta\leqslant Cm^{-2n}\|A\|_\beta,\quad \|a_{lm}\|_\beta\leqslant Cm^{-3n/2}\|A\|_\beta,$$
$$\text{and}\quad a_{lm}=\gamma_m^{-1}b_{lm}\quad\text{where}\quad |\gamma_m^{-1}|\leqslant Cm^{n/2}.$$

Finally it is known that

(6.4) $$|Y_{lm}| = O(m^{(n-2)/2}).$$

Thus the series for $\sigma(A)$ converges uniformly and hence

(6.5) $$Au(x) = a(x)u(x)+\sum_{l;m} a_{lm}(x)R_{lm}u(x)$$

where R_{lm} is the translation invariant singular integral operator with kernel $Y_{lm}(z)|z|^{-n}$. The operator norm of R_{lm} on L^p is bounded independent of m.

We define the approximation A_h as the operator with symbol

$$\sigma(A_h)(x,\xi) = a(x) + \hat{T}_h(x,\xi)$$

where for each x, $\hat{T}_h(x,\xi)$ has period $2\pi/h$ in ξ and $\hat{T}_h(x,\xi) = \hat{T}(x,\xi)$ for $\xi \in Q_h$. That is, for $u \in S(R^n)$,

$$A_h u(x) = a(x)u(x) + [T_h(x) * u](x)$$

where the distribution $T_h(x)$ has support on hZ^n and

$$T_h(x) = h^n(2\pi)^{-n/2} \sum_{\mu \in Z^n} \int_{Q_h} \hat{T}_h(x,\xi) e^{ih\langle\mu,\xi\rangle} d\xi \, \delta_{-h\mu}.$$

THEOREM 6.1. *For $h > 0$ and $0 \leqslant M \leqslant r$, A_h may be extended to a bounded operator on W_p^M, and there exists a constant C independent of h such that for $u \in W_p^M$,*

$$\|A_h u\|_{p,M} \leqslant C\|A\|_\beta \|u\|_{p,M}.$$

Proof. Since the series for $\sigma(A)$ converges uniformly, we have

(6.6) $$A_h u(x) = a(x)u(x) + \sum_{l,m} a_{lm}(x) R_{lmh} u(x)$$

where R_{lmh} is the approximation to R_{lm} considered in Section 3. It is easy to see that for $M \leqslant r$, $u \in W_p^M$, and $f \in B_r$,

(6.7) $$\|fu\|_{p,M} \leqslant C\|f\|_r \|u\|_{p,M}.$$

Now the result follows easily from (6.6), (6.7), and (6.3).

Now we prove an error estimate for $u \in W_p^M$.

THEOREM 6.2. *Let $0 \leqslant K \leqslant M \leqslant r$. There exists a constant C such that for $u \in W_p^M$ and $0 < h \leqslant 1$,*

$$\|Au - A_h u\|_{p,K} \leqslant C\|A\|_\beta h^{M-K} |u|_{p,M}.$$

Proof. Since

$$Au - A_h u = \sum_{l,m} a_{lm}(x)(R_{lm}u - R_{lmh}u),$$

the result follows easily from (6.3) and Corollary 4.2.

Finally we consider the Besov space $B_p^{\beta,\infty}$ and show that A and A_h are bounded on $B_p^{\beta,\infty}$ and prove estimates for $Au - A_h u$, $u \in B_p^{\beta,\infty}$.

THEOREM 6.3. *Let A be a operator of type β and A_h the approximation considered above. Then A and A_h are bounded on $B_p^{\beta,\infty}$ and there exist constants C_1 and C_2 such that for $u \in B_p^{\beta,\infty}$,*

$$\|Au\|_{B_p^{\beta,\infty}} \leqslant C_1 \|A\|_\beta \|u\|_{B_p^{\beta,\infty}}$$

and for $h > 0$,

$$\|A_h u\|_{B_p^{\beta,\infty}} \leqslant C_2 \|A\|_\beta \|u\|_{B_p^{\beta,\infty}}.$$

Proof. We shall use the fact that if $f \in B_\beta$ and $u \in B_p^{\beta,\infty}$ then $fu \in B_p^{\beta,\infty}$ and there is a constant C independent of f and u such that

(6.8) $$\|fu\|_{B_p^{\beta,\infty}} \leqslant C\|f\|_{B_\beta} \|u\|_{B_p^{\beta,\infty}}.$$

(6.8) follows easily from the definition of $B_p^{\beta,\infty}$, (6.7), and the following identities. Let $\tau_y u(x) = u(x+y)$ and let 1 denote the identity operator. Then

$$(\tau_y - 1)(fu)(x) = (\tau_y - 1)f(x)\tau_y u(x) + f(x)(\tau_y - 1)u(x)$$

and

$$(\tau_y - 2 + \tau_{-y})(fu)(x) = f(x)(\tau_y - 2 + \tau_{-y})u(x) + \tau_y u(x)(\tau_y - 1)f(x) - \\ - \tau_{-y}u(x)(1 - \tau_{-y})f(x).$$

The theorem is an immediate consequence of (6.5), (6.6), (6.8), and (6.3).

THEOREM 6.4. *There is a constant C depending only on p, n, and β such that for $u \in B_p^{\beta,\infty}$ and $h > 0$,*

$$\|Au - A_h u\|_p \leqslant C\|A\|_\beta h^\beta \|u\|_{\dot{B}_p^{\beta,\infty}}.$$

Proof. The estimate follows from the expansion for $Au - A_h u$, (6.3), and Theorem 5.1.

THEOREM 6.5. *Let $0 < \alpha \leqslant \beta$. Then there is a constant C such that for $0 < h \leqslant 1$ and $u \in B_p^{\beta,\infty}$,*

$$\|Au - A_h u\|_{B_p^{\alpha,\infty}} \leqslant C\|A\|_\beta h^{\beta-\alpha} \|u\|_{\dot{B}_p^{\beta,\infty}}.$$

Proof. It is clear that for $0 < \alpha \leqslant \beta$, $B_\beta \subset B_\alpha$ and $B_p^{\beta,\infty} \subset B_p^{\alpha,\infty}$. The estimate now follows easily from the expansion for $Au - A_h u$, (6.8), (6.3), and Corollary 5.3.

It is clear that the results of Section 4, 5, and 6 are also valid if the approximation A_h is constructed in the following way. Let η be a C_0^∞ function which is one in a neighborhood of the origin and vanishes outside Q_1. Define $\hat{T}_h$ to be periodic with period $2\pi/h$ in ξ and $\hat{T}_h = \eta(\xi h)\hat{T}$, $\xi \in Q_h$. Then $A_h u = T_h * u$. The operator A_h is bounded as before and the estimates for $Au - A_h u$ are easily established.

References

[0] A. P. Calderón, *Integrales singulares y sus aplicaciones a equaciones diferenciales hiperbólicas*, Cursos y seminarios de Mathemática, Fasc. 3, Univ. of Buenos Aires.

[1] K. de Leeuw, *On L^p multipliers*, Annals of Math. (2) 81 (1965), pp. 364–379.

[2] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), pp. 93–140.
[3] M. A. Jodeit, Jr., *Restrictions and extensions of Fourier multipliers*, Studia Math. 34 (1970), pp. 215–226.
[4] J. Löfström, *Besov spaces in theory of approximation*, Annali di Mat. pura ed appl. (IV) 85 (1970), pp. 93–184.
[5] D. C. Shreve, *Approximation of translation invariant operators*, Studia Math. 41 (1972), pp. 35–43.

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