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Power series spaces $\Lambda_k(\alpha)$ of finite type and related nuclearities

by

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Abstract. Associated with the space $\Lambda_k(\alpha)$ of the space of power series with radius of convergence k, the notion of $\tilde{\Lambda}_k$-nuclearity is defined and the $\tilde{\Lambda}_k$-nuclearity of smooth sequence spaces of the finite and infinite types are considered. The notion of $\Lambda_N(\alpha)$-nuclearity is formulated and distinction between $\Lambda_\infty(\alpha)$ and $\Lambda_N(\alpha)$-nuclearities is brought out; for $\alpha_n = \log(n+1)$ these correspond to s-nuclearity and nuclearity. $\Lambda_\infty(\alpha)$ is exhibited as a single generator of the variety of $\Lambda_N(\alpha)$-nuclear spaces, when α is stable.

Introduction. We consider nuclear power series spaces $\Lambda_k(\alpha)$ of the finite type and make a detailed study of locally convex spaces which are $\tilde{\Lambda}_k(\alpha)$-nuclear. Section 1 of the paper contains various definitions and preliminary results; in Section 2 $\tilde{\Lambda}_k(\alpha)$-nuclearity of Köthe spaces and in particular, of the smooth sequence spaces of the finite or infinite type are considered; also the concepts of $\Lambda_N(\alpha)$ and of uniform $\tilde{\Lambda}_k(\alpha)$-nuclearity are defined and the inter-relation between these various concepts are established. In Section 3 we prove some permanence properties for $\Lambda_N(\alpha)$-nuclear spaces and obtain $\Lambda_\infty(\alpha)$ as a single generator for the variety of $\Lambda_N(\alpha)$-nuclear spaces whenever α is stable exponent sequence.

1. Definitions and preliminary results. For terms not explicitly defined here we refer to Köthe [4], Pietsch [7], Dubinsky and Ramanujan [2] and Terzioglu [11].

Let E and F be Banach spaces, λ a normal sequence space and $\lambda^\times$ its Köthe-dual. For a map $T \in \mathscr{L}(E, F)$, suppose there exists a representation

$$Tx = \sum_{n=0}^{\infty} \gamma_n \langle x, a_n \rangle y_n, \qquad \forall x \in E;$$

T is said to be λ-*nuclear* (written, $T \in N_\lambda(E, F)$), if $(\gamma_n) \in \lambda$, $a_n \in E'$, $\|a_n\| \leqslant 1$ and $y_n \in F$, $(\langle y_n, b \rangle) \in \lambda^\times$ for each $b \in F'$. T is said to be *pseudo-λ-nuclear* or $\tilde{\lambda}$-*nuclear*, $(T \in \tilde{N}_\lambda)$, if $(\gamma_n) \in \lambda$, $a_n \in E'$, $\|a_n\| \leqslant 1$, $y_n \in F$ and $\|y_n\| \leqslant 1$.

T is defined to be *quasi-λ-nuclear*, ($T \in N_\lambda^Q$), if there exist $(\gamma_n) \in \lambda$, $a_n \in E'$, $\|a_n\| \leqslant 1$ such that for each $x \in E$,

$$\|Tx\| \leqslant \sum^{\infty} |\gamma_n| \, |\langle x, a_n \rangle|.$$

We shall assume that the definition of a Köthe sequence space $\lambda(P)$ with its natural locally convex topology is known.

Throughout this paper $a = (a_n)$ is a sequence of reals and $0 \leqslant a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \uparrow \infty$. Now for a as above and fixed real number $k > 0$, define

$$\Lambda_k(a) = \{x = (x_n) \colon \sum^{\infty} |x_n| R^{a_n} < \infty \text{ for each } R < k\}$$

and the space $\Lambda_k(a)$ is considered as a Köthe space with its natural (Fréchet) topology. We shall assume also that $\Lambda_k(a)$ is a nuclear space or equivalently, $\sum R^{-a_n} < \infty$ for each R, $0 < R < 1$. As is well known, the power series spaces $\Lambda_k(a)$, $\Lambda_l(a)$, $0 < k$, $l < \infty$, are isomorphic.

For a locally convex space (l.c.s.) E, we shall let $\mathscr{U}$ denote a basis of absolutely convex sets with U, V denoting typical members of $\mathscr{U}$; U^0 will denote the polar of U, $\hat{E}_U$ will denote the completion of the normed space $E/p_U^{-1}(0)$.

We reserve the symbol $\delta_n(\cdot,\cdot)$ for the Kolmogorov diameters and $\Delta(E)$ for the diametral dimension of the l.c.s. E; for more details see Pietsch [7] or Terzioglu [11].

Consider now a Köthe space $\lambda(P)$ with its generating Köthe set P.

The Köthe set P is called a *power set of infinite type* if it satisfies the following (additional) conditions:

(∞.1) for each $a \in P$, $0 < a_n \leqslant a_{n+1}$, $\forall n$; and

(∞.2) for each $a \in P$, there exists a $b \in P$ with $a_2^n \leqslant b_n$ for each n.

The corresponding space $\lambda(P)$ is called a *smooth sequence space of infinite type* or a *G_∞-space*. The nuclearity and related concepts of such spaces is discussed in [2], [11], [12].

The Köthe set P is called a *power set of finite type* if

(1.1) for each $a \in P$, $0 < a_{n+1} \leqslant a_n$, $\forall n$; and

(1.2) for each $a \in P$, there exists a $b \in P$ such that $\sqrt{a_n} \leqslant b_n$, for each n.

In this case $\lambda(P)$ is called a *smooth sequence space of finite type* or a *G_1-space*.

We start with the following preliminary results. Let k and a be fixed with $k > 1$ and $\Lambda_k = \Lambda_k(a)$.

LEMMA 1.1. *Suppose E and F are Hilbert spaces and $T \in \mathscr{L}(E, F)$. The following conditions on T are equivalent:*

(a) *T is quasi-Λ_k-nuclear;*

(b) *T is of type Λ_k;*

(c) *T is $\tilde{\Lambda}_k$-nuclear;*

(a)′ *T' is quasi-Λ_k-nuclear;*

(b)′ *T' is of type Λ_k;*

(c)′ *T' is $\tilde{\Lambda}_k$-nuclear.*

Proof. First we recall (see for instance, Pietsch [7], 8.3.1) that for a compact linear map T between Hilbert spaces E, F respectively with unit balls U and V, T has the representation

$$Tx = \sum \lambda_n (x, e_n) f_n$$

for suitable orthonormal sequences (e_n), (f_n) in E and F respectively and for a suitable non-increasing, non-negative sequence λ_n with $\lim \lambda_n = 0$; also

$$\lambda_n = \delta_n\big(T(U), V\big) = \delta_n\big(T'(V^0), U^0\big) = a_n(T) = a_n(T').$$

Now in view of $a_n(T) = a_n(T')$, it suffices to prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a), of which the non-trivial part (a) $\Rightarrow$ (b) is proved below.

We first observe that if (β_n) is a non-negative decreasing sequence and $(\beta_n) \in \Lambda_k$ then the sequence $(\gamma_n) \in \Lambda_k$ where $\gamma_n = \sum_{m \geqslant n} \beta_m$; this is easily verified by using the nuclearity of Λ_k. If now T is quasi-Λ_k-nuclear, then $\|Tx\| \leqslant \sum_0^\infty \beta_n |(x, u_n)|$ where $u_n \in E'$, $\|u_n\| \leqslant 1$, $\beta_n \downarrow 0$ and $\beta \in \Lambda_k$. Then γ, as defined above, is also in Λ_k. Let $M_n = \{x \in E \colon \langle x, u_i \rangle = 0, i = 0, 1, 2, \ldots \ldots, n-1\}$; if $x \in M_n$, then by the above inequality,

$$\|Tx\| \leqslant \sum_{m=n}^{\infty} \beta_m |(x, u_m)| \leqslant \gamma_n \|x\|.$$

Hence $T(U \cap M_n) \subset \gamma_n V$; by taking polars we get $T'(V^0) \subset \gamma_n U^0 + M_n^\perp$ (see [11]; p. 65); hence $a_n(T) = \delta_n\big(T'(V^0), U^0\big) \leqslant \gamma_n$ and so T is of type Λ_k.

PROPOSITION 1.2. *On a l.c.s. E the following conditions are equivalent:*

(a) $\forall U \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V < U$ *such that the canonical map $K(V, U)$ on $\hat{E}_V$ to $\hat{E}_U$ is quasi-Λ_k-nuclear;*

(b) $\forall U \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V < U$ *such that* $K(V, U) \in \tilde{N}_{\Lambda_k}$;

(c) $\forall U \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V < U$ *such that* $\big(\delta_n(V, U)\big) \in \Lambda_k$;

(d) $\forall U \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V < U$ *such that* $\big(\delta_n(U^0, V^0)\big) \in \Lambda_k$;

(e) $\forall U \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V < U$ *such that for the canonical map* $K' \colon E'_{U^0} \to E'_{V^0}$, $K' \in \tilde{N}_{\Lambda_k}$;

(f) $\forall U \in \mathscr{U}$, $V \in \mathscr{U}$, $V < U$ *such that* $K' \in N_{\Lambda_k}^Q$.

The above proposition is easily proved by using Lemma 1.1 since each of the above conditions implies that E is nuclear and therefore there is a base $\mathscr{U}'$ of neighborhoods (U) such that $\hat{E}_U$ and E'_{U^0} are Hilbert spaces.

DEFINITION 1.3. A l.c.s. E is defined to be $\tilde{\Lambda}_k$-*nuclear* if it satisfies any of the six equivalent conditions of Proposition 1.2.

Clearly each $\tilde{\Lambda}_k$-nuclear space is nuclear. Referring now to conditions (c) and (d) of Proposition 1.2 and the earlier results of the second author [11] one easily obtains the following permanence properties.

PROPOSITION 1.4. *Subspaces, quotient spaces by closed subspaces, completions and natural biduals of a $\tilde{\Lambda}_k$-nuclear space are all $\tilde{\Lambda}_k$-nuclear.*

2. Köthe spaces and $\tilde{\Lambda}_k$-nuclearity. Consider now a Köthe space $\lambda(P) \equiv E$. A neighborhood base $\mathscr{U}$ of absolutely convex sets is made of sets $\{U_a: a \in P\}$ where $U_a = \{x \in \lambda(P): p_a(x) = \sum |x_n| a_n \leqslant 1\}$; the canonical map $K(b, a): U_a \to U_b$ can be identified with the diagonal map $D = (b_n/a_n)$ from l_1 to l_1; as has been shown by Köthe [6] the approximation numbers $a_n(D)$ of D are precisely the decreasing rearrangement of (b_n/a_n) and since $\delta_n(D) \leqslant a_n(D)$ and $\Lambda_k(a)$ is normal, one can now easily establish the following Grothendieck–Pietsch–Köthe type criterion.

PROPOSITION 2.1. *A Köthe space $\lambda(P)$ is $\tilde{\Lambda}_k$-nuclear if and only if for each $a \in P$ there exists a $b \in P$ with $a \leqslant b$ and an injection $\sigma: \boldsymbol{N} \to \boldsymbol{N}$* such that $\sigma(\boldsymbol{N}) = \{n: a_n \neq 0\}$ *and* $(a_{\sigma(n)}/b_{\sigma(n)}) \in \Lambda_k$.

DEFINITION 2.2. Following Köthe [6] we shall call a Köthe space $\lambda(P)$ to be *uniformly $\tilde{\Lambda}_k$-nuclear* if there exists a "universal permutation" σ such that for each $a \in P$ there exists a $b \in P$ such that $a < b$ and $(a_{\sigma(n)}) \in (b_{\sigma(n)}) \cdot \Lambda_k$.

DEFINITION 2.3. A l.c.s. E is said to be $\Lambda_{\boldsymbol{N}}(a)$-*nuclear* if it is $\tilde{\Lambda}_k(a)$-nuclear for each $k > 1$.

We also recall that a l.c.s. E is $\Lambda_\infty(a)$-nuclear if for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$, $V < U$, so that $K(V, U)$ is $\Lambda_\infty(a)$-nuclear, due essentially to Spuhler [10], is that for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ so that $\delta_n(V, U) \in \Lambda_\infty(a)$.

Remarks 2.4. We emphasize at this stage the difference between $\Lambda_\infty(a)$-nuclearity and $\Lambda_{\boldsymbol{N}}(a)$-nuclearity of l.c.s. E. Since $\Lambda_\infty(a) = \bigcap \Lambda_k(a)$, the above characterization of $\Lambda_\infty(a)$-nuclearity says that for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$, V *not dependent on* k, so that $\delta_n(V, U) \in \Lambda_k(a)$ for each k; on the other hand, an application of Proposition 1.2 gives that E is $\Lambda_{\boldsymbol{N}}(a)$-nuclear if and only if for each $U \in \mathscr{U}$ and each k there exists a $V = V(k) \in \mathscr{U}$ so that $\delta_n(V, U) \in \Lambda_k(a)$. Thus, clearly E is $\Lambda_\infty(a)$-nuclear $\Rightarrow E$ is $\Lambda_{\boldsymbol{N}}(a)$-nuclear $\Rightarrow E$ is $\tilde{\Lambda}_k(a)$-nuclear, $k > 1$.

We shall point out the falsity of the reverse of the first implications later in the paper and also obtain partial results on the reverse of the second of the above implications.

The case $a_n = \log(n+1)$ is of special significance and motivates much of the discussions in this paper; in this case $\Lambda_\infty(a) = s$, the space of rapidly decreasing sequences and $\Lambda_\infty(a)$-nuclearity is the so-called strong nuclearity [6]; however, $\Lambda_k(a)$ is not nuclear in this case; despite this violation of our assumptions on $\Lambda_k(a)$ (in general), the condition that "given $U \in \mathscr{U}$ and $k > 1$, $\exists V \in \mathscr{U}$, $V = V(k)$, so that $\delta_n(V, U) \in \Lambda_k(a)$" is exactly the condition that E is nuclear (see [11], p. 75) and roughly speaking $\Lambda_{\boldsymbol{N}}(a)$-nuclearity of l.c.s. in this case is ordinary nuclearity.

One might also consider the $\Lambda_k(a)$-nuclearity of l.c.s. E for a fixed $k \geqslant 1$ or for each fixed $k \geqslant 1$; as is easily seen, there is no difference between $\Lambda_k(a)$-nuclearity and $\Lambda_l(a)$-nuclearity for k, $l < \infty$ since each of the spaces Λ_k, Λ_l is a diagonal transform of the other; also $\Lambda_1(a)$-nuclearity is discussed in detail by Robinson [9]. Moreover, for $k > 1$ and fixed, each $\tilde{\Lambda}_k(a)$-nuclear map is $\Lambda_k(a)$-nuclear.

In the context of the above remarks we state below a proposition whose proof we omit and one may supply a proof by comparing the result with Proposition 8.6.2 in Pietsch [7] and taking into consideration the remarks 2.4.

PROPOSITION 2.5. *If E is a $\Lambda_{\boldsymbol{N}}(a)$-nuclear DF-space then E is $\Lambda_\infty(a)$-nuclear.*

PROPOSITION 2.6. *For a G_∞-space $\lambda(P)$ the following statements are equivalent:*

(a) $\lambda(P)$ *is $\tilde{\Lambda}_k$-nuclear;*

(b) $\lambda(P)$ *is uniformly $\tilde{\Lambda}_k$-nuclear;*

(c) *for each $b \in P$ there exists an $a \in P$ with $b < a$ and $1/a \in \Lambda_k$;*

(d) *there exists an $a \in P$ with $1/a \in \Lambda_k$.*

Proof. (a) $\Rightarrow$ (c): Recalling the discussion preceding Proposition 2.1 and using Proposition 1.2 we see that if (a) is true then for each $b \in P$ there is an $a \in P$ with $b < a$ such that $\delta_n(U_a, U_b) \in \Lambda_k(a)$. Let L_n denote the span of $e_0, e_1, \ldots, e_n$ (the standard unit vectors); then for $x \in L_n$ we have $x \in \lambda(P)$ and

$$p_a(x) = \sum |x_i| b_i \frac{a_i}{b_i} \leqslant \frac{a_n}{b_0} p_b(x).$$

Therefore $(b_0/a_n)(U_a \cap L_n) \subset U_b$; then by a known theorem of Tikhomirov (see [7]) one has $b_0/a_n \leqslant \delta_n(U_a, U_b)$ and since Λ_k is normal and $\delta_n(U_a, U_b) \in \Lambda_k$, it now follows $(1/a) \in \Lambda_k$.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (b): Assuming (d), we have $(1/a) \in \Lambda_k$. Now for $b \in P$ choose $d \in P$ so that $a < d$ and $b < d$. Then $(1/d) \in \Lambda_k$. Next choose $c \in P$ so that $d_n^2 \leqslant c_n$, $\forall n$. Then since $b_n/c_n \leqslant b_n/d_n^2 \leqslant 1/d_n$ we get $(b/c) \in \Lambda_k$. Thus $\lambda(P)$ is uniformly $\tilde{\Lambda}_k$-nuclear.

The rest of the proof, viz. (b) $\Rightarrow$ (a), is trivial.

Corollary 2.7. *The power series space $\Lambda_\infty(\beta)$ is $\tilde{\Lambda}_k(\alpha)$-nuclear $\Leftrightarrow$ (α_n/β_n) is bounded.*

Proof. ($\Rightarrow$) If $\Lambda_\infty(\beta)$ is $\tilde{\Lambda}_k(\alpha)$-nuclear then by Proposition 2.6, there exists a number S, $0<S<1$, such that $(S^{\beta_n})\in\Lambda_k(\alpha)$, and this implies $(\alpha_n/\beta_n)\in l_\infty$.

($\Leftarrow$) If $(\alpha_n/\beta_n)\in l_\infty$, then there exists $M>1$ such that $\alpha_n\leqslant M\beta_n$, $\forall n$; also $(\alpha_n/\beta_n)\in l_\infty\Rightarrow\Lambda_\infty(\beta)$ is nuclear; thus there exists a $T>1$ so that $\sum 1/T^{\beta_n}<\infty$; choose now $S>k^M T$; then $S>1$ and

$$\sum\frac{k^{\alpha_n}}{S^{\beta_n}}\leqslant\sum\left(\frac{k^M}{S}\right)^{\beta_n}\leqslant\sum\frac{1}{T^{\beta_n}}<\infty \quad\text{and so}\quad \left(\frac{1}{S^{\beta_n}}\right)\in\Lambda_k(\alpha).$$

Now the result follows from Proposition 2.3.

Proposition 2.8. *For a G_1-space $\lambda(Q)$ the following statements are equivalent:*

(a) *$\lambda(Q)$ is $\tilde{\Lambda}_k$-nuclear;*

(b) *$\lambda(Q)$ is uniformly $\tilde{\Lambda}_k$-nuclear;*

(c) *$Q\subset\Lambda_k(\alpha)$.*

We omit the proof of the proposition with the remark that it is similar to that of Proposition 2.3.

$\Lambda_1(\beta)$ is a G_1-space (see, for instance, [11]); we now apply Proposition 2.8 to $\Lambda_1(\beta)=\lambda(Q)$.

Corollary 2.9. *$\Lambda_1(\beta)$ is $\tilde{\Lambda}_k(\alpha)$-nuclear $\Leftrightarrow$ $(\alpha_n/\beta_n)\in c_0$.*

Proof. ($\Rightarrow$) If $\Lambda_1(\beta)$ is $\tilde{\Lambda}_k(\alpha)$-nuclear then by Proposition 2.8, $(R^{\beta_n})\in\Lambda_k(\alpha)$ for each $R<1$ and this implies $(\alpha_n/\beta_n)\to 0$.

($\Leftarrow$) $(\alpha_n/\beta_n)\to 0$ implies, by Grothendieck–Pietsch criterion that $\Lambda_1(\beta)$ is nuclear and so $\sum 1/T^{\beta_n}<\infty$ for each $T>1$. Now given $R>1$ and given S, $1<S<k$, choose R', $1<R'<R$, and n_0 such that $S^{\alpha_n/\beta_n}<R'$ for $n>n_0$. Then

$$\sum_{n_0}^{\infty}\frac{S^{\alpha_n}}{R^{\beta_n}}<\sum_{n_0}^{\infty}\left(\frac{R'}{R}\right)^{\beta_n}<\infty \quad\text{and so}\quad \left(\frac{1}{R^{\beta_n}}\right)\in\Lambda_k$$

and by Proposition 2.6 the proof is complete.

Remark 2.10. In both Corollaries 2.7 and 2.9 the condition on α_n/β_n is independent of the index k and so we have, in fact, that $\Lambda_\infty(\beta)$ or $\Lambda_1(\beta)$ is $\tilde{\Lambda}_k(\alpha)$-nuclear for some $k>1$ if and only if it is $\Lambda_N(\alpha)$-nuclear.

The next propotion extends the above remark.

Proposition 2.11.

(i) *If the Köthe space $\lambda(P)$ is uniformly $\tilde{\Lambda}_{k_0}(\alpha)$-nuclear for some $k_0>1$ then it is $\Lambda_N(\alpha)$-nuclear.*

(ii) *If the G_1-space $\lambda(Q)$ is $\tilde{\Lambda}_{k_0}(\alpha)$-nuclear for some $k_0>1$ then it is $\Lambda_\infty(\alpha)$-nuclear.*

Proof. The proof of (i) follows from Proposition 2.1, Definition 2.2 and the fact that if $a,b\in\Lambda_{k_0}(\alpha)$ then $a\cdot b=(a_nb_n)\in\Lambda_{\sqrt{k_0}}(\alpha)$.

Proof of (ii). By part (i) and Proposition 2.8, $Q\subset\Lambda_l(\alpha)$, for each $l>1$ and so $Q\subset\Lambda_\infty(\alpha)$; then $\lambda(Q)$ is $\Lambda_\infty(\alpha)$-nuclear by a known theorem ([12], Theorem 3.2).

The following proposition is a summary of the results of Corollaries 2.7, 2.9 and Remark 2.10 of this paper, Corollary 3.3 of [12], Propositions 3.3, 3.4 of Robinson [9] and the result of Dubinsky [1].

Proposition 2.12.

(i) $\dfrac{\alpha}{\beta}\in l_\infty$ $\Leftrightarrow\Lambda_\infty(\beta)$ *is $\Lambda_N(\alpha)$-nuclear*

$\Leftrightarrow\Lambda_\infty(\beta)$ *is $\Lambda_1(\alpha)$-nuclear*

$\Leftrightarrow\Lambda_1(\beta)$ *is $\Lambda_1(\alpha)$-nuclear;*

(ii) $\dfrac{\alpha}{\beta}\in c_0$ $\Leftrightarrow\Lambda_1(\beta)$ *is $\Lambda_N(\alpha)$-nuclear*

$\Leftrightarrow\Lambda_1(\beta)$ *is $\Lambda_\infty(\alpha)$-nuclear*

$\Leftrightarrow\Lambda_\infty(\beta)$ *is $\Lambda_\infty(\alpha)$-nuclear.*

Corollary 2.13.

(i) *$\Lambda_\infty(\alpha)$ is $\Lambda_N(\alpha)$-nuclear;*

(ii) *$\Lambda_\infty(\alpha)$ is not $\Lambda_\infty(\alpha)$-nuclear.*

The above corollary reveals that $\Lambda_\infty(\alpha)$-nuclearity is a stronger assertion that $\Lambda_N(\alpha)$-nuclearity.

Motivated by Proposition 2.11 we ask the question: Is there a locally convex space E which is $\Lambda_k(\alpha)$-nuclear for some $k>1$ but is not $\Lambda_N(\alpha)$-nuclear? At this time we only have a partial answer to the above question.

Proposition 2.14. *Suppose there exists a ϱ, $0<\varrho<2$, such that $\alpha_{2n+1}\leqslant\varrho\alpha_n$ for sufficiently large n. Then if E is $\tilde{\Lambda}_k(\alpha)$-nuclear for some $k>1$ then it is $\Lambda_N(\alpha)$-nuclear.*

Proof. Without loss of generality we assume $\alpha_{2n+1}\leqslant\varrho\alpha_n$ $\forall n$. Clearly it is sufficient to show that for each $U\in\mathscr{U}(E)$ there exists a $W\in\mathscr{U}(E)$ such that $\big(\delta_n(W,U)\big)\in\Lambda_{k^{2/\varrho}}(\alpha)$. Given U, find V, W such that $\big(\delta_n(V,U)\big)\in\Lambda_k$ and $\big(\delta_n(W,V)\big)\in\Lambda_k$. Then for $1<R<k$,

$$\sum_{n=0}^{\infty}\delta_n(W,U)R^{(2/\varrho)\alpha_n}\leqslant 2\sum_{n=0}^{\infty}\delta_{2n}(W,U)R^{(2/\varrho)\alpha_{2n+1}}$$

since

$$\delta_{2n}(W,U)\geqslant\delta_{2n+1}(W,U)\quad\text{and}\quad R^{\alpha_{2n+1}}\geqslant R^{\alpha_{2n}}.$$

Since

$$\delta_{2n}(W, U) \leqslant \delta_n(W, V)\delta_n(V, U) \quad \text{and} \quad R^{(2/\varrho)a_{2n+1}} \leqslant R^{2a_n},$$

we have

$$\sum \delta_n(W, U)R^{(2/\varrho)a_n} \leqslant 2\sum \delta_n(W, V)R^{a_n}\delta_n(V, U)R^{a_n} < \infty.$$

This completes the proof.

Note. The condition $a_{2n+1} \leqslant \varrho_{a_n}$, $0 < \varrho < 2$, for large n implies that (a_n) is stable, [2].

We consider next $\tilde{\Lambda}_k(a)$-nuclearity and duals of locally convex spaces. We first prove

PROPOSITION 2.15. *The strong dual E_b' of a locally convex E is $\tilde{\Lambda}_k(a)$-nuclear $\Leftrightarrow$ for each bounded set A in E there exists another, B, such that $A < B$ and $(\delta_n(A, B)) \in \Lambda_k$.*

Proof. ($\Leftarrow$) Given $U \in \mathscr{U}(E_b')$, find bounded set A in E such that $A^0 \subset U$ and B such that $\delta_n(A, B) \in \Lambda_k$. Set $V = B^0 \in \mathscr{U}(E_b')$. Then

$$\delta_n(U^0, V^0) \leqslant \delta_n(A^{00}, B^{00}) \leqslant \delta_n(A, B).$$

Now the proof follows from Proposition 1.2.

($\Rightarrow$) Given A, choose $U, V \in \mathscr{U}(E_b')$ so that $A^0 \subset U$ and E_V', E_U' are pre-Hilbert spaces and the canonical map $\hat{E}_V' \to \hat{E}_U'$ is $\tilde{\Lambda}_k$-nuclear. Then, by Lemma 1.1, its adjoint map is also $\tilde{\Lambda}_k$-nuclear and its restriction to E_{U^0}, which is the canonical imbedding of E_{U^0} in E_{V^0}, is quasi-Λ_k-nuclear. Since $\hat{E}_{U^0}$ and $\hat{E}_{V^0}$ are Hilbert spaces, this map is, by Lemma 1.1, $\tilde{\Lambda}_k$-nuclear. Let now $C = U^0$ and $B = V^0$. Then, by Lemma 1.1, $\delta_n(C, \hat{B}) \leqslant \lambda_n$ where $(\lambda_n) \in \Lambda_k$ and $\hat{B}$ is the closure of B in $\hat{E}_B$, i.e., the unit ball of E_B. Thus, for each $\varepsilon > 0$, there exist $Z_1, Z_2, \ldots, Z_n \in \hat{E}_B$ such that

$$C \subset (\lambda_n + \varepsilon)\hat{B} + \Gamma\{Z_1, \ldots, Z_n\}$$

(as shown in [11]). Choose next $x_1, \ldots, x_n \in E_B$ so that

$$(x_i - Z_i) \in \frac{\varepsilon}{2^i}\hat{B}.$$

Then

$$C \subset (\lambda_n + 2\varepsilon)\hat{B} + \Gamma\{x_1, \ldots, x_n\}.$$

If $L = \operatorname{span}\{x_1, \ldots, x_n\} \subset E_B$, then for each $x \in C$ there is a $y \in L$ such that $p_B(x-y) \leqslant \lambda_n + 2\varepsilon$. Hence $C \subset (\lambda_n + 2\varepsilon)B + L$ so that $\delta_n(C, B) \leqslant \lambda_n$. But $\delta_n(A, B) \leqslant \delta_n(C, B)$, since $A \subset C$. Thus $\delta_n(A, B) \in \Lambda_k$.

PROPOSITION 2.16. *If E or E_b' is $\Lambda_N(a)$-nuclear then for each bounded set $A \subset E$ and each $U \in \mathscr{U}(E)$, $\delta_n(A, U) \in \Lambda_\infty(a)$.*

Proof. Suppose E in Λ_N-nuclear; then given $k > 1$, and $U \in \mathscr{U}$, find $V_k \in \mathscr{U}$ such that $(\delta_n(V_k, U)) \in \Lambda_k$. Since $\delta_n(A, U) \leqslant \delta_0(A, V_k)\,\delta_n(V_k, U)$, we get $(\delta_n(A, U)) \in \Lambda_k$. This being true for each $k > 1$, we get the result claimed.

If E_b' is Λ_N-nuclear a similar proof can be given using Proposition 2.15.

We conclude this section with the following result on the Λ_N-nuclearity of the dual of a uniformly $\tilde{\Lambda}_k$-nuclear Köthe space $\lambda(P)$.

PROPOSITION 2.17. *Suppose the Köthe space $\lambda(P)$ is uniformly $\tilde{\Lambda}_k(a)$-nuclear for some $k > 1$; then its strong dual $[\lambda(P)]_b'$ is a dense subspace of a uniformly $\Lambda_l(a)$-nuclear space $\lambda(L)$, where $1 < l < k$; hence $[\lambda(P)]_b'$ is $\Lambda_N(a)$-nuclear.*

Proof. Let $L = \{x \in \lambda(P)\colon x_n \geqslant 0\}$. By Köthe's lemma [5], $[\lambda(P)]_b'$ is a dense subspace of $\lambda(L)$. Since $\lambda(P)$ is uniformly $\tilde{\Lambda}_k$-nuclear, there exists a permutation σ such that for each $a \in P$ there exists a $b \in P$ and $(d_n) \in \Lambda_k$ so that $a_{\sigma(n)} \leqslant d_n b_{\sigma(n)}$. Let $x \in L$. Then

$$\sup l^{a_n} x_{\sigma(n)} a_{\sigma(n)} \leqslant \sup l^{a_n} d_n \sup x_{\sigma(n)} b_{\sigma(n)} < \infty.$$

Hence if $y_n = l^{a_{\sigma^{-1}(n)}} x_n$, then $(y_n) \in L$; also $x_{\sigma(n)} = y_{\sigma(n)}/l^{a_n}$. But $(1/l^{a_n}) \in \Lambda_l(a)$. So $\lambda(L)$ is uniformly $\tilde{\Lambda}_l$-nuclear and hence, by Proposition 2.11(i), is $\Lambda_N(a)$-nuclear.

COROLLARY 2.18. *Let $\lambda(P)$ be a barrelled, nuclear, Köthe space. Then $\lambda(P)$ is uniformly $\tilde{\Lambda}_k$-nuclear if and only if $[\lambda(P)]_b'$ is uniformly $\tilde{\Lambda}_k$-nuclear.*

Proof. Let L be as in the above proof. Since $\lambda(P)$ is barrelled and nuclear, $[\lambda(P)]_b' = \lambda(L)$; since $\lambda(P)$ is reflexive, $[\lambda(L)]_b' = \lambda(P)$ and so by the above proposition, $[\lambda(L)]_b'$ is uniformly $\tilde{\Lambda}_k$-nuclear if $\lambda(L)$ is so.

3. Universal $\Lambda_N(a)$-nuclear space. Let $a = (a_n)$ be a fixed exponent sequence so that $\Lambda_k(a)$ is a nuclear space (for each k). Proposition 1.4 gives us that subspaces, quotients and biduals of Λ_N-nuclear spaces are also Λ_N-nuclear. We know also from [2] and [8] that $([\Lambda_\infty(a)]_b')^I$ is a universal $\Lambda_\infty(a)$-nuclear space whenever $(a_{2n}/a_n) \in l_\infty$ or equivalently a is a stable exponent sequence and that $\Lambda_\infty(a)$ is not $\Lambda_\infty(a)$-nuclear. So we ask now what the model of a universal $\Lambda_N(a)$-nuclear space is. Our main result in this section is that whenever a is a stable exponent sequence, $\Lambda_\infty(a)$ is a single generator for the variety of $\Lambda_N(a)$-nuclear spaces.

We start with the following lemma, ([2]), Theorem 2.10)

LEMMA 3.1. *For the exponent sequence a_n, $\sup(a_{2n}/a_n) < \infty \Leftrightarrow$ there exists a bijection $\beta\colon \boldsymbol{N}\to\boldsymbol{N}\times\boldsymbol{N}$ such that for each $k\in\boldsymbol{N}$,*

$$\sup_m \frac{a_{\beta^{-1}(k,m)}}{a_m} < \infty.$$

We now refer the reader to [2] and particularly to the details of the proofs of Lemma 2.7, Theorem 2.8 and Theorem 2.9. In the proof of the proposition stated below we shall give part of its proof in as far it is different from that of Theorem 2.8 of [2] and leave the rest of the routine details.

PROPOSITION 3.2. *Suppose a is a stable exponent sequence, i.e. $\sup(a_{2n}/a_n) < \infty$. Then* (i) *closed subspaces,* (ii) *quotients by closed subspaces,* (iii) *completions,* (iv) *natural biduals,* (v) *countable direct sums, and* (vi) *arbitrary products, of $\Lambda_{\boldsymbol{N}}(a)$-nuclear spaces are also $\Lambda_{\boldsymbol{N}}(a)$-nuclear.*

Proof. We shall now indicate a partial proof of (v) above.

Let E_k, $k\in\boldsymbol{N}$ be $\Lambda_{\boldsymbol{N}}(a)$-nuclear and $E = \bigoplus_{k=1}^{\infty} E_k$. Consider in E the fundamental system of neighborhoods of the form $U = \Gamma\big((U_k)_{k\in\boldsymbol{N}}\big)$ where each U_k is a barrelled neighborhood of 0 in E_k and Γ represents the closed convex hull of the union.

Now by the stability of (a_n) and Lemma 3.1, we get $c_k > 0$, $k\in\boldsymbol{N}$, such that $a_{\beta^{-1}(k,m)} \leqslant c_k a_m$ for each k, $m\in\boldsymbol{N}$. Let $d_k = [c_k]+1$ and $p_k = (k+1)^{d_k}$; let $\varrho_k = k^{c_k} < p_k$.

Let $R > 1$ be given; R is fixed; for each $k\in\boldsymbol{N}$ and $k \geqslant [R]+1$ we can obtain from the hypothesis that each E_k is a $\Lambda_{\boldsymbol{N}}(a)$-nuclear space, a barrelled neighborhood W_k of 0 in E_k, $\xi^k\in\Lambda_{p_k}(a)$ and sequences $(\tilde a^k_m)_m$ in the unit ball of $\big(\widehat{(E_k)}_{W_k}\big)'$ and $(y^k_m)_m$ in the unit ball of $\widehat{(E_k)}_{U_k}$ such that the canonical map $\hat K_k\colon \widehat{(E_k)}_{W_k}\to\widehat{(E_k)}_{U_k}$ can be represented by

$$\hat K_k(\hat x_k) = \sum_m \xi^k_m\langle \hat x_k, \tilde a^k_m\rangle y^k_m.$$

Now for each $k\leqslant [R]$, $k\in\boldsymbol{N}$, get ξ^k, $(\tilde a^k_m)$, (y^k_m) as above except that $\xi_k\in\Lambda_{q_k}$, $q_k = ([R]+1)^{c_k}$; if now $P_R(x)$ denotes $\sum_{n=1}^{\infty}|x_n|R^{a_n}$, choose (t_k), $k = 1, 2, \ldots$, $t_k > 0$ such that

$$P_{\varrho_k}(t_k\xi^k) \leqslant \frac{1}{2^k} \quad \text{for} \quad k\geqslant [R]+1$$

and

$$P_{R^{c_k}}(t_k\xi^k) \leqslant \frac{1}{2^k} \quad \text{for} \quad k\leqslant [R].$$

Now using the notation that for the bijection $\beta\colon \boldsymbol{N}\to\boldsymbol{N}\times\boldsymbol{N}$, $\beta(n) = \big(\beta_1(n), \beta_2(n)\big)$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} t_{\beta_1(n)}|\xi^{\beta_1(n)}_{\beta_2(n)}|R^{a_n} &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} t_k|\xi^k_m|R^{a_{\beta^{-1}(k,m)}}\\ &\leqslant \sum_{k=1}^{\infty} t_k\sum_{m=1}^{\infty}|\xi^k_m|R^{c_k a_m}\\ &\leqslant \sum_{k=1}^{[R]} t_k P_{R^{c_k}}(\xi^k) + \sum_{k=[R]+1}^{\infty} t_k \sum_{m=1}^{\infty}|\xi^k_m|k^{c_k a_m}\\ &= \sum_{k=1}^{[R]}\frac{1}{2^k} + \sum_{k=[R]+1}^{\infty} t_k P_{\varrho_k}(\xi_k) < \infty.\end{aligned}$$

Thus, we have shown that $\big(t_{\beta_1(n)}\xi^{\beta_1(n)}_{\beta_2(n)}\big)\in\Lambda_R(a)$. If we now construct, for each k, $V_k = t_kW_k$, $a^k_m = \tilde a^k_m/t_k$ and $V = \Gamma\big((V_k), k\in\boldsymbol{N}\big)$ we can show as in the proof of Theorem 2.8 of [2] that the canonical map $\hat T\colon \hat E_V\to\hat E_U$ is $\tilde\Lambda_R(a)$-nuclear; therefore E is $\tilde\Lambda_R(a)$-nuclear and since we can do this for each $R > 1$, E is $\Lambda_{\boldsymbol{N}}(a)$-nuclear.

We isolate the following corollary because of its significance in what follows later.

COROLLARY 3.3. *The arbitrary I-fold product $[\Lambda_\infty(a)]^I$ is $\Lambda_{\boldsymbol{N}}(a)$-nuclear, whenever a is stable.*

Our next result is in the direction of obtaining $\Lambda_\infty(a)$ as the single generator for the variety of $\Lambda_{\boldsymbol{N}}(a)$-nuclear spaces whenever a is stable. It is modelled after Komuras's imbedding theorem for nuclear spaces [3].

PROPOSITION 3.4. *Let $a = (a_n)$ be a stable exponent sequence and $\Lambda_\infty(a)$ be a nuclear space. Suppose $\big(\Lambda_\infty(a)\big)^\times \subset \Delta(E)$. Then E is isomorphic to a subspace of $[\Lambda_\infty(a)]^I$ for a suitable I.*

Proof. Let $k\geqslant 1$ be fixed. Since a is stable and $\big(\Lambda_\infty(a)\big)^\times \subset \Delta(E)$, it follows that $(k^{a_{2^k n}})_n\in\Delta(E)$. By hypothesis, clearly E is nuclear and so there exists an absolutely convex closed neighborhood U in E such that E'_{U^0} is a Hilbert space; now by Proposition IV.1 of [11] there exists an orthonormal basis $(e^k_n)_n$ of E'_{U^0} so that the set A_k,

$$A_k = \Big\{\sum_n \xi_n k^{a_{2^k n}} e^k_n\colon \sum_n|\xi_n|^2\leqslant 1\Big\}$$

is an equicontinuous subset of E'.

Order the set $\{e^k_n\colon k, n = 1, 2, \ldots\}$ into a sequence by using the bijection $\beta\colon \boldsymbol{N}\to\boldsymbol{N}\times\boldsymbol{N}$ defined by $\beta^{-1}(k, n) = 2^{k-1}(2n-1)$; apply the Gram–Schmidt process to this sequence to obtain a new orthonormal basis (e_m) of E'_{U^0}.

Fix $k \in N$; if $m, n \in N$ are such that $2^k n < m$, then $\langle e_m, e_n^k \rangle = 0$ since $2^{k-1}(2n-1) < m$; expanding e_m in terms of $(e_n^k)_n$ we then have

$$e_m = \sum_{n \geqslant m/2^k} (e_m, e_n^k) e_n^k.$$

Now, from the inequality

$$\sum_{n \geqslant m/2^k} \frac{|(e_m, e_n^k)|^2}{(k^{a_{2^k n}})^2} \leqslant \Big(\sum_{n=1}^{\infty} |(e_m, e_n^k)|^2 \Big) \frac{1}{(k^{a_m})^2} \leqslant \frac{1}{(k^{a_m})^2}.$$

We get $k^{a_m} e_m \in A_k$.

Thus we have shown that there exists an orthonormal basis (e_m) of E'_{U^0} such that $\{k^{a_m} e_m:\ m = 1, 2, \ldots\}$ is equicontinuous in E', for each fixed k.

Now let $\mathscr{U} = \{U_i:\ i \in I\}$ be a base of neighborhoods of E such that each U_i is a barrel and $E'_{U_i^0}$ is a Hilbert space. Choose an orthonormal basis (e_m^i) of $E'_{U_i^0}$ such that the sets

$$(*) \qquad B_{i,k} = \{k^{a_m} e_m^i:\ m = 1, 2, \ldots\}.$$

are equicontinuous for each fixed $k \geqslant 1$; for each $i \in I$, define the map $T_i: E \to \Lambda_\infty(a)$ by $T_i x = (\langle x, e_m^i \rangle)_m$; from (*) it follows that $T_i x \in \Lambda_\infty(a)$ and T_i is continuous. Define next $T: E \to [\Lambda_\infty(a)]^I$ by $Tx = (T_i x)_{i \in I}$. Then T is continuous and one-one.

We shall complete the proof by showing that $T^{-1}: T(E) \to E$ is continuous. Let $V = \{y = (y^i)_{i \in I} \in [\Lambda_\infty(a)]^I:\ \sup_n R^{a_n} |y_n^j| \leqslant 1,\ j \in I$ is fixed and for $i \neq j$, $y^i \in \Lambda_\infty(a)$ is arbitrary, where R is chosen so that $R > 1$ and $\sum R^{-2a_n} < 1\}$. Then V is a neighborhood of 0 in $[\Lambda_\infty(a)]^I$. If $Tx \in V$ then $\sup_n R^{a_n} |\langle x, e_n^j \rangle| \leqslant 1$. If now $u \in U_j^0$, then

$$|u(x)| = \Big| \sum_n \langle e_n^j, u \rangle \langle x, e_n^j \rangle \Big| \leqslant \Big(\sum_n |\langle e_n^j, u \rangle|^2 \Big)^{1/2} \Big(\sum_n |\langle x, e_n^j \rangle|^2 \Big)^{1/2}$$

$$\leqslant 1 \sum \frac{1}{R^{2a_n}} < 1,$$

so that $x \in U_j$ and $V \cap T(E) \subset U_j$; so T^{-1} is continuous.

Corollary 3.5. *Suppose a is stable. Each $\Lambda_N(a)$-nuclear space E is isomorphic to a subspace of $[\Lambda_\infty(a)]^I$.*

Proof. E is $\Lambda_N(a)$-nuclear

$\Leftrightarrow$ for each $k > 1$, and each $U \in \mathscr{U}$, there exists a $V_k \in \mathscr{U}$ such that $(\delta_n(V_k, U)) \in \Lambda_k(a)$,

$\Leftrightarrow$ for each $R > 1$ and each $U \in \mathscr{U}$ there exists a $V = V_R \in \mathscr{U}$ such that $R^{a_n} \delta_n(V_R, U) \to 0$,

$\Leftrightarrow R^{a_n} \in \Delta(E)$ for each $R > 1$,

$\Leftrightarrow [\Lambda_\infty(a)]^\times \subset \Delta(E)$.

Now apply Proposition 3.4.

Corollary 3.6. (Komura–Komura [3]). *Each nuclear space E is isomorphic to a subspace of $(s)^I$, s being the space of rapidly decreasing sequences.*

Proof. Note that $s = \Lambda_\infty(a)$, $a_n = \log(n+1)$. The proof now follows from the fact that E is nuclear $\Leftrightarrow s^\times \subset \Delta(E)$, as shown in [11] (Prop. II 4.7).

Corollary 3.7. *Let a be stable. If the Fréchet space E is $\Lambda_N(a)$-nuclear then E'_b is $\Lambda_\infty(a)$-nuclear.*

Proof. E is a Fréchet, $\Lambda_N(a)$-nuclear space $\Rightarrow E'_b$ is isomorphic to a quotient space of the direct sum $\bigoplus_{n=1}^{\infty} (\Lambda_\infty(a))'_b$ and this direct sum is $\Lambda_\infty(a)$-nuclear whenever a is stable.

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