where C_{XY} is a connecting copula of X and Y (which may be chosen arbitrarily and may depend on X and Y).

Next, since $T \neq \mathrm{Min}$, there exist numbers a and b, $0 < a, b < 1$ such that $T(a, b) \neq \mathrm{Min}(a, b)$. Let X be a random variable assuming the value 0 with probability a and the value 1 with probability $1-a$. Then $F_X = G_a$. Similarly, let Y be such that $F_Y = G_b$. Then (3.16) yields:

$$\tau_T(G_a, G_b) = \sigma_{C_{ab}}(G_a, G_b), \tag{3.17}$$

where C_{ab} is some connecting copula for X and Y. Now, using (3.14) and (3.15), we have:

$$\tau_T(G_a, G_b)(1/2) = T(a, b) = \sigma_{C_{ab}}(G_a, G_b)(1/2) = C_{ab}(a, b),$$

and

$$\tau_T(G_a, G_b)(3/2) = \mathrm{Max}(a, b) = \sigma_{C_{ab}}(G_a, G_b)(3/2) = a+b-C_{ab}(a, b).$$

Thus,

$$\mathrm{Min}(a, b) \neq T(a, b) = C_{ab}(a, b) = a+b-\mathrm{Max}(a, b) = \mathrm{Min}(a, b).$$

This is a contradiction, whence g cannot exist, and the theorem is proved.

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Brownian motion, approximation of functions, and Fourier analysis

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Abstract. Quantitative approximation theory, initiated by Kolmogorov, is used to show almost-sure properties of *all* mappings $F \circ X$. Here X is Brownian motion, and F is a diffeomorphism of class Lip^a, for example. The problems considered touch on Hausdorff dimension, Kronecker sets, Salem sets, and Diophantine approximation. In some cases a critical exponent of smoothness can be found by the category method.

Introduction. In this paper we apply the quantitative approximation theory of Kolmogorov to certain questions on Fourier–Stieltjes transforms and Brownian motion. For example, let E be a compact subset of $(0, +\infty)$ of positive Hausdorff dimension; Kahane proved that $X(E)$ is an M_0-set for almost all Brownian paths X. Therefore the same is true of X of $f(E)$ whenever f is a C^1-diffeomorphism of $(0, +\infty)$ into itself. How large a class S of diffeomorphisms f can be named, so that $X \circ f(E)$ is an M_0-set for *all* f in S, almost surely? An answer is contained in the first chapter. A similar question for transforms $f \circ X(E)$ is considered next; for these sets we obtain strong bounds on certain Fourier transforms. Here matters become distinctly non-linear, but we obtain some precise estimates by simple devices.

In the course of the paper we refer to constructions and inequalities in scattered sources; we list some of these now, as a guide to the flavor of the work.

(a) Lipschitz spaces Λ^a and λ^a, and Kolmogorov's estimates of the sizes of sets in these spaces, under the name "ε-entropy" [13] and [16, 17, 18 Ch. 10].

(b) Hausdorff measures, Hausdorff dimension, and construction of special "dyadic" sets [6 I, II].

(c) Gaussian processes and Brownian motion [4 XI, XIV].

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(d) Special sets introduced in abstract harmonic analysis, in particular, Kronecker sets [3], [5, VIII, 14].

(e) Classical theorems on sets of uniqueness and sets of multiplicity [6, V], [23, IX (6, 7, 8)].

(f) Banach space methods in the construction of examples to (d) and (e) [5, VII, 8, 9, 12].

I. THEOREM 1. *Let E be a compact subset of $(0, +\infty)$ of positive Hausdorff β-measure $0<\beta<1$, and let S be the class of diffeomorphisms of $[0, +\infty)$ into itself of class $\Lambda^{1/\beta}$. Then it is almost sure that all sets $X\circ f(E)$ are M_0-sets, $f\in S$.*

To each β in $(0,1)$ there is a compact set F of dimension β, with this property: for almost all paths X, there is a random mapping f_1 of class $\lambda^{1/\beta}\cap S$, such that $X\circ f_1(F)$ is a Kronecker set.

In explanation of Theorem 1, we recall Frostman's theorem on Hausdorff measures [6, I, II]; in consequence of this theorem E carries a probability measure μ whose primitive belongs to Λ^{β}; $\mu(a, a+h)\leqslant h^{\beta}$ for all intervals $(a, a+h)$. As the conclusion is rather trivial if E has positive Lebesgue measure, we suppose the opposite. The class S is the union of sets $\bigcup_1^{\infty} S_n$, each bounded in the space $\Lambda^{1/\beta}$ on an interval $[0, L]$ containing E; moreover $f'>n^{-1}$ on $[0, L]$ for each f in S_n.

1. For each f mapping $[0, +\infty)$ into itself, there is a probability measure $\sigma\equiv\sigma(X,f)$ carried by $X\circ f(E)$; its transform is

$$\hat{\sigma}(u)=\int\exp-2\pi iuX\big(f(t)\big)\mu(dt),\qquad -\infty<u<\infty.$$

We shall prove that $\hat{\sigma}(u)\to 0$ as $u\to+\infty$ along the sequence $1, \sqrt{2}, \ldots$ $\ldots, k^{1/2}, \ldots,$ *uniformly* for all f in S_n. For this we require a very precise bound for the moments of $\hat{\sigma}(u)$ [4, p. 168]

$$E\big(|\hat{\sigma}(u)|^{2q}\big)<(C_n q u^{-2\beta})^{q}, \tag{1}$$

wherein $q=1, 2, 3, \ldots$ and C_n is valid for all f in S_n. To obtain this estimate from the one cited, we have merely to observe that the probability measures $\mu\circ f^{-1}$, with $f'\geqslant n^{-1}$, are subject to a uniform Lipschitz condition in dimension β. For fixed $\delta>0$, inequality (1) leads to

$$P\big(|\sigma(u)|>\delta\big)\leqslant\exp-C_n' u^{-2\beta}. \tag{2}$$

2. To each η in $(0,1)$ we choose a finite set $T_n(\eta)\subseteq S_n$, such that each function f in S_n has distance $<\eta$ from $T_n(\eta)$, in the *uniform metric over E*. From Kolmogorov's estimates [13], [18, Ch. 10] it is known that the cardinality $|T_n(\eta)|$ can be brought down to $\exp C_n''\eta^{-\beta}$; but E has measure 0, so the observations of Vosburg [22] allow a bound of the form $\exp\eta^{-\beta}\varphi_n(\eta)$, $\varphi_n(0+)=0$. Thus we can define a function $q(\eta)$ such that $q(\eta)=o(\eta)$ as $\eta\to 0+$, while $\log\big|T_n\big(q(\eta)\big)\big|=o(\eta^{-\beta})$. Let us write $T_n^*(\eta)\equiv T_n\big(q(\eta)\big)$.

Comparing the size of $T_n^*(u^{-2})$ with our estimate for $P(|\hat{\sigma}(u)|>\delta)$ we can conclude that almost surely we have

$$\sup{}^*\Big|\int\exp-2\pi i k^{1/2}X\big(f(t)\big)\mu(dt)\Big|\to 0,$$

where $\sup^*$ means the supremum for f in $T_n^*(u^{-2})$.

3. In the last step of the proof we need controls on the oscillation of X; these do not follow from Lévy's celebrated work. Let us write g for an element of $T_n^*(u^{-2})$ and f for an element S_n, such that $|f(t)-g(t)|\leqslant q(u^{-2})$ uniformly in E. We seek a bound for

$$\int\min\Big(1, \big|uX\big(f(t)\big)-uX\big(g(t)\big)\big|\Big)\mu(dt).$$

Before passing to this final part of the proof, we notice that a bound $o(1)$, valid uniformly with respect g in $T_n^*(u^{-2})$, as $u\to+\infty$ through values $k^{1/2}$, will prove the first assertion in Theorem 1.

Let us fix g and estimate the μ-measure of the set on the t-axis B_1: X oscillates more than $u^{-1}\delta$ on the interval $[g(t)-q(u^{-2}), g(t)+q(u^{-2})]$. This is nothing but the $\mu'=\mu\circ g^{-1}$ measure of the set on the s-axis B_2: X oscillates more than $u^{-1}\delta$ on the interval $[s-q(u^{-2}), s+q(u^{-2})]$.

Let us divide the s-axis into adjacent intervals $(I_p)_{-\infty}^{\infty}$ of length exactly $q(u^{-2})$ and let Σ^* be the sum $\{\Sigma\mu(I_p)$: X oscillates by more than $u^{-1}\delta$ over $I_p\}$. Now Σ^* is a sum of independent random variables y_p, such that $0\leqslant y_p\leqslant\mu'(I_p)$ and $P(y_p\neq 0)<c(u)\to 0$, because $q(u^{-2})=o(u^{-2})$. Let λ be the positive number defined by the inequality $\lambda\max\mu'(I_p)=1$. Elementary inequalities yield $E(\exp\lambda\Sigma^*)\leqslant\exp\frac{1}{2}\lambda c(u)$, $P(\Sigma^*>\delta)\leqslant\exp\frac{1}{2}\lambda[c(u)-2\delta]$. To finish the argument we recall that all the measures $\mu'=\mu\circ g^{-1}$ satisfy a uniform Lipschitz condition in exponent β, so $\mu'(I_p)\leqslant|I_p|^{\beta}=o(u^{-2\beta})$; thus $P(\Sigma^*>\delta)<\exp-u^{+2\beta}$ for large u, while $\log|T_n^*(u^{-2})|=o(u^{+2\beta})$. Setting $u=k^{1/2}$, we obtain a convergent sequence for each $\delta>0$. The same argument applies of course to the oscillation of $X\big(t+q(u^{-2})\big)$ and $X\big(t-q(u^{-2})\big)$. With the aid of a simple diagram, we see that the μ' measures of sets B_2 tend to 0, uniformly with respect to g in $T_n^*(u^{-2})$.

A moment's reflection shows that it would be sufficient to obtain the bounds of this paragraph along the sequence $u=2^{-k}$, and for this we need much weaker bounds on μ, for example $\mu(a, a+h)\leqslant(\log\log h^{-1})^{-1}$ for $0<h<e^{-e}$.

4. The set F is a "dyadic" set determined by a strictly increasing sequence $M=(m_k)_1^{\infty}$ of positive integers: F is the set of all sums $\Sigma\varepsilon_k 2^{-m_k}$, $\varepsilon_k=0$ or 1. The sequence M must have two properties

(i) $m_k \leqslant \beta^{-1}k + o(k)$ for all $k \geqslant 1$,

(ii) $m_{k+1} > \beta^{-1}k + 10\log k$, for all k in an infinite sequence N of positive integers.

Property (i) ensures that F has Hausdorff dimension β, and indeed F cannot be expressed as a countable union of sets F_i with $\dim F_i < \beta$. Let Y be the Banach space $\lambda^{1/\beta}[0,1]$, or $Y = C^r[0,1]$ in case $\beta^{-1} = r$, an integer. Let Y^+ be the open subspace of Y containing positive functions; Y^+ like Y is separable. The second part of Theorem 1 is a consequence of this statement:

For all paths X except a set of probability 0, the set of functions $f \in Y^+$ such that $X \circ f(F)$ is a Kronecker set, is a dense G_δ-set in Y.

Observing that Y^+ contains an open set Y_0^+, defined by the inequalities $f > 0$, $f' > 0$ on $[0,1]$, and that each element of Y_0^+ admits an extension to a diffeomorphism of class $\lambda^{1/\beta}$ on $[0, +\infty)$, we obtain the asserted properties of f.

Let V be an open set in Y^+, g a continuous real function on $[0,1]$ and $\varepsilon > 0$. We shall prove that for almost all X, there is a function f in V, and a number $u > 0$, such that $|uX \circ f(t) - g(t)| < \varepsilon$ (modulo 1) for all t in F. Since $C[0,1]$ and $\lambda^{1/\beta}$ are separable, this leads by a familiar path to the result stated above [8, 9, 12].

5. Let $k \in N$ so that $m_{k+1} > \beta^{-1}k + 10\log k > m_k + 1$, and let $F = \bigcup F_v$ be the splitting of F determined by $\varepsilon_1, \ldots, \varepsilon_k$. Thus each F_v has diameter $\leqslant a_k = 2 \cdot 2^{-m_{k+1}}$ and the sets F_v have mutual distances $\geqslant b_k = 2^{-1}2^{-m_k}$. For large k we find $b_k \geqslant a_k^\beta k^6$. We choose a function f_0 in $V \subseteq Y^+$ and elements x_v of F_v.

We assume now that the oscillation of $X(t)$ over each t-interval $|t - f_0(x_v)| \leqslant C_k = k^{-1}b_k^{1/\beta}$ exceeds $d_k = k^{-2}b_k^{1/2\beta}$. Later we show that this holds for sufficiently large $k \in N$, almost surely. Setting $u = d_k^{-1}$, we can then choose numbers $f(x_v)$ such that $|f(x_v) - f_0(x_v)| \leqslant C_k$ while $uX \circ f(x_v) \equiv g(x_v)$ (modulo 1), $1 \leqslant v \leqslant 2^k$. Now let f in $\lambda^{1/\beta}$ be defined so that it takes the specified values $f(x_v)$ and $f - f_0$ is constant over each F_v. This can be done with a function f such that $\|f - f\| \ll C_k(1 + b_k^{-1/\beta}) = o(1)$ [8]; thus for large k $f > 0$ and $X \circ f$ is defined on F, while $f \in V \subseteq Y^+$. Moreover, f oscillates no more than f_0 on each F_v, thus f oscillates $\leqslant C(f_0)a_k$ there. By Lévy's estimates for the modulus of continuity of X, we see that almost surely $X \circ f$ oscillates $\leqslant C(f_0)(a_k \log a_k^{-1})^{1/2} \ll C(f_0)a_k^{1/2}k^{1/2}$ (because $m_{k+1} \ll k$). To obtain a uniform estimate for $uX \circ f(t) - g(t)$ (modulo 1), over $F = \bigcup F_v$, we have only to verify that $ua_k^{1/2}k^{1/2} \to 0$, or $a_k k = o(d_k^2)$. But $d_k^2 = k^{-4}b_k^{1/\beta}$, while $b_k = a_k^\beta k^6$, and all is proved.

Returning to the oscillation of X over an interval of length C_k, we must estimate the probability of the event $|X(t)| \leqslant k^{-3/2}$, $0 \leqslant t \leqslant 1$. This inequality implies $|X(mk^{-3}) - X(mk^{-3} + k^{-3})| \leqslant 2k^{-3/2}$ for integers $m \geqslant 0$, $m \leqslant k^3 - 1$. Each of these inequalities has $P = C_1 < 1$, so their intersection has $P < \exp - C_2 k^3$. Moreover, the number of intervals in question is 2^k, and $\sum 2^k \exp - C_2 k^3 < \infty$, so the necessary oscillations are obtained almost surely for all large k.

There is an odd variant of the theorem above whose proof is almost identical with the one just completed: everything can be accomplished in the set Y^{++} defined by $f > 0$ and $f' = 1$ on F. This shows that Y^{++} is almost as massive as Y^+, as a subset of $C(F)$.

II. Let us say that a set E has C^{1+}-multiplicity if $X \circ f(E)$ is an M_0-set for every f in any space Λ^a, $a > 1$, with $f > 0$, $f' > 0$ on $[0, \infty)$.

THEOREM 2. *Let E be the dyadic set based on a sequence M such that $m_{k+1} = m_k + o(m_k/\log m_k)$. Then E has C^{1+}-multiplicity.*

The cited condition on the sequence M is simply what is required to balance an inequality in the course of the proof; its interest is in examples leading to dyadic sets of Hausdorff dimension 0, for example, $m_k = k^2$. The method of proof also applies to sequences M with positive density d: $\lim k^{-1}m_k = d^{-1} < \infty$.

For large numbers $u > 1$ we define $k = k(u)$ by the inequalities $2^{-m_{k+1}} \leqslant u^{-2} < 2^{-m_k}$. Thus, if β is fixed in $(0,1)$ then, since $m_{k+1}/m_k \to 1$, $m(k(u)) \cong m(k(u^\beta)) \cdot \beta^{-1}$, and the hypotheses on $m(k)$ yield $[k(u) - k(u^\beta)]/\log m(k(u^\beta)) \to +\infty$.

To apply these calculations we take a bounded subset $S \subseteq \lambda^a[0,1]$, and choose β so that $1 < a\beta < a$. The product measure on E admits a factorization for each $u > 1$: in μ_1 we group the factors corresponding to indices k in $[1, k(u^\beta)]$, and in μ_2 we take indices $k > k(u^\beta)$. An interval of length u^{-2} has μ_2-measure $2^{k(u)} - 2^{k(u^\beta)}$, and the support of μ_2 has length $L(u) = 2 \cdot 2^{m(k(u^\beta)+1)} < 2u^{-2\beta}$.

The integral

$$\int \exp - 2\pi i u X \circ f(t) \mu(dt)$$

can be bounded by the maximum of integrals

$$\int_0^L \exp - 2\pi i u X \circ g(t) \mu_2(dt) \equiv I(g, u),$$

say, where the functions g are subject to inequalities

$$|g| \leqslant B, \quad |g'| \leqslant B, \quad |g'(x) - g'(y)| \leqslant B|x - y|^{a-1},$$

and $g' \geqslant C > 0$ as well if we impose the last inequality on the functions in S. On $[0, L]$ we have by the mean-value theorem, and $a\beta > 1$,

$$g(t) = g(0) + tg'(0) + O(t^a) = g(0) + tg'(0) + O(u^{-2}u^{-\delta}),$$

$$X \circ g(t) = X(g(0) + tg'(0)) + o(u^{-1}), \quad \text{almost surely},$$

by Lévy's bounds on the modulus of continuity. Moreover, if $|g(0)-g^*(0)| < u^{-3}$ and $|g'(0)-g^{*\prime}(0)| < u^{-3}$, then $|g-g^*| = O(u^{-3})+O(u^{-2}u^{-\delta})$ on $[0, L]$. Thus to bound the supremum with respect to g in S, we require u^6 inequalities on integrals of integrals $I(g, u)$.

Now the μ_2-measure, of intervals of length u^{-2}, is at most 2^{-s}, $s = k(u) - k(u^\beta)$. Moreover, $s/\log m(k(u^\beta)) \to +\infty$, or $s/\log\log u \to +\infty$, so $2^{-s} = o(\log u)^{-1}$. Using Kahane's inequality for the moments of $I(g, u)$ [4, p. 168], much asi n the proof of Theorem 1, we find for any $\delta > 0$ and any $A > 1$,

$$P(|I(g, u)| > \delta) < u^{-A}, \qquad u > u(\delta, A),\ g \in S.$$

As we required only u^6 inequalities on individual integrals $I(g, u)$, the proof is complete.

There is a much more powerful method for uniform approximation; unfortunately in this situation it leads to exactly the same term, namely $o(m_k/\log m_k)$. The idea is to find a covering of E by intervals I of length $u^{2\beta}$, whose number N is estimated by means of the integer $k(u^{-2\beta})$. On each interval I we replace $f \in S$ by its secant, obtaining a function f_0, and then $|f-f_0| = O(u^{-2\alpha\beta})$. Now the functions f_0 belong to a subspace of $C(E)$, of dimension $2N$, and here we can use an inequality between widths and entropy [18, p. 163] in Banach spaces; the resulting inequality is obtained for α in $(0, 1)$ by Vosburg [22]. A better estimate, in $L^1(\mu)$ instead of $C(E)$, would allow an improvement in the term $m_k/\log m_k$. However, most results about approximation of functions in A^α, in the usual spaces $L^p(0, 1)$, suggest that no improvement can be expected in passing from $C(E)$ to $L^1(\mu)$ [1, 15].

III. The construction in this chapter is complementary to the foregoing, leading to random exceptional functions in Banach spaces of C^∞ functions of very high smoothness. In defining these spaces we follow the classical method of Denjoy–Carleman [7,V, 20 ch. 19] so that each element of any space considered will be determined by its sequence of derivatives at an arbitrary real number. Let $1 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n \to +\infty$, let $M_n = \lambda_1, \ldots, \lambda_n$, and let Y be the Banach space of C^∞ functions defined by $\|f\| = \sup_n \sup_t M_n^{-1}|f^{(n)}(t)|$. To obtain a separable subspace, we define Y_0 to be the closed linear span of integrable functions f such that $\hat{f}$ has compact support. Finally, we require

(i) $$\sum \lambda_n^{-1} < \infty,$$

(ii) $$n\log n < \lambda_n \quad (n > 10).$$

The first of these ensures the *quasi-analyticity* of the class Y [7, 20], while the second is imposed for technical reasons.

THEOREM 3. *Suppose that in the sequence (m_k), we have $m_{k+1} > m_k^{-1/2}2^{m_k}$ infinitely often. Then for almost all paths X, there is a mapping f in Y_0, such that $f > 0$ and $f' > 0$ on the dyadic set F, and $X\circ f(F)$ is a Kronecker set.*

The condition imposed on (m_k) is consistent with $m_k = o(2^k)$; under this restriction the modulus of continuity, $\sup \mu(a, a+h)$, of the product measure μ, is $o(\log h^{-1})^{-1}$ for small h, so $X(F)$ is almost surely an M_0-set [4, p. 165].

1. Before assembling all the elements, we list some technical points necessary in the proof.

a) Given an interval $[0, L]$ in $[0, \infty)$ and an integer $k \geqslant 1$ we consider a condition on the oscillation of $X(t)$ over $[qk^{-1}L, qk^{-1}L+k^{-1}L]$, $q = 0, 1, 2, \ldots$ We require that

$$|X(qk^{-1}L+k^{-1}L)-X(qk^{-1}L)| > k^{-1}L^{1/2},$$

and

$$|X(s)-X(t)| < k\cdot(k^{-1}L)^{1/6}|t-s|^{1/3}$$

for all numbers s and t in our interval of length $k^{-1}L$.

The probability of each part tends to 1 as k increases, and of course is independent of L. Hence the probability, that the condition holds for three consecutive values of q in $0, \ldots, k-1$, exceeds $1-\varepsilon(k)^k$, $\varepsilon(k) \to 0$. We apply this to find a middle interval I over which $X(t)$ fills an interval of length $k^{-1}L^{1/2}$, and then majorize the oscillation of X over small intervals that meet I — these intervals of course are contained in the union of the three intervals, and the oscillation can be controlled by tripling the bounds above.

b) The function of a real variable $\varphi(s) = \sup s^n/M_n$ can be estimated $\varphi(s) < \exp(Bs/\log s)$ for large s. This becomes clear once we verify that the supremum is attained with n of magnitude near $s/\log s$.

c) The function $G_p(x) = (\sin x/x)^p$, $(p = 1, 2, \ldots)$ has derivative $D^n G_p \leqslant p^n$ in absolute value. This follows from the representation of G_p as a Fourier integral.

2. Now we follow as closely as possible the second part of Theorem 1, introducing again the splitting $F = \bigcup F_v$ and the 2^k intervals $|t-f_0(x_v)| \leqslant k^{10}2^{-m_{k+1}}$. The event described in (a) holds for each v, when k is large. Thus the interval of length $L = k^{10}2^{-m_{k+1}}$ about $f_0(x_v)$ contains a subinterval of length $k^{-1}L = k^9 2^{-m_{k+1}}$ on which $X(t)$ oscillates at least $k^4 2^{\frac{1}{2}m_{k+1}}$, and on each interval of length $k2^{-m_{k+1}}$, intersecting I, $X(t)$ oscillates at most $3k^3 2^{-\frac{1}{3}m_{k+1}}$. Therefore we choose a frequency $u = u_k = k^{-3/2}2^{\frac{1}{2}m_{k+1}}$, and corresponding displacements ξ_v, of absolute value $\leqslant k^{10}2^{-m_{k+1}}$.

For the displacement of f_0 we choose $g = \Sigma \xi_v G_p(T(x-x_v))$, with $p = \log k$ and $T = A2^{m_k}$, A being a large constant.

To bound g we first investigate the sum $\Sigma' |G_p(T(x-x_v))|$, where the dash means that we omit the index v at which $|x-x_v|$ attains its smallest value. The numbers $T|x-x_v|$ generated by the remaining indices v can be estimated from below by the sequence $\frac{1}{2}qA$, $q = 1, 2, \ldots$, taken twice. As $|G_p(y)| \leqslant |y|^{-p}$ and $p = \log k$, the resulting sum has order of magnitude k^{-A_1}, with $A_1 = \log A - \log 2$. (Therefore we choose A so that $\log A_1 > 20$, say.) This shows that the supremum of $|g|$ is comparable with $\max|\xi_v|$, hence it tends to 0. We also require a strong inequality on the error $g-\xi_v$ on F_v. Now F_v has diameter $2\cdot 2^{-m_{k+1}}$, so $|G_p(T(x-x_v))-1| \ll pT|x-x_v| \ll \log k 2^{-m_k} 2^{-m_{k+1}}$. Adding to this the bounds obtained before, we obtain an inequality $|g-\xi_v| < 2^{-m_{k+1}}$ on F_v.

For the norm of g in Y we have an upper bound $2^k \cdot \max|\xi_v| \times \times \max(pT)^n/M_n = 2^k \max|\xi_v|\cdot\varphi(pT)$, following (b) and (c). Now $pT = A2^{m_k}\log k$, $pT/\log pT < A_1 2^{m_k}\log k/m_k = o(m_{k+1})$. In view of the large gap between m_k and m_{k+1}, we have $2^k\max|\xi_v| < \exp-\delta m_{k+1}$ (for δ near $\log 2$), while $\log\varphi(pT) \ll pT/\log pT = o(m_{k+1})$. This accomplishes the estimation of norms.

To show now the efficacy of the displaced function $f_0+g = f_0 + + \Sigma\xi_v G_p(T(x-x_v))$, we recall that on F_v it differs from $f_0+\xi_v$, the correct value, by $o(2^{-m_{k+1}})$. Moreover $f_0 \in C^1$, so we are considering the oscillation of $X(t)$ over a certain interval of length $\ll 2^{-m_{k+1}}$, the diameter of F_v. But $f_0(x_v)+\xi_v$ has the special property that $X(t)$ oscillates at most $3k^3 2^{-\frac{1}{2}m_{k+1}}$, when t varies within $k2^{-m_{k+1}}$ of $f_0(x_v)+\xi_v$. Since the oscillation of $X(t)$ is thus $o(k^{3/2}2^{-m_{k+1/2}}) = o(u^{-1})$, the construction is complete.

IV. In this chapter we study sets $f\circ X(E)$, using a method of estimating Fourier–Stieltjes coefficients slightly less precise than the one used for Theorem 1, but more flexible. Other methods for obtaining estimates, involving differentiable transformations f, were introduced in [10, 11].

1. THEOREM 4. *Let μ be a probability measure on $[0,1]$ whose modulus of continuity of $\sup\mu(a, a+h) = o(\log h^{-1})^{-1}$, and let f be a function of class $C^1(-\infty, \infty)$ and $f' > 0$. Then $f\circ X(E)$ is almost surely an M_0-set. This remains true if f is monotonic and $f' > 0$ almost everywhere.*

Proof. Because $X(E)$ is almost surely a bounded set, we can suppose that f' is bounded between two constants, $0 < C_1 \leqslant f' \leqslant C_2$, and uniformly continuous. For each $u > 1$ we divide $[0,1]$ into $k = k(u) \cong u$ adjacent intervals and divide the integral

$$\int_0^1 \exp -2\pi iuf\circ X(t)\,\mu(dt) \tag{1}$$

into the corresponding partial integrals J_n, $0 \leqslant n \leqslant k-1$. Then we denote by I_0 and I_1 the sums of these for n even and odd, respectively. Focussing on I_0 for simplicity, we denote by J_n^* the conditional expectation of J_n, relative to the field of $X(t)$, $0 \leqslant t \leqslant (n-2)k^{-1}$. By the Markov property, this is the expectation given $X((n-1)k^{-1})$. We shall prove that $|J_n^*| = o(\mu(nk^{-1}, (n+1)k^{-1}))$ uniformly with respect to n, almost everywhere. This last statement in fact does not involve μ at all; we need only estimate the supremum of all integrals

$$(2\pi)^{-1/2}\int \exp -2\pi iuf(b+\lambda x)\cdot\exp -\tfrac{1}{2}x^2\,dx, \tag{2}$$

for real b and $\lambda \geqslant u^{-1/2}$. We obtain this formulation from the conditional distribution of $X(t)$ given $X(s)$, with $t-s \geqslant u^{-1}$. To accomplish the qualitative study of the integrals we show that the functions $\exp -2\pi iuf(b+\lambda x)$, of x, tend weakly to 0 in L^∞, uniformly with respect to b and $\lambda \geqslant u^{-1/2}$. In fact, if I is an interval over which $f(b+\lambda x)$ increases by exactly u^{-1}, then $|I| \approx u^{-1}\lambda^{-1} = o(\lambda^{-1})$. Hence it differs from its secant line over I, by $o(\lambda)\cdot|I| = o(u^{-1})$. Thus

$$\int_I \exp -2\pi iuf(b+\lambda x)\,dx = o(|I|),$$

and since $|I| \ll u^{-1/2}$, this proves the weak convergence needed.

In analyzing the sum $\Sigma J_{2n} - J_{2n}^* \equiv \Sigma Y_n$, say, we have first $E(Y_n|Y_1, \ldots \ldots, Y_{n-1}) = 0$. Putting $b_n = \|Y_n\|_\infty$ we observe that $\Sigma b_n \leqslant 2$, as μ is a probability measure, and $b = \sup_n b_n = o(\log u)^{-1}$, as $k \cong u^{-1}$. Thus, if $y > 0$ and $yb \leqslant 1$, we obtain by Taylor's formula $E(y|\Sigma Y_n|) < 4\exp Ay^2 b$, with A an absolute constant. For small $\varepsilon > 0$ we get

$$P(|\Sigma Y_n| > \varepsilon) < 4\exp -(4A)^{-1}\varepsilon^2 b^{-1}.$$

Putting in this inequality, $b = b(u) = o(\log u)^{-1}$ and $u = 1, 2^{1/2}, 3^{1/2}, \ldots$, we obtain a convergent series for every $\varepsilon > 0$. Since we saw before that $\|J_1^*\|_\infty + \|J_2^*\| + \ldots = o(1)$, the proof is complete.

The extension to functions f not necessarily of class C^1, follows the technique of [10]; the essential point is this: to each $\varepsilon > 0$ there is a C^1 function g, with $g' > 0$ and $m(f \neq g) < \varepsilon$ [23, II, pp. 73–77]. Now $X(t)$ has an absolutely continuous distribution when $t > 0$, and so if S denotes a measurable set in $(-\infty, \infty)$ then by Fubini's theorem,

$$E(\mu\{t\colon X(t)\in S\}) \to 0 \quad \text{when } m(S) \to 0.$$

Applying this when $S = (f(t) \neq g(t))$, we obtain the theorem for the more general function f mentioned in the last sentence.

2. To obtain quantitative conclusions about integrals (1) containing $f\circ X(t)$ it is clear that we need bounds on expected values of the form (2). These are given in the following statement.

LEMMA. *Suppose that all functions f in a set S are subject to inequalities*

$$0 < C_1 \leqslant f' \leqslant C_2 < \infty, \quad |f^{(n)}| \leqslant C_3 < \infty, \quad 1 \leqslant n \leqslant r+1.$$

Then $I = \int \exp -2\pi i u f(x) \cdot \exp -\frac{1}{2}x^2\, dx = O(|u|^{-r})$ *for large* $u > 1$, *uniformly for* f *in* S.

Proof. We can assume that $f(o) = 0$, and write F for the inverse mapping of $(-\infty, \infty)$ onto itself, so the functions satisfy inequalities analogous to the functions f; moreover

$$I = \int \exp -2\pi i u x \cdot \exp -\tfrac{1}{2}F^2(x) \cdot F'(x)\, dx.$$

To obtain the bound $O(|u|^{-r})$ we shall show that the cofactor of $\exp -2\pi i u x$ has r derivatives, uniformly bounded in $L^1(-\infty, \infty)$. By Leibniz' formula, we only need L^1-estimates for the derivatives of $\exp -\frac{1}{2}F^2(x)$. It is clear, however, that each derivative can be expressed by $p(F(x), F'(x), \ldots) \times \exp -\frac{1}{2}F^2(x)$ for certain polynomials p. Since $F', \ldots, F^{(r)}$ are subject to uniform bounds, the L^1-estimates follow from the presence of the term $\exp -\frac{1}{2}F^2(x)$.

The integral (2) can be treated by the lemma when $u \geqslant u\lambda > 1$, by the substitution $f_0(x) = \lambda^{-1} f(b + \lambda x)$, for the nth derivative of f_0 is $\leqslant \lambda^{-1}\lambda^n \leqslant 1$.

THEOREM 5. *Let μ be a probability measure, in E, satisfying a Lipschitz condition in exponent β, $0 \leqslant \beta < \frac{1}{2}$, and let $0 < a < \beta$. Then we have almost surely*

$$\int \exp -2\pi i u f X(t)\, \mu(dt) = O(u^{-a})$$

for all f in $C^\infty(-\infty, \infty)$ with $f' > 0$.

Theorem 5 expresses a property of the sets $f \circ X(E)$: each has "Fourier-dimension" $\geqslant 2a$. As we can choose E to have Hausdorff dimension exactly β, and then apply the conclusion with a sequence $a_n \to \beta$; $f \circ X(E)$ is a Salem set [21] of dimension 2β. A similar purpose was achieved in [11], but the sets constructed there had a very special structure.

In the proof of Theorem 5 we use the fact that $X(E)$ is almost surely a bounded linear set, so it suffices to make the argument for functions f that are linear outside a finite interval. Indeed, for each f in $C^\infty(-\infty, \infty)$, and each interval $[-L, L]$, we can find a special function f_0, coincident with f on $[-L, L]$, and $f_0' > 0$ everywhere if $f' > 0$ on $[-L, L]$. The same is of course true for the spaces Λ^s, $s > 1$. Given $a < \beta$ we choose b so that $0 < b < 2$ and $2a < b\beta$, and then an integer s so that $s(2a - b\beta) > 2$ and $(s-1)(1-b/2) > 1$.

Then we apply the method used before, dividing the compact set E into adjacent intervals of length approximately $V \cong u^{-b}$. In the remainder of the proof we omit details similar to these encountered in the proof of Theorem 1, merely indicating the estimations necessary to obtain the bound u^{-a}. The expectations in this case take the form mentioned after the lemma, with $\lambda \geqslant u^{-b/2}$, so that $u\lambda > u^{1-b/2}$ and $(s-1)(1-b/2) > 1$. Thus the expectations are all $o(u^{-a})$. Now the μ-measure of each interval is $O(u^{-b\beta})$, so we obtain bounds of the type $E(\exp|yI|) < \exp Ay^2u^{-b\beta}$, $0 \leqslant y \leqslant u^{b\beta}$. The resulting estimate for $(P|I| > u^{-a})$ then becomes $\exp -A' u^{2a-b\beta}$. To obtain approximation of $\exp -2\pi i u f \circ X(t)$ within u^{-1}, say, we require approximation to f within u^{-2}. Since $2s^{-1} < 2a - b\beta$, and we are operating in $C^s \supseteq C^\infty$, the estimates of entropy used before are still adequate. To obtain the rate of decrease u^{-a} for *real* numbers u we use the device from harmonic analysis described in [4, p. 165].

The following lemma uses the idea of "widths" in a Banach space, as set forth in [17], [18, Ch. 9].

3. LEMMA. *Let S be a bounded subset of $\Lambda^a[a, b]$; $1 < a < 2$. Then for all large numbers y, and integers $N \geqslant 1$, we can find a set $S(u) \subseteq S$ containing $\exp AN\log y$ elements with the following property: to each set $F \supseteq [a, b]$ covered by N intervals of length y^{-1}, and each f in S, there is an f_0 in $S(u)$, such that $|f - f_0| < y^{-a}$ everywhere on F.*

Proof. First we divide $[a, b]$ into adjacent intervals J of length y^{-1} (and one odd interval at most), so the number of intervals is $\leqslant (b-a)y + 1 < 2(b-a)y$ for large y. Each set H, covered by N intervals of length y^{-1}, meets at most $2N$ intervals J just constructed, and the number of ways in which these $2N$ can be chosen is $< \exp AN\log y$. Thus, if we construct subsets $T(u) \subseteq S$, containing $\exp An\log y$ elements, for *each* selection G of $2N$ intervals J, their union is the set $S(u)$ sought. Let G be a fixed union of that type.

For each f in S we denote by f_0 the function, defined on G, linear on each constituent J of G and coincident with f on the end-points of those intervals. Then $|f - f_0| \leqslant y^{-a}$ by the mean-value theorem. The functions f_0 belong to a bounded subset of $C(G)$, namely a ball with radius independent of y. Moreover, the functions f_0 belong to a subspace of dimension $4N$, so the largest collection of functions f_0, with mutual distances $\geqslant y^{-a}$ — that is, a maximal "y^{-a}-distinguishable" subset — contains $\exp AN\log y$ elements. This means that at most $\exp AN\log y$ elements of S can be My^{-a}-distinguishable, with an M depending on S alone. Thus, a set $S(u) \subseteq S$, containing $\exp AN\log y$ elements gives approximation within My^{-a}, and a slight adjustment yields the statement of the lemma. (In case $a > 1$ we use a more complicated interpolation than the secant but the main ideas remain similar.) Compare [18, p. 163].

THEOREM 6. *Let E be the dyadic set constructed in Theorem 2. Then it is almost sure that all sets $f \circ X(E)$, with $f \in \Lambda^a$ and $a > 1$, $f' > 0$, are M_0-sets.*

Proof. It is sufficient to make the proof for a fixed $a > 1$, and for this we choose C in the interval $a^{-1} < C < 1$. Let N be the number of intervals of length u^{-2C} required to cover E. Then N can be estimated by 2^l, where $m_{l+1}\log 2 \geqslant 2C\log u \geqslant m_l \log 2$. Thus it is almost sure that N intervals of length $o(u^{-1/a})$ suffice to cover $X(E)$, so we can apply the lemma with $y = u^a$, and note that $\exp A n \log y \leqslant \exp A' N \log u$. This gives a bound on the number of integrals to be estimated, for a fixed $u > 1$.

However, the estimates of probabilities take the form $\exp - \delta 2^k$, where $m_{k+1}\log 2 \geqslant 2\log u \geqslant m_k \log 2$. Thus $m_k \cong C^{-1} m_l$, so that $\log m_k = o(k-l)$, whence finally, $N\log u = 2^l \log u = o(2^k \log u/m_k) = o(2^k)$, and this is sufficient: the remainder of the argument follows Theorem 2.

4. Theorem 5a. *Let μ be a probability measure in a set $E \subseteq (0, +\infty)$, satisfying a Lipschitz condition with exponent β in $(0, \frac{1}{2})$. Then it is almost sure that all sets $f\circ X(E)$, with $f \in \Lambda^{\frac{1}{2}\beta}, f' > 0$, are M_0-sets.*

(a) Since $0 < \beta < \frac{1}{2}$, we can assume that E has Hausdorff dimension exactly β; otherwise E would carry a measure with a higher Lipschitz condition, and then the proof is much easier. We use this condition on E only to ensure that $X(E)$ has Lebesgue measure 0, and then $\mu\circ X^{-1}$ is almost surely singular.

(b) For each fixed $\eta > 0$, there is a sequence (k_n) of compact sets, such that $m(k_n) < \eta$ for each n, and $\sup_n \lambda(k_n) = 1$ for every singular probability measure λ on $(-\infty, \infty)$. Applying this to the measures $\mu\circ X^{-1}$ for a sequence $\eta_1 > \eta_2 > \ldots > \eta_n > \ldots$, we can find a subset Ω_1 of the probability space Ω, with $P(\Omega_1)$ near 1, with this property: for each $\eta > 0$, there are compact sets $k_1, \ldots, k_n$, with $m(k_1) < \eta, \ldots, m(k_n) < \eta$ and $\sup_j \mu\circ X^{-1}(k_j) > 1-\eta$ for each $X \in \Omega_1$.

(c) Now we consider the problem of approximating the integrals $\int \exp - 2\pi i u f\circ X(t)\cdot \mu(dt)$, where the elements $f \in S$, a bounded subset of $\Lambda^{\frac{1}{2}\beta}(-\infty, \infty)$. To obtain approximation with error $< \eta$, it is sufficient to approximate f with an error $< \eta/8|u|$, except on a set of $\mu\circ X^{-1}$-measure $< \eta/2$. But when $X \in \Omega_1$, this can be accomplished by approximating f on certain compact sets $L_1, \ldots, L_s$, all of Lebesgue measure $< \eta$. This the number of elements of S, needed to "match" f in the sense of the Lemma to Theorem 6, is $\exp o(|u|^{2\beta})$ as $|u| \to +\infty$. For this, see [22].

(d) If we divide $(0, +\infty)$ into intervals of length u^{-2}, then the estimates of probabilities become $\exp - Cu^{2\beta}$, since μ fulfills the Lipschitz condition of order β. However, if we express the bound $\exp o(|u|^{2\beta})$ obtained before in the form $\exp u^{2\beta} h(u)$, with $h(+\infty) = 0$, then we can employ a division of $(0, +\infty)$ into intervals of length $u^{-2}/h(u)$, because the probabilities then admit a bound $\exp - Cu^{2\beta} h^{\beta}(u)$.

The factor $1/h(u) \to +\infty$ makes the expected values tend to 0, because $u(u^{-2}/h(u))^{1/2} \to +\infty$. Since the estimate of probabilities is small enough in comparison with the number of functions needed in approximation, Theorem 5a is proved.

In the extreme case $\beta = \frac{1}{2}$, the result seems to hold for C^1 functions, after some variations in the argument; it is useful to note that C^1 is a Souslin set in the metric space $C(-\infty, \infty)$. The theorem may indeed dail for Λ^1; the absolute continuity of the measures $\mu\circ X^{-1}$ is a difficult problem.

V. Our aim is to show that the order of smoothness $(2\beta)^{-1}$ in Theorem 5a is best possible. This is not very difficult if we choose a weak condition on $f\circ X(E)$ that nevertheless prevents $f\circ X(E)$ from being an M_0-set. If, however, we seek properties close to the Kronecker property, the arguments become much more subtle. Therefore our plan is to find geometric properties of $X(E)$, for certain dyadic sets E of Hausdorff dimension arbitrarily close to β, and then study transformations of special sets by diffeomorphisms of class $\lambda^{1/2\beta}$. For definiteness we state a complement to Theorem 5a in a simple form.

Theorem 7. *Let E be a dyadic set constructed over a sequence $(m_k)_1^\infty$ such that $m_{k+1}/m_k \geqslant d > \beta^{-1} \geqslant 2$ for infinitely many integers k. Then for almost all paths X, there is a dense G_δ-set $\lambda_X \subseteq \lambda^{1/2\beta}$, such that $f\circ X(E)$ is not an M_0-set, for each f in λ_X.*

As before we use a splitting of E into sets E_p, $1 \leqslant p \leqslant 2^k$, and choose $x_p \in E_p$ arbitrarily. Later we shall choose k to be a special index, but the next statement is valid for all k.

1. Lemma. $\Sigma(|t-x_p| + 2^{-m_k})^{-1/2} \ll 2^{m_k}$ *for all real t.*

Proof. Those summands in which $|t-x_p| > 1/2$, say, contribute $\ll 2^k \leqslant 2^{m_k}$ to the sum. The number of solutions of the inequality $2^{-m_{l+1}} \leqslant |t-x_p| \leqslant 2^{-m_l}$, is $\ll 2^{k-l}$, and $m_l \leqslant m_k - k + l$, so the contribution for $l = 1, 2, \ldots, k$ doesn't exceed $2^{m_k/2}\sqrt{2}^{k-l}$ in order of magnitude; summing for $1 \leqslant l \leqslant k$ we find the bound 2^{m_k}.

Let γ be a real number in the interval $1 < \gamma < d\beta$, and let r be an integer such that $(r-1)(\gamma-1) > 1$. We claim now that with probability near 1 for large k:

any interval of length $M_k \equiv 2^{-m_k\gamma}$ contains at most $r-1$ of the images $X(x_p)$.

To see this we consider increasing r-tuples $y_1 < \ldots < y_r$ chosen from the numbers x_p, and the corresponding event: $|X(y_{i+1}) - X(y_i)| \leqslant M_k$, $1 \leqslant k < r$. There are 2^k possibilities for y_1 and the lemma shows that the total probability of all these events has magnitude $\ll 2^k(M_k 2^{m_k})^{(r-1)}$. Because $(r-1)(\gamma-1) > 1$, the probability is less than $2^{-\eta k}$ for some $\eta > 0$, and the claim is established. By Lévy's modulus of continuity, the sets $X(E_p)$

have length $< 2^{-m_k+1/2} \cdot m_{k+1}$ for large k, and this length is $o(M_k)$ because $\frac{1}{2}d > \gamma$.

Let us now summarize the property of $X(E) = F$ that is used in the remaining steps of the argument: there is a sequence $D_k \to 0$ and corresponding decompositions $X(E) = \bigcup F_p$, with $\operatorname{diam} F_p \leqslant L_k$, such that no interval of length D_k meets more than r of the sets F_p, and finally $L_k = o(D_k^{1/2\beta})$. The last relation follows from the inequalities $1 < \gamma < d\beta$.

2. In using these properties of $X(E)$ we need a finer splitting than $\bigcup F_p = F$. Let ε_k be defined by the formula $L_k = (\varepsilon_k^{3r} D_k)^{1/2\beta}$, and let Γ_1 be a maximal selection of sets F_p, having mutual distances $\geqslant \varepsilon_k D_k$. Let Γ_2 be a maximal selection of sets F_p not in Γ_1, having mutual distances $\geqslant \varepsilon_k^2 D_k$, etc. Then $\Gamma_1 \cup \ldots \cup \Gamma_r$ exhausts $\{F_p\}$ if ε_k is small, because any set F_q, not selected after r steps, has distance $< \varepsilon_k D_k$ from r distinct sets F_p.

Let $g \in \lambda^{1/2\beta}$, let ψ be continuous on F, and $\varepsilon > 0$. We shall choose a function g_0, with small norm in $\lambda^{1/2\beta}$ such that

$$|(\varepsilon_k^{-s/\beta} D_k^{-1/2\beta})(g-g_0)-\psi| < \varepsilon \text{ modulo } 1 \quad \text{on } \Gamma_s, 1 \leqslant s \leqslant r.$$

First we define h_1 on the intervals F_p occurring in Γ_1, so that the inequality above is an equality modulo 1 at some point in each F_p; this can be accomplished with $|h_1| \leqslant \varepsilon_k^{1/\beta} D_k^{1/2\beta}$ and thus h_1 has small norm in $\lambda^{1/2\beta}$, since the intervals in Γ_1 have mutual distances $\varepsilon_k D_k$. Also, $L_k \cdot \varepsilon_k^{-r/\beta} D_k^{-1/2\beta} \to 0$, so the inequality actually holds on the intervals F_p, whose length is L_k at most. Next we construct h_2, so that $g+h_1+h_2$ has the necessary properties on each interval F_p in Γ_2. The critical point here is that $|h_2| \leqslant \varepsilon_k^{2/\beta} D_k^{1/2\beta}$, so the addition of h_2 does not interfere substantially with the inequality attained on Γ_1. Continuing in this manner we construct $g_0 = h_1 + \ldots + h_r$.

Writing $a_s \equiv \varepsilon_k^{-s/\beta} D_k^{-1/2\beta}$, we note that $a_1 \to +\infty$, while $a_{s+1} > \varepsilon^{-1} a_s$, $1 \leqslant s < r$. Thus, taking $\psi = 0$, we have a dense G_δ-set $\lambda \subseteq \lambda^{1/2\beta}$ whose members f have the following properties:

To each $\varepsilon > 0$, we can find numbers $b_1 < \ldots < b_r$, so that $b_1 > \varepsilon^{-1}$ and $b_{s+1} > \varepsilon^{-1} b_s$, and one of the inequalities,

$$|b_s f(x)| < \varepsilon \text{ modulo } 1, \qquad s = 1, 2, \ldots, r,$$

holds for each x in F.

Thus $f(F)$ is a set $H^{(r)}$ of Pyateckii–Sapiro [23, p. 346; 6, p. 58] and is a set of uniqueness for trigonometric series: $f(F)$ does not carry a (Schwartz) distribution $\psi \neq 0$, whose Fourier transform $\hat{\psi}$ is in L^∞ and tends to zero at infinity. Thus $f(F)$ is not an M_0-set, since finite measures are distributions.

It is very plausible that $\lambda^{1/\beta}$ contains a dense G_δ-set of functions f, such that $f(F)$ is a Helson set [5, IV] but the proof of this would involve machinery from harmonic analysis [2; 19, VI A].

Instead of introducing ideas divergent from our main topics, we shall show how the construction just completed can be improved. Let us fix an m-tuple $\psi_1, \ldots, \psi_m$ of continuous functions on F, and $\varepsilon > 0$. We shall construct g_0^* with similar functional-analytic properties to g_0. To each $j = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, r$ there will be a number $C(s, j)$ so that

$$|C(s,j)\, a_s \cdot (g+g_0) - \psi_j| < \varepsilon \text{ modulo } 1, \qquad \text{on } \Gamma_s.$$

Moreover, $C(s, j)$ is not too large: $1 \leqslant C(s,j) < C(m, \varepsilon)$.

In fact, functions g_0^* and numbers $C(s, j)$ are found by a simple device. Let $a_1, \ldots, a_j, \ldots, a_m$ be rationally independent numbers in $[1, 2]$, and let R be so large that the vectors $(pa_1, \ldots, pa_m)$, $1 \leqslant p \leqslant R$ form an "ε-net" modulo 1 in the m-cube: each point in I^m has distance $< \varepsilon$ from some m-tuple $(pa_1, \ldots, pa_m)$, $1 \leqslant p \leqslant R$. Then we define $C(s,j) = a_j$ for all s, j and construct g_0^* at the sth step by making adjustments equal to some number pa_s^{-1}, $1 \leqslant p \leqslant R$. (For details, see [12].)

Applying Baire's theorem in this more complicated situation, we find functions f such that $f(F)$ resembles the union of r Kronecker sets; hence we conjecture that $f(F)$ is a Helson set for all f in $\lambda^{1/2\beta}$ except a set of first category.

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An inequality for the distribution of a sum of certain Banach space valued random variables

by

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Abstract. We prove an inequality for the distribution of a sum of independent Banach space valued random variables provided they take values in a space having a norm with a smooth second directional derivative and the random variables have $2+\delta$ moments. This inequality is applied to obtain the central limit theorem and the law of the iterated logarithm, and it is shown that these results apply to the L^p spaces, $2 \leqslant p < \infty$.

1. Introduction. Throughout the paper B is a real separable Banach space with norm $\|\cdot\|$, and all measures on B are assumed to be defined on the Borel subsets of B generated by the norm open sets. We denote the topological dual of B by B^*.

A measure μ on B is called a *mean zero Gaussian measure* if every continuous linear function f on B has a mean zero Gaussian distribution with variance $\int_B [f(x)]^2\mu(dx)$. The bilinear function T defined on B^* by

$$T(f, g) = \int_B f(x)g(x)\mu(dx) \qquad (f, g \in B^*)$$

is called the *covariance function of* μ. It is well known that a mean zero Gaussian measure on B is uniquely determined by its covariance function. This is so because T uniquely determines μ on the Borel subsets of B generated by the weakly open sets, and since B is separable, the Borel sets generated by the weakly open sets are the same as those generated by the norm open sets.

However, a mean zero Gaussian measure μ on B is also determined by a unique subspace H_μ of B which has a Hilbert space structure. The norm on H_μ will be denoted by $\|\cdot\|_\mu$ and it is well known that the B norm $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ on H_μ. In fact, $\|\cdot\|$ is a measurable norm on H_μ in the sense of [7]. Since $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ it follows that B^* can be linearly embedded (by the restriction map) into the dual of H_μ, call it

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