STUDIA MATHEMATICA, T. LII. (1974)

Operations on distribution functions not derivable from operations on random variables

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Abstract. A binary operation β on the space of probability distribution functions is *derivable from an operation on random variables* if there exists a Borel-measurable 2-place real function g such that, for any 2 distribution functions F, G there exist random variables X, Y such that the distribution function of X is F, that of Y is G, and that of $g(X, Y)$ is $\beta(F, G)$. Thus, for example, convolution is derivable from the operation of addition on random variables. The purpose of this paper is to exhibit a class of very simple and very natural operations on distribution functions that are *not* derivable from *any* operation on random variables. To facilitate this purpose, we introduce and establish some of the salient facts concerning 2-dimensional copulas, i. e., the functions that connect a 2-dimensional joint distribution function with its one-dimensional margins.

1. Introduction. The triangle inequality for a probabilistic metric space, in the formulation due to A. N. Šerstnev [7, 8], reads

$$F_{pr} \geqslant \tau(F_{pq}, F_{qr}). \tag{1.1}$$

Here F_{pr}, F_{pq} and F_{qr} belong to the space Δ of (one-dimensional) probability distribution functions([1]) and τ is a suitable binary operation on Δ. The most common choices for τ are *convolution* and the operations τ_T defined for any F, G in Δ and any real x by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T\big(F(u), G(v)\big), \tag{1.2}$$

where T is a *t-norm*, i. e., a suitable binary operation on the unit interval. The first choice yields Wald's inequality, while the second leads to the family of Menger inequalities (for a discussion, see [4], [6], [8]).

* Supported in part by the National Science Foundation, grants GP-22515 and GP-34526.

([1]) The elements of Δ are thus non-decreasing functions F defined on the extended real line $\overline{\boldsymbol{R}} = [-\infty, +\infty]$, with $F(-\infty) = 0$ and $F(+\infty) = 1$. For convenience of exposition, we shall normalize Δ by requiring its elements to be left-continuous on the unextended real line $\boldsymbol{R} = (-\infty, +\infty)$; as will be seen, nothing of importance hinges on this particular choice of normalization.

It is well known that convolution of distribution functions corresponds to addition of random variables, in the sense that if F and G are distribution functions then there exist (independent) random variables X and Y, defined on the same probability space, such that F is the distribution function of X, G the distribution function of Y, and their convolution $F*G$ the distribution function of $X+Y$. It is therefore natural to ask: What operations on random variables correspond to the operations τ_T? The primary purpose of this paper is to show that the general answer to this question is: None. More precisely, we shall do the following: We begin with:

Definition 1. Let β be a binary operation on the space Δ. Then β is *derivable from an operation on random variables* if there exists a Borel-measurable 2-place real function g such that, for any F, G in Δ, there exist random variables X, Y, defined on a common probability space, such that the distribution function of X is F, that of Y is G, and that of $g(X, Y)$ is $\beta(F, G)$.

Definition 2. The set $\mathscr{T}$ consists of all functions T such that: (a) T maps the closed unit square $[0,1]^2$ into the closed unit interval $[0,1]$, i. e., T is a binary operation on $[0,1]$; (b) T is non-decreasing in each argument; (c) T satisfies

$$(1.3)\qquad T(a,0)=T(0,a)=0,\qquad T(a,1)=T(1,a)=a$$

for every a in $[0,1]$.

Then our major result is:

Theorem 1. *Let T be any function in $\mathscr{T}$ other than* Min. *Let τ_T be the binary operation on Δ defined by* (1.2). *Then τ_T is not derivable from any operation on random variables.*

This result clearly shows that the distinction between working directly with distribution functions (as we generally do in the theory of probabilistic metric spaces) rather than with random variables, is intrinsic and not just a matter of taste. It further shows that there are topics in probability which are not encompassed by the standard measure-theoretic model of the theory.

The easiest way to prove Theorem 1 is with the aid of copulas. These are the functions which connect an n-dimensional distribution function with its one-dimensional margins. Copulas were introduced by one of us in [9], and are discussed at length in [10]. However, these papers contain no proofs. Accordingly, the next section of this paper is devoted to a recapitulation of the salient facts concerning copulas, complete with proofs. However, since more is not required in the sequel, we confine ourselves to the two-dimensional case. The third section is then devoted to the proof of Theorem 1 and to several allied results.

2. Copulas. Let $X, Y, \ldots$, be real-valued random variables defined on a common probability space. For each such random variable X, let F_X denote the distribution function of X, so that for any real x, $F_X(x)$ is the probability that X is less than x. Similarly, for each pair (X, Y) of random variables, let H_{XY} denote the joint distribution function of X and Y, so that for any real u and v, $H_{XY}(u,v)$ is the probability that $X<u$ and $Y<v$. Clearly, for all u, v, we have:

$$(2.1)\qquad H_{XY}(u,+\infty)=F_X(u)\quad\text{and}\quad H_{XY}(+\infty,v)=F_Y(v),$$

i. e., the individual distribution functions F_X and F_Y are the margins of the joint distribution function H_{XY}.

It is well-known (see, e.g., [3], pp. 148–149) that the function H_{XY} has the following properties:

$$(2.2)\qquad H_{XY}\ \text{is left-continuous in each place};$$

$$(2.3)\qquad H_{XY}(u,-\infty)=H_{XY}(-\infty,v)=0,\ \text{for all}\ u,v;$$

$$(2.4)\qquad H_{XY}(+\infty,+\infty)=1;$$

$$(2.5)\qquad H_{XY}(s,t)-H_{XY}(s,v)-H_{XY}(u,t)+H_{XY}(u,v)\geqslant 0,\quad\text{whenever}\ s\leqslant u,\ t\leqslant v.$$

Conversely, any two-place real function H satisfying the conditions (2.2)–(2.5) is the joint distribution function of a pair of random variables defined on a common probability space. The condition that a joint distribution function H is non-decreasing in each place follows readily from (2.3) and (2.5), as does the inequality: For any s, t, u, v,

$$(2.6)\qquad |H(s,t)-H(u,v)|\leqslant|F(s)-F(u)|+|G(t)-G(v)|,$$

where F and G are the margins of H (see [1], p. 290). It follows at once from (2.6) that if $F(s)=F(u)$ and $G(t)=G(v)$ then $H(s,t)=H(u,v)$. Hence the set of ordered pairs,

$$\{((F(u),G(v)),H(u,v))\,|\,u,v\in\mathbf{R}\}$$

defines a two-place real function whose domain is the Cartesian product $(\operatorname{Ran}F)\times(\operatorname{Ran}G)$. Thus we have:

Lemma 1. *If X, Y are random variables with distribution functions F_X, F_Y and joint distribution function H_{XY}, then there exists a unique two-place real function C^*_{XY} such that*

$$(2.7)\qquad \operatorname{Dom}C^*_{XY}=(\operatorname{Ran}F_X)\times(\operatorname{Ran}F_Y),\qquad \operatorname{Ran}C^*_{XY}=\operatorname{Ran}H_{XY},$$

$$(2.8)\qquad H_{XY}(u,v)=C^*_{XY}(F_X(u),F_Y(v)),\quad\textit{for all}\ u,v.$$

In order to study functions such as C^*_{XY} it is convenient to introduce several definitions. Let I denote the closed unit interval $[0,1]$.

Definition 3. A *subcopula* is a function C^* with the following properties:

(2.9) $\operatorname{Dom} C^* = D_1 \times D_2$, where D_1 and D_2 are subsets of I containing 0 and 1;

(2.10) $$C^*(a, 0) = C^*(0, b) = 0, \quad \text{for all } a \in D_1,\ b \in D_2;$$

(2.11) $$C^*(a, 1) = a, \quad C^*(1, b) = b, \quad \text{for all } a \in D_1,\ b \in D_2;$$

(2.12) $$C^*(a, b) - C^*(a, d) - C^*(c, b) + C^*(c, d) \geqslant 0,$$
$$\text{whenever } a, c \in D_1,\ b, d \in D_2 \text{ and } a \leqslant c,\ b \leqslant d.$$

Definition 4. A *copula* is a subcopula whose domain is the entire unit square I^2.

We now readily obtain:

Lemma 2. *Every copula is in the set $\mathcal{T}$ of Definition 2.*

Lemma 3. *The function C^*_{XY} defined by (2.7) and (2.8) is always a subcopula and is a copula if and only if* $\operatorname{Ran} F_X = \operatorname{Ran} F_Y = I$.

Lemma 4. *If C^* is a subcopula then C^* is non-decreasing in each place,* $\operatorname{Ran} C^* \subseteq I$, *and*

(2.13) $$|C^*(a, b) - C^*(c, d)| \leqslant |a - c| + |b - d|,$$

for all $a, c \in D_1$, $b, d \in D_2$, whence C^* is continuous on its domain.

The next result is crucial.

Lemma 5. *Let C^* be a subcopula. Then there is a copula C such that:*

(2.14) $$C(a, b) = C^*(a, b) \quad \text{for all } (a, b) \in \operatorname{Dom} C^*,$$

i.e., any subcopula can be extended to a copula. The extension is generally non-unique.

Proof. Let $\operatorname{Dom} C^* = D_1 \times D_2$. It follows at once from Lemma 4 that C^* can be extended by continuity to a function $\bar{C}$ with domain $\bar{D}_1 \times \bar{D}_2$, where $\bar{D}_1$ is the closure of D_1, and $\bar{D}_2$ that of D_2. Clearly $\bar{C}$ is also a subcopula. We next extend $\bar{C}$ to a function C with domain I^2 as follows: For any $(a, b) \in I^2$, let a_1 and a_2, respectively, be the greatest and least elements of $\bar{D}_1$ that satisfy $a_1 \leqslant a \leqslant a_2$; and b_1, b_2, respectively, the greatest and least elements of $\bar{D}_2$ that satisfy $b_1 \leqslant b \leqslant b_2$. Note that $a_1 = a = a_2$ if $a \in \bar{D}_1$, $b_1 = b = b_2$ if $b \in \bar{D}_2$. Set

$$\lambda_1 = \begin{cases} (a - a_1)/(a_2 - a_1), & \text{if } a_1 < a_2, \\ 1, & \text{if } a_1 = a_2; \end{cases}$$

$$\mu_1 = \begin{cases} (b - b_1)/(b_2 - b_1), & \text{if } b_1 < b_2, \\ 1, & \text{if } b_1 = b_2; \end{cases}$$

and define:

(2.15) $$C(a, b) = (1-\lambda_1)(1-\mu_1)\bar{C}(a_1, b_1) + (1-\lambda_1)\mu_1\bar{C}(a_1, b_2) + \lambda_1(1-\mu_1)\bar{C}(a_2, b_1) + \lambda_1\mu_1\bar{C}(a_2, b_2).$$

It is immediate that $\operatorname{Dom} C = I^2$, that

$$C(a, b) = \bar{C}(a, b) \quad \text{for any } (a, b) \in \operatorname{Dom} \bar{C},$$

and that C satisfies (2.10) and (2.11). It therefore only remains to show that C satisfies (2.12). To this end, let (c, d) be another point in I^2 with $c \geqslant a$, $d \geqslant b$, and let $c_1, d_1, c_2, d_2, \lambda_2, \mu_2$ be related to c, d as $a_1, b_1, a_2, b_2, \lambda_1, \mu_1$ are to a, b above. Let $M(a, b; c, d)$ denote the second-order difference

(2.16) $$C(a, b) - C(a, d) - C(c, b) + C(c, d),$$

with corresponding expressions for other such differences. There are now several cases to consider, depending upon whether or not a and c belong to the same component interval of $I \setminus \bar{D}_1$, and b and d to the same component interval of $I \setminus \bar{D}_2$. The most involved of these cases is: $a < a_2 \leqslant c_1 < c$ and $b < b_2 \leqslant d_1 < d$, which is illustrated in Fig. 1. In this case, applying (2.15) to each of the four terms in $M(a, b; c, d)$ and rearranging yields:

$$\begin{aligned} M(a, b; c, d) = {} & (1-\lambda_1)\mu_2 M(a_1, d_1; a_2, d_2) + \mu_2 M(a_2, d_1; c_1, d_2) + \\ & + \lambda_2\mu_2 M(c_1, d_1; c_2, d_2) + (1-\lambda_1) M(a_1, b_2; a_2, d_1) + \\ & + M(a_2, b_2; c_1, d_1) + \lambda_2 M(c_1, b_2; c_2, d_1) + \\ & + (1-\lambda_1)(1-\mu_1) M(a_1, b_1; a_2, b_2) + \\ & + (1-\mu_1) M(a_2, b_1; c_1, b_2) + \lambda_2(1-\mu_1) M(c_1, b_1; c_2, b_2). \end{aligned}$$

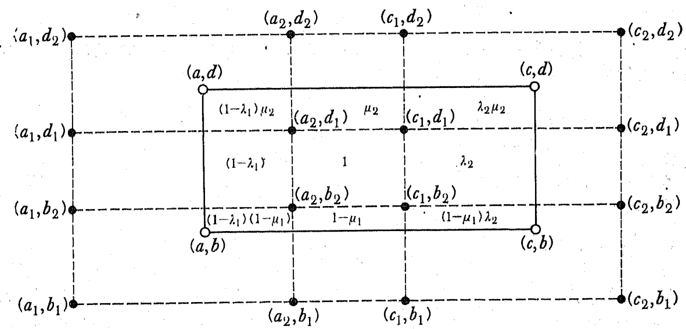

Fig. 1

Since $\bar{C}$ is a subcopula, the right-hand side of (2.17) is a combination of non-negative quantities with non-negative coefficients, and hence non-negative. Thus $M(a, b; c, d) \geqslant 0$ in this case. The remaining cases lead to similar combinations for $M(a, b; c, d)$. Hence $M(a, b; c, d) \geqslant 0$ in all cases and the lemma is proved.

As an immediate consequence of the preceding lemmas, we have:

THEOREM 2. *Let X and Y be real-valued random variables defined on a common probability space, with distribution functions F_X and F_Y respectively, and joint distribution function H_{XY}. Then there exists a copula C_{XY} such that*

$$H_{XY}(u, v) = C_{XY}(F_X(u), F_Y(v)), \quad \text{for all } u, v \text{ in } \bar{\mathbf{R}}. \tag{2.18}$$

If, furthermore, $\operatorname{Ran} F_X = \operatorname{Ran} F_Y = I$, i. e., if both F_X and F_Y are continuous on $\bar{\mathbf{R}}$, then C_{XY} is unique.

For any two random variables X, Y, the function C_{XY} in (2.18) will be called a *connecting copula* of X and Y.

The functions on I^2 given by $\operatorname{Min}(a, b)$, $\operatorname{Prod}(a, b) = ab$ and

$$T_m(a, b) = \operatorname{Max}(a+b-1, 0), \tag{2.19}$$

respectively, are copulas. Furthermore, we have:

THEOREM 3. *Any copula C satisfies the inequalities*

$$T_m(a, b) \leqslant C(a, b) \leqslant \operatorname{Min}(a, b), \quad \text{for all } (a, b) \in I^2. \tag{2.20}$$

In the other direction, any function on I^2 that satisfies (2.12) *and* (2.20) *is a copula.*

Proof. Let C be a copula. From (2.12) with $c = d = 1$ and (2.11) we obtain

$$C(a, b) - a - b + 1 \geqslant 0, \quad \text{or} \quad C(a, b) \geqslant a + b - 1.$$

Thus, since $C(a, b) \geqslant 0$ for any (a, b), we have $C(a, b) \geqslant T_m(a, b)$. Similarly, from (2.11) and Lemma 4, we obtain

$$C(a, b) \leqslant C(a, 1) = a \quad \text{and} \quad C(a, b) \leqslant C(1, b) = b,$$

whence $C(a, b) \leqslant \operatorname{Min}(a, b)$. This proves (2.20). Finally, both T_m and Min satisfy (2.10) and (2.11), whence any function C on I^2 satisfying (2.20) inherits these properties.

3. Operations on Δ. As in the preceding section, let X, Y be random variables with distribution functions F_X, F_Y and joint distribution function H_{XY}. Let C_{XY} be a connecting copula of X and Y. Then we have:

THEOREM 4. *Let g be a Borel-measurable, two-place real function. Then $g(X, Y)$ is a real random variable whose distribution function $F_{g(X,Y)}$ is given by:*

$$F_{g(X,Y)}(x) = \iint_{g(u,v)<x} dC_{XY}(F_X(u), F_Y(v)), \quad \text{for all } x. \tag{3.1}$$

Proof. It is well known (see, e. g. [3], p. 170) that $g(X, Y)$ is a random variable, and that

$$F_{g(X,Y)}(x) = \iint_{g(u,v)<x} dH_{XY}(u, v). \tag{3.2}$$

Substituting (2.18) into (3.2) yields (3.1).

The fact that the integral in (3.1) depends on the distribution functions F_X, F_Y, the copula C_{XY} and the function g, but not directly on the random variables X and Y, motivates the following:

DEFINITION 5. For any Borel-measurable, two-place real function g, any copula C, and any pair of distribution functions F and G, $\mathscr{I}(g, C)(F, G)$ is the real function given by:

$$[\mathscr{I}(g, C)(F, G)](x) = \iint_{g(u,v)<x} dC(F(u), G(v)), \quad \text{for all } x. \tag{3.3}$$

It follows readily from (2.12) and the definition of a copula that $0 \leqslant [\mathscr{I}(g, C)(F, G)](x) \leqslant 1$ and that $\mathscr{I}(g, C)(F, G)$ is non-decreasing on $\mathbf{R}$. Thus (when normalized to be left-continuous, if necessary) $\mathscr{I}(g, C)(F, G)$ belongs to Δ; or, in other words, for fixed g and C, $\mathscr{I}(g, C)$ is a binary operation on Δ. Furthermore, if g is non-negative on the first quadrant then it follows from (2.10) that the restriction of $\mathscr{I}(g, C)$ to Δ^+ is a binary operation on Δ^+, where Δ^+ denotes the subset of Δ consisting of all distribution functions F with $F(0) = 0$.

Combining (3.1) and (3.3) allows us to write

$$F_{g(X,Y)} = \mathscr{I}(g, C_{XY})(F_X, F_Y); \tag{3.4}$$

and from this display we see at once that every binary operation on random variables gives rise to a family of binary operations (one for each possible choice of connecting copula) on distribution functions. In the special case of addition, i. e., $g(x, y) = x + y$, we write σ_C for $\mathscr{I}(g, C)$, so that

$$F_{X+Y} = \sigma_{C_{XY}}(F_X, F_Y). \tag{3.5}$$

It is well known that σ_{Prod} is convolution. Apart from this case, however, the operations σ_C have been little studied. The most extensive investigation to date is by M. J. Frank [2] who, among other things, has shown that σ_C is associative if and only if C is Prod, Min, or an ordinal sum of Prod and Min.

DEFINITION 6. The set $\mathscr{T}_0$ consists of all functions T satisfying conditions (a) and (b) of Definition 2, and (c′) (a weaker version of condition

(c) of Definition 2):

$$T(0,0) = 0, \qquad T(1,1) = 1. \tag{3.6}$$

Thus $\mathcal{T}_0$ contains $\mathcal{T}$ as a subset, and indeed a proper subset, as is seen by the fact that each function $C^\wedge$ defined on I^2 by

$$C^\wedge(a,b) = a+b-C(a,b), \tag{3.7}$$

where C is a copula, belongs to $\mathcal{T}_0$ but not to $\mathcal{T}$.

It is readily established (cf. [5]) that every function in $\mathcal{T}_0$ gives rise to binary operations π_T and τ_T on Δ (or Δ^+) via:

DEFINITION 7. Let T be in $\mathcal{T}_0$ and F, G be in Δ (resp., Δ^+). Then $\pi_T(F,G)$ is the function in Δ (resp., Δ^+) given, at every point x of continuity, by:

$$\big(\pi_T(F,G)\big)(x) = T\big(F(x), G(x)\big); \tag{3.8}$$

and $\tau_T(F,G)$ is the function in Δ (resp., Δ^+) given, at every point x of continuity, by (1.2).

THEOREM 5. *Let C be a copula and let $C^\wedge$ be the function given by* (3.7). *Then*:

$$\pi_C = \mathcal{I}(\max, C) \quad \textit{and} \quad \pi_{C^\wedge} = \mathcal{I}(\min, C). \tag{3.9}$$

Proof. Let $F, G \in \Delta$. Then for any $x \in \mathbf{R}$, we have

$$\big(\pi_C(F,G)\big)(x) = C\big(F(x), G(x)\big) = \iint\limits_{\substack{u<x\\ v<x}} dC\big(F(u), G(v)\big)$$

$$= \iint\limits_{\max(u,v)<x} dC\big(F(u), G(v)\big) = \big(\mathcal{I}(\max, C)(F,G)\big)(x)$$

and

$$\big(\pi_{C^\wedge}(F,G)\big)(x) = C^\wedge\big(F(x), G(x)\big) = F(x)+G(x)-C\big(F(x), G(x)\big)$$

$$= \iint\limits_{u<x} dC\big(F(u), G(v)\big) + \iint\limits_{v<x} dC\big(F(u), G(v)\big) - \iint\limits_{\substack{u<x\\ v<x}} dC\big(F(u), G(v)\big)$$

$$= \iint\limits_{\min(u,v)<x} dC\big(F(u), G(v)\big) = \big(\mathcal{I}(\min, C)(F,G)\big)(x).$$

Thus each of the operations π_C, $\pi_{C^\wedge}$ is derivable from an operation on random variables. Similarly, since it can be shown (see [2]) that

$$\tau_{\mathrm{Min}} = \sigma_{\mathrm{Min}} = \mathcal{I}(\mathrm{sum}, \mathrm{Min}), \tag{3.10}$$

the operation τ_{Min} is also derivable from an operation on random variables, namely addition.

DEFINITION 8. For any $s \in \mathbf{R}$ and any $a \in I$, let ε_s and G_a be the distribution functions defined by:

$$\varepsilon_s(x) = \begin{cases} 0, & s \leqslant x, \\ 1, & s > x; \end{cases} \tag{3.11}$$

$$G_a(x) = \begin{cases} 0, & x \leqslant 0, \\ a, & 0 < x \leqslant 1, \\ 1, & 1 < x. \end{cases} \tag{3.12}$$

Clearly $G_a \in \Delta^+$ for any $a \in I$ and $\varepsilon_s \in \Delta^+$ for any $s \geqslant 0$.

LEMMA 6. *Let $T \in \mathcal{T}$. Then for any $s, t \in \mathbf{R}$, we have*:

$$\tau_T(\varepsilon_s, \varepsilon_t) = \varepsilon_{s+t}. \tag{3.13}$$

Proof. This follows at once from the fact that $\tau_T\big(\varepsilon_s(u), \varepsilon_t(v)\big) = 1$ if and only if $u > s$ and $v > t$ and is equal to 0 otherwise.

LEMMA 7. *Let $T \in \mathcal{T}$, and let C be a copula. Then, for any $a, b \in I$, we have*:

$$\tau_T(G_a, G_b)(x) = \begin{cases} 0, & x \leqslant 0, \\ T(a,b), & 0 < x \leqslant 1, \\ \mathrm{Max}(a,b), & 1 < x \leqslant 2, \\ 1, & 2 < x; \end{cases} \tag{3.14}$$

and

$$\sigma_C(G_a, G_b)(x) = \begin{cases} 0, & x \leqslant 0, \\ C(a,b), & 0 < x \leqslant 1, \\ a+b-C(a,b), & 1 < x \leqslant 2, \\ 1, & 2 < x. \end{cases} \tag{3.15}$$

Proof. The display (3.14) is an easy consequence of (1.2) and (1.3). Next, note that $C(G_a, G_b)$ is the joint distribution function corresponding to four point masses situated as follows: a mass of size $C(a,b)$ at $(0,0)$, one of size $a-C(a,b)$ at $(0,1)$, one of size $b-C(a,b)$ at $(1,0)$ and one of size $1-a-b+C(a,b)$ at $(1,1)$. From this observation and the definition of σ_C (see (3.5)), (3.15) follows at once.

We can now prove Theorem 1.

Proof. Assume that τ_T is derivable, i. e., that a suitable function g exists. Let X and Y be random variables that are constant almost everywhere, with respective values x and y. Then $F_X = \varepsilon_x$ and $F_Y = \varepsilon_y$, whence, by Lemma 6, $\tau_T(F_X, F_Y) = \varepsilon_{x+y}$. Thus, by Definition 1, $F_{g(X,Y)} = \varepsilon_{x+y}$, which means that $g(X, Y)$ assumes the value $x+y$ almost everywhere. Therefore $g(x,y) = x+y$ and, since x, y are arbitrary, g is addition. Hence, by Definition 1 and (3.5), for all random variables X and Y, we have

$$\tau_T(F_X, F_Y) = F_{g(X,Y)} = F_{X+Y} = \sigma_{C_{XY}}(F_X, F_Y), \tag{3.16}$$

where C_{XY} is a connecting copula of X and Y (which may be chosen arbitrarily and may depend on X and Y).

Next, since $T \neq \mathrm{Min}$, there exist numbers a and b, $0 < a, b < 1$ such that $T(a, b) \neq \mathrm{Min}(a, b)$. Let X be a random variable assuming the value 0 with probability a and the value 1 with probability $1-a$. Then $F_X = G_a$. Similarly, let Y be such that $F_Y = G_b$. Then (3.16) yields:

$$\tau_T(G_a, G_b) = \sigma_{C_{ab}}(G_a, G_b), \tag{3.17}$$

where C_{ab} is some connecting copula for X and Y. Now, using (3.14) and (3.15), we have:

$$\tau_T(G_a, G_b)(1/2) = T(a, b) = \sigma_{C_{ab}}(G_a, G_b)(1/2) = C_{ab}(a, b),$$

and

$$\tau_T(G_a, G_b)(3/2) = \mathrm{Max}(a, b) = \sigma_{C_{ab}}(G_a, G_b)(3/2) = a+b-C_{ab}(a, b).$$

Thus,

$$\mathrm{Min}(a, b) \neq T(a, b) = C_{ab}(a, b) = a+b-\mathrm{Max}(a, b) = \mathrm{Min}(a, b).$$

This is a contradiction, whence g cannot exist, and the theorem is proved.

References

[1] M. Eisen, *Introduction to mathematical probability theory*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[2] M. J. Frank, *Associativity in a class of operations on spaces of distribution functions* (to appear in Aequationes Math.).

[3] A. Rényi, *Wahrscheinlichkeitsrechnung*, VEB Deutscher Verlag der Wissenschaften, Berlin 1962.

[4] B. Schweizer, *Probabilistic metric spaces — the first 25 years*, The New York Statistician 19 (1967), pp. 3–6.

[5] — *Multiplications on the space of probability distribution functions* (to appear).

[6] — and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10 (1960), pp. 313–334.

[7] A. N. Šerstnev, *The notion of a random normed space*, Doklady Akad. Nauk SSSR 149 (2) (1963), pp. 280–283. Translated in: Soviet Math. Doklady 4 (2), pp. 338–391.

[8] — *Probabilistic generalization of metric spaces*, Kazan. Gos. Univ. Učen. Zap. 124, Kn. 2 (1964), pp. 3–11.

[9] A. Sklar *Fonctions de répartition à n dimensions et leurs marges*, Public. Inst. Statist. Univ. Paris 8 (1959), pp. 229–231.

[10] — *Random variables, joint distribution functions and copulas*, Kybernetika 9 (1973), pp. 449–460.

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Received August 5, 1973 (723)

STUDIA MATHEMATICA, T. LII. (1974)

Brownian motion, approximation of functions, and Fourier analysis

by

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Abstract. Quantitative approximation theory, initiated by Kolmogorov, is used to show almost-sure properties of *all* mappings $F\circ X$. Here X is Brownian motion, and F is a diffeomorphism of class Lip^{α}, for example. The problems considered touch on Hausdorff dimension, Kronecker sets, Salem sets, and Diophantine approximation. In some cases a critical exponent of smoothness can be found by the category method.

Introduction. In this paper we apply the quantitative approximation theory of Kolmogorov to certain questions on Fourier–Stieltjes transforms and Brownian motion. For example, let E be a compact subset of $(0, +\infty)$ of positive Hausdorff dimension; Kahane proved that $X(E)$ is an M_0-set for almost all Brownian paths X. Therefore the same is true of X of $f(E)$ whenever f is a C^1-diffeomorphism of $(0, +\infty)$ into itself. How large a class S of diffeomorphisms f can be named, so that $X\circ f(E)$ is an M_0-set for *all* f in S, almost surely? An answer is contained in the first chapter. A similar question for transforms $f\circ X(E)$ is considered next; for these sets we obtain strong bounds on certain Fourier transforms. Here matters become distinctly non-linear, but we obtain some precise estimates by simple devices.

In the course of the paper we refer to constructions and inequalities in scattered sources; we list some of these now, as a guide to the flavor of the work.

(a) Lipschitz spaces Λ^{α} and λ^{α}, and Kolmogorov's estimates of the sizes of sets in these spaces, under the name "ε-entropy" [13] and [16, 17, 18 Ch. 10].

(b) Hausdorff measures, Hausdorff dimension, and construction of special "dyadic" sets [6 I, II].

(c) Gaussian processes and Brownian motion [4 XI, XIV].

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