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The hermitian operators on some Banach spaces

by

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Abstract. The hermitian operators on certain types of Banach spaces are described. It is shown that the hermitian operators on an l^p-direct sum $(1 \leqslant p < \infty, p \neq 2)$ of a sequence of Banach spaces are precisely the direct sums of hermitian operators on the summand spaces. The spaces $AC[0,1]$, $C^1[0,1]$, $\mathrm{Lip}[0,1]$, and $\mathrm{lip}\,\alpha$, $0 < \alpha < 1$, admit only trivial hermitian operators, i. e., real multiples of the identity operator. It is further shown that the set of hermitian operators on the dual space of a C^*-algebra A is the closure in the strong operator topology of the set of all adjoints of hermitian operators on A.

1. Introduction. Let X be a Banach space (we use complex scalars throughout), and let T be a bounded linear operator mapping X into X. T is said to be *hermitian* if and only if $\|\exp(itT)\| = 1$ for all real t. For the background and basic features of the notion of hermitian operator, due to G. Lumer and I. Vidav, the reader is referred to [3]. Let $\mathscr{B}(X)$ denote the algebra of bounded operators on X, and let $\mathscr{H}(X)$ be the set of hermitian operators on X. In this paper we characterize $\mathscr{H}(X)$ for some special spaces X-specifically, for l^p-direct sums of Banach spaces (§ 2), for the spaces $AC[0,1]$, $C^1[0,1]$, $\mathrm{Lip}[0,1]$, and $\mathrm{lip}\,\alpha$, $0<\alpha<1$ (§ 3), and for the dual space of a C^*-algebra (§ 4). It turns out that all the spaces considered in § 3 admit only trivial bounded hermitian operators, i. e., real multiples of the identity operator I.

2. Direct sums. Denote by X^* the dual space of the arbitrary Banach space X. For $x \in X$, let $\mathscr{S}(x)$ be the set $\{x^* \in X^*\colon \|x^*\| = 1, x^*(x) = \|x\|\}$. For T in $\mathscr{B}(X)$ it is well-known [3, p. 84] that $T \in \mathscr{H}(X)$ if and only if x^*Tx is real whenever $x \in X$ and $x^* \in \mathscr{S}(x)$. We shall make frequent use of this fact.

In what follows, the l^p-direct sum $(1 \leqslant p < \infty)$ of a sequence $\{X_n\}$ of Banach spaces will be the space of all sequences $x = \{x_n\}$ in $\prod_n X_n$ such that $\sum_n \|x_n\|^p < \infty$, with $\|x\| = \big(\sum_n \|x_n\|^p\big)^{1/p}$. The l^p-direct sum will be

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denoted by $\oplus_p X_n$. We denote by $\mathscr{M}(\{X_n\})$ the space of all sequences $x = \{x_n\}$ in $\prod_n X_n$ such that $\{\|x_n\|\}$ is bounded, with $\|x\| = \sup_n \|x_n\|$.

(2.1) THEOREM. *Let $\{X_n\}$ be a finite or infinite sequence of Banach spaces, and let $X = \oplus_p X_n$, $1 \leqslant p < \infty$, $p \neq 2$. Then an operator T in $\mathscr{B}(X)$ is hermitian if and only if for each n $T(X_n) \subseteq X_n$ and the restriction $T|X_n$ belongs to $\mathscr{H}(X_n)$. The same result holds if $\{X_n\}$ is a finite sequence and X is taken to be $\mathscr{M}(\{X_n\})$.*

Proof. We observe at the outset that if T in $\mathscr{B}(X)$ is a direct sum of hermitian operators on the spaces X_n, then it is easy to see that for each real t, $\exp(itT)$ is an isometry, and hence $T \in \mathscr{H}(X)$.

Suppose first that $p > 1$, and let q be the index conjugate to p. Identify X^* with $\oplus_q X_n^*$ under the natural isometry. Let us note that (the case $p = 2$ included) for each non-zero vector $x = \{x_n\}$ in X $\mathscr{S}(x)$ consists of all sequences $\{x_n^*\} \in \oplus_q X_n^*$ such that $x_n^* \in (\|x_n\|/\|x\|)^{p-1}\mathscr{S}(x_n)$ for each n. It is easy to see that such a sequence belongs to $\mathscr{S}(x)$ (we shall not need the converse of this fact for the proof of the theorem, but we include it for the sake of completeness). Suppose $x^* = \{x_n^*\}$ belongs to $\mathscr{S}(x)$. Then we have

$$\|x\| = x^*(x) = \sum_n x_n^*(x_n) \leqslant \sum_n \|x_n^*\|\,\|x_n\| \leqslant \|x^*\|\,\|x\| = \|x\|.$$

Clearly, for each n, $x_n^*(x_n) = \|x_n^*\|\,\|x_n\|$ and $x_n^*(x_n) = 0$ if and only if $\|x_n^*\| = \|x_n\| = 0$. Moreover, by virtue of [15, p. 17], there is a constant $a > 0$ such that $\|x_n^*\|^q = a\|x_n\|^p$ for all n. a must be $\|x\|^{-p}$, and it follows that $\|x_n^*\| = (\|x_n\|/\|x\|)^{p-1}$.

Next we show that if $T \in \mathscr{H}(X)$, then T has the required form. For each i, j let $T_{ij} = (P_i T)|X_j$, where P_i is the i^{th}-coordinate projection of X onto X_i. For fixed k, let $x = \{x_n\}$ be an arbitrary vector such that $x_n = 0$ for $n \neq k$, and $x_k \neq 0$. Then $x^* = \{x_n^*\} \in \mathscr{S}(x)$ if and only if $x_n^* = 0$ for $n \neq k$ and $x_k^* \in \mathscr{S}(x_k)$. Thus, as x_k^* runs through $\mathscr{S}(x_k)$, $x_k^* T_{kk} x_k = x^* T x$ is real. Hence $T_{kk} \in \mathscr{H}(X_k)$. Next, let k, m be distinct indices, and let $x = \{x_n\}$ be a vector in X with x_k an arbitrary non-zero vector in X_k, x_m an arbitrary non-zero vector in X_m, and $x_n = 0$ for $n \neq k, m$. For arbitrary $y_k^* \in \mathscr{S}(x_k)$ and $y_m^* \in \mathscr{S}(x_m)$ define $x^* = \{x_n^*\}$ (in $\mathscr{S}(x)$) by $x_n^* = 0$ for $n \neq k, m$, and $x_n^* = (\|x_n\|/\|x\|)^{p-1} y_n^*$ for $n = k, m$. We have:

$$x^* T x = x_k^* T_{kk} x_k + x_m^* T_{mm} x_m + x_k^* T_{km} x_m + x_m^* T_{mk} x_k. \tag{2.2}$$

The left-hand side of (2.2) and the first two summands on the right being real, we conclude that

$$(\|x_k\|^{p-1} y_k^* T_{km} x_m + \|x_m\|^{p-1} y_m^* T_{mk} x_k) \quad \text{is real.} \tag{2.3}$$

Keeping x_m fixed, replace x_k in (2.3) by $2x_k$ (note that $\mathscr{S}(2x_k) = \mathscr{S}(x_k)$). This gives

$$(2^{p-1}\|x_k\|^{p-1} y_k^* T_{km} x_m + 2\|x_m\|^{p-1} y_m^* T_{mk} x_k) \quad \text{is real.} \tag{2.4}$$

By subtracting twice the expression in (2.3) from the expression in (2.4) we see that $y_k^* T_{km} x_m$ is real. Replace x_m by ix_m in the last conclusion and get that $y_k^* T_{km} x_m = 0$. It follows easily that $T_{km} = 0$.

Next, we consider the case $p = 1$. In this case $X^* = \mathscr{M}(\{X_n^*\})$ (under a natural isometry). We observe that if $x = \{x_n\}$ is a non-zero vector in X, then $x^* = \{x_n^*\}$ is in $\mathscr{S}(x)$ if and only if $x_n^* \in \mathscr{S}(x_n)$ for $x_n \neq 0$, and $\|x_n^*\| \leqslant 1$ for $x_n = 0$. Indeed, the "if" part of the assertion is obvious. Conversely, if $x^* \in \mathscr{S}(x)$, then

$$\|x\| = \sum_n x_n^*(x_n) \leqslant \sum_n |x_n^*(x_n)| \leqslant \sum_n \|x_n^*\|\,\|x_n\| \leqslant \|x\|.$$

It follows that for each n, $x_n^*(x_n) = \|x_n^*\|\,\|x_n\| = \|x_n\|$, and hence $x_n^* \in \mathscr{S}(x_n)$ if $x_n \neq 0$.

Now if $T \in \mathscr{H}(X)$, with T_{ij} as above, the same argument as before shows that every T_{kk} is hermitian. Let m, k be distinct indices, and let $x = \{x_n\}$ be a non-zero vector in X with $x_n = 0$ for $n \neq k$. For arbitrary y_k^* in $\mathscr{S}(x_k)$ and y_m^* in the unit ball of X_m^*, define $x^* = \{x_n^*\}$ (in $\mathscr{S}(x)$) by setting $x_n^* = 0$ for $n \neq k, m$, and $x_n^* = y_n^*$ for $n = k, m$. $x^* T x = y_k^* T_{kk} x_k + y_m^* T_{mk} x_k$. Hence $y_m^* T_{mk} x_k$ is real. As before, $T_{mk} = 0$.

Finally, suppose $\{X_n\}$ is a finite sequence and $X = \mathscr{M}(\{X_n\})$. In this case $X^* = \oplus_1 X_n^*$. Note that an operator on a Banach space is hermitian if and only if its adjoint is hermitian. Thus, in the case at hand, given an operator $T \in \mathscr{H}(X)$, the proof of the theorem is easily concluded by applying the foregoing for the case $p = 1$ to the operator T^* on X^*.

Remarks. (i) It is known that if X is one of the sequence spaces l^p, $1 \leqslant p \leqslant \infty$, $p \neq 2$, then $\mathscr{H}(X)$ consists of the multiplication operators induced by bounded sequences of real numbers ([13], [14]). Except for the case $p = \infty$, Theorem (2.1) generalizes this known fact. (ii) The description of $\mathscr{H}(X)$ in the statement of Theorem (2.1) is known to hold for a certain type of Banach space X which is required to be the direct sum of a sequence of Hilbert spaces and to satisfy some additional conditions [9, Theorem (2.6)].

We shall touch briefly on the situation when X is the l^2-direct sum of a sequence $\{X_n\}$ of Banach spaces. The conclusion of Theorem (2.1) is, of course, no longer valid for $p = 2$ if each X_n is a Hilbert space. In this connection, the following theorem is available.

(2.5) THEOREM. *Let Y be a Banach space, and let X be the l^2-direct sum, $X = Y \oplus Y$. Let T be the element of $\mathscr{B}(X)$ whose matrix (relative to the* *given direct sum decomposition of* X) *is* $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. *Then* $T \in \mathscr{H}(X)$ *if and only if* Y *is a Hilbert space.*

Proof. The "if" part of the assertion is obvious. Conversely, suppose $T \in \mathscr{H}(X)$. We note that for each ordered pair $\langle y_1, y_2 \rangle$ of elements of Y, $T\langle y_1, y_2 \rangle = \langle y_2, y_1 \rangle$. For each $y \in Y$, let $\mathscr{C}(y)$ be the set $\|y\|\mathscr{S}(y)$. Since $T \in \mathscr{H}(X)$, it is easy to see that $\{y_1^*(y_2) + y_2^*(y_1)\}$ is real for y_1, y_2 in Y, y_1^* in $\mathscr{C}(y_1)$, y_2^* in $\mathscr{C}(y_2)$. In particular, if f_1 and f_2 are in $\mathscr{C}(y_1)$, then $(f_1 - f_2)(y_2)$ is real for all y_2 in Y. Thus each $\mathscr{C}(y)$ for $y \in Y$ is a singleton. Let φ_y denote the unique element of $\mathscr{C}(y)$, and observe that for λ a complex number and y in Y, $\varphi_{\lambda y} = \bar{\lambda}\varphi_y$ (where the bar denotes complex conjugation). On the Cartesian product $Y \times Y$ define the function $[\ ,\ ]$ by setting $[z, y] = \varphi_y(z)$ ($[\ ,\ ]$ is a "semiinner-product" for Y [3, § 9]). Clearly, $[\ ,\ ]$ is left linear, and, for all y in Y, $[y, y] = \|y\|^2$. To complete the proof it suffices to show that $[\ ,\ ]$ is conjugate commutative. For $y, z \in Y$, we showed above that $\{[z, y] + [y, z]\}$ is real. Replacing z by iz in this last expression gives the conclusion that $i\{[z, y] - [y, z]\}$ is real. It follows readily that $[y, z] = \overline{[z, y]}$.

3. Certain concrete spaces with only trivial hermitian operators. In this section we shall be concerned with the following Banach spaces:

(i) $C^1[0, 1]$, the space of continuously differentiable complex-valued functions on $[0, 1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$,

(ii) $\mathrm{Lip}[0, 1]$, the space of all complex-valued functions on $[0, 1]$ satisfying a Lipschitz condition of order 1 with $\|f\| = \|f\|_\infty + \operatorname{ess\,sup}|f'|$,

(iii) $AC[0, 1]$, the space of absolutely continuous functions on $[0, 1]$ with $\|f\| = \|f\|_\infty + \|f'\|_1$,

(iv) $\mathrm{lip}\,\alpha$, $0 < \alpha < 1$, the space of all complex-valued functions f on the real line R of period 1 such that $\sup_{x \in R} |f(x+h) - f(x)| = o(|h|^\alpha)$, as $h \to 0$, with

$$\|f\| = \sup_{x,y,h} \{|f(x)|, |h|^{-\alpha}|f(y+h) - f(y)|\}.$$

In the foregoing, ess sup and $\|\cdot\|_1$ are, of course, taken with respect to Lebesgue measure.

(3.1) Theorem. *If* X *is one of the spaces* $C^1[0, 1]$, $\mathrm{Lip}[0, 1]$, $AC[0, 1]$, *or* $\mathrm{lip}\,\alpha$ $(0 < \alpha < 1)$, *then* $\mathscr{H}(X) = \{rI : r \in R\}$.

Proof. Let $A \in \mathscr{H}(X)$. Then $\{\exp itA\}$, $t \in R$, is a one-parameter group of isometries of X onto X, continuous with respect to the uniform operator topology. Let $T_t = \exp itA$, for $t \in R$. We consider first the case where X is one of the spaces $C^1[0, 1]$, $\mathrm{Lip}[0, 1]$, $AC[0, 1]$. By [10, Theorems 2.5, 3.3, and 4.1] for each $t \in R$, T_t has the form $(T_t f)(x) = \lambda_t f(\tau_t(x))$,

$x \in [0, 1]$, where λ_t is a unimodular complex constant, and $\tau_t(\cdot)$ is a monotone one-to-one absolutely continuous mapping of $[0, 1]$ onto itself (if X is $C^1[0, 1]$ or $\mathrm{Lip}[0, 1]$, then $\tau_t(x)$ is, in fact, identically x or identically $1 - x$). Let φ_0 (resp., φ_1) be the element of X defined for each $x \in [0, 1]$ by $\varphi_0(x) = 1$ (resp., $\varphi_1(x) = x$). We observe that $T_t\varphi_0$ has the constant value λ_t, and that $\tau_t = \bar{\lambda}_t T_t \varphi_1$. Thus λ_t and τ_t are uniquely determined by T_t, and λ_t and τ_t, as functions of t, are continuous mappings of R into the set of unimodular complex numbers and X, respectively. From the uniqueness of representation for the operators of the group $\{T_t\}$, $t \in R$, we have $\lambda_{s+t} = \lambda_s\lambda_t$ for all $s, t \in R$. Thus $\{\lambda_t\}$ is a one-parameter continuous group of unimodular complex numbers, and hence there is a real constant r such that $\lambda_t = e^{irt}$ for all $t \in R$. It suffices for the proof to assume that $\lambda_t = 1$ for all $t \in R$, and show that A must be 0 (since the result could then be applied to the group $\{e^{-irt}T_t\}$, $t \in R$). If X is $C^1[0, 1]$ or $\mathrm{Lip}[0, 1]$, then, as noted earlier, each τ_t belongs to the doubleton set $\{\varphi_1, (1 - \varphi_1)\}$. Since $\lim_{t \to 0} \|\tau_t - \varphi_1\| = 0$ (by the continuity of $t \mapsto \tau_t$ as a map from R into X), there is a real neighborhood N of 0 such that $\tau_t = \varphi_1$ (and hence $T_t = I$) for all $t \in N$. Thus $iA = \left.\frac{dT_t}{dt}\right|_{t=0} = 0$. If X is $AC[0, 1]$, then each continuously differentiable f on $[0, 1]$ belongs to X, and, for each fixed x in $[0, 1]$, $\left.\frac{df(\tau_t(x))}{dt}\right|_{t=0}$ exists and is equal to $(iAf)(x)$. In particular, taking $f = \varphi_1$, we get $\left.\frac{d\tau_t(x)}{dt}\right|_{t=0} = (iA\varphi_1)(x)$, for $x \in [0, 1]$. Application of the chain rule now gives for all continuously differentiable f, and all x in $[0, 1]$, $f'(x)[(iA\varphi_1)(x)] = (iAf)(x)$. If $A\varphi_1$ were not the zero function, then this last equation would give the absurd conclusion that for every continuously differentiable function f on $[0, 1]$, there is a set of positive Lebesgue measure at each point of which f'' exists. It follows readily that $A = 0$.

Suppose now that $X = \mathrm{lip}\,\alpha$. [4, Theorem 4.1] states that a linear isometry U of $\mathrm{lip}\,\alpha$ onto itself has the form

$$(3.2) \qquad (Uf)(x) = \lambda f(a + \sigma x) \quad \text{for all } x \in R,\ f \in \mathrm{lip}\,\alpha,$$

where λ, a, σ are constants such that λ is complex of modulus one, $a \in R$, and σ is 1 or -1. It follows from this fact (applied to $T_{t/2}$) and the equation $T_t = (T_{t/2})^2$ that each T_t can be represented in the form (3.2) with σ equal to 1. Let us choose such a representation for each T_t, denoting the constants which occur by λ_t and a_t. Define the sequence $\{g_n\}_{n=0}^\infty \subseteq \mathrm{lip}\,\alpha$ by

$$g_n(t) = \exp(2\pi int), \quad \text{for } t \in R.$$

Since for each $t \in R$, $T_t g_0$ has the constant value λ_t, it is clear that λ_t is uniquely determined by t, and that (as a function of t) λ_t is a continuous character of the additive group of R. As before, it suffices for the proof to assume that $\lambda_t = 1$ for all $t \in R$ and show that $A = 0$. We remark in passing that it was necessary to choose (as we have done) a definite value of a_t for each $t \in R$, since it follows from the periodicity of the functions in lip α that a_t could not be uniquely determined by t and (3.2). Without loss of generality we let a_0 be 0. In view of the fact that $iAg_n = \left.\frac{dT_t g_n}{dt}\right|_{t=0}$, we have:

$$iAg_n = \left[\left.\frac{d\exp(2\pi i n a_t)}{dt}\right|_{t=0}\right] g_n, \quad n = 0, 1, 2, \ldots \tag{3.3}$$

Define the complex constant β by setting $2\pi i\beta = \left.\frac{d\exp(2\pi i a_t)}{dt}\right|_{t=0}$. Then it follows from (3.3) (for $n = 1$) that for each $t \in R$, $T_t g_1 = [\exp(2\pi i\beta t)] g_1$, and consequently $a_t - \beta t$ is an integer. Thus without loss of generality we can take $a_t = \beta t$ for each $t \in R$. Now (3.3) gives $iAg_n = 2\pi i n\beta g_n$, $n = 0, 1, 2, \ldots$ Since A is bounded, β must be 0. Hence for all $t \in R$, $a_t = 0$ and $T_t = I$. This concludes the proof.

4. Hermitian operators on the dual space of a C^*-algebra. Throughout this section a C^*-algebra $\mathscr{A}$ will be a Banach *-algebra with identity such that $\|x^*x\| = \|x\|^2$ for all $x \in \mathscr{A}$. A W^*-algebra will be a C^*-algebra which is (linearly isometric to) the dual space of a Banach space. It will be convenient henceforth to denote dual spaces and adjoints of operators on Banach spaces by prime superscripts.

In the scholium which follows we record a known result in a form convenient for our purposes.

(4.1) Scholium (A. M. Sinclair). *If X is a W^*-algebra, then $\mathscr{H}(X)$ consists of all operators $T \in \mathscr{B}(X)$ for which there exist self-adjoint elements u and v of X such that $Tx = ux + xv$ for all $x \in X$.*

Proof. By [12], Remark 3.5 and [6], Theorem 1, p. 311.

If A is a C^*-algebra, and U its universal representation, then it is well-known that A'', the second dual space of A, can be identified with the closure in the weak operator topology of $U(A)$ so as to make U the canonical embedding of A in A'' [7, 12.1.3-(iv)]. We shall make free use of this fact; in particular, we shall regard A'' as a W^*-algebra in the sense of this identification. (4.1) allows us to deduce as a corollary an unpublished result of G. Lumer, which we state next for later convenience.

(4.2) Corollary (G. Lumer). *Let A be a commutative C^*-algebra, and let L be the regular representation of A (i. e., $L_a x = ax$ for $a, x \in A$). Then $\mathscr{H}(A) = \{L_a : a \in A,\ a = a^*\}$.*

Proof. If $T \in \mathscr{H}(A)$, then $T'' \in \mathscr{H}(A'')$. Since $U(A)$ is commutative, A'' is also commutative. By (4.1) there is a self-adjoint element $c \in A''$ such that $T''x = cx$ for all $x \in A''$. Thus $c = T''(U1) = U(T1) \in U(A)$, by a standard property of second adjoint operators. Thus $\mathscr{H}(A) \subseteq \{L_a : a \in A,\ a = a^*\}$. The reverse inclusion is easy.

(4.3) Theorem. *Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{H}(\mathscr{A}')$ is the closure in the strong operator topology of $\{T' : T \in \mathscr{H}(\mathscr{A})\}$.*

Proof. Since for any Banach space X, $\mathscr{H}(X)$ is closed in the strong operator topology of $\mathscr{B}(X)$, and a convex subset of $\mathscr{B}(X)$ has the same closure in the weak operator topology as in the strong operator topology [1, Lemma 3.3], it suffices for the proof of the theorem to show that if $T \in \mathscr{H}(\mathscr{A}')$, then there is a net $\{T_\gamma\} \subseteq \mathscr{H}(\mathscr{A})$ such that $z(Ty) = \lim_\gamma z(T'_\gamma y)$ for all $z \in \mathscr{A}''$ and all $y \in \mathscr{A}'$. We note first that by (4.1) there are self-adjoint elements u, v of $\mathscr{A}''$ such that $T'z = uz + zv$ for all $z \in \mathscr{A}''$. By Goldstine's theorem [8, V.4.5], there are (bounded) nets $\{u_\gamma\}_{\gamma \in \Gamma}$, $\{v_\gamma\}_{\gamma \in \Gamma}$ in $\mathscr{A}$ such that $\{U(u_\gamma)\}$ (resp., $\{U(v_\gamma)\}$) converges to u (resp., v) in the weak*-topology of $\mathscr{A}''$. (We remark that $\mathscr{A}''$, being a W^*-algebra, has a unique weak*-topology [11, p. 30].) Since the involution of $\mathscr{A}''$ is weak*-continuous [11, Theorem 1.7.8], we can, without loss of generality, take u_γ and v_γ to be self-adjoint in $\mathscr{A}$ for all γ. Moreover, multiplication in $\mathscr{A}''$ is weak*-continuous in each variable separately [11, Theorem 1.7.8], and so for all $z \in \mathscr{A}''$ and all $y \in \mathscr{A}'$,

$$(T'z)(y) = \lim_\gamma [U(u_\gamma)z + zU(v_\gamma)](y). \tag{4.4}$$

Denote by Λ (resp., P) the left (resp., right) regular representation of $\mathscr{A}$, i. e., $\Lambda_a x$ (resp., $P_a x$) is ax (resp., xa) for all $a, x \in \mathscr{A}$. Then for all $a, x \in \mathscr{A}$, $(\Lambda_a)'' U(x)$ (resp., $(P_a)'' U(x)$) is equal to $U(a)U(x)$ (resp., $U(x)U(a)$). Since mutliplication in $\mathscr{A}''$ is weak*-continuous in each variable separately, and $U(\mathscr{A})$ is weak*-dense in $\mathscr{A}''$, it is obvious that, on all of $\mathscr{A}''$, $(\Lambda_a)''$ (resp., $(P_a)''$) is left (resp., right) multiplication by $U(a)$. Combining this last observation with (4.4) completes the proof.

Remark. We know of no Banach space X such that $\mathscr{H}(X')$ is not the closure in the strong operator topology of $\{T' : T \in \mathscr{H}(X)\}$.

Let Ω be a compact Hausdorff space, and let $C(\Omega)$ (resp., $B(\Omega)$) be the algebra of all complex-valued continuous (resp., bounded Borel) functions on Ω. With the usual involution and with the norm of f given by $\sup\{|f(x)| : x \in \Omega\}$, $C(\Omega)$ and $B(\Omega)$ are C^*-algebras. By the Riesz representation theorem $[C(\Omega)]' = M(\Omega)$, the space of all regular Borel measures on Ω. For each $f \in B(\Omega)$ define $S_f \in \mathscr{B}(M(\Omega))$ by $S_f(\mu) = \int_{(\cdot)} f\,d\mu$, for all $\mu \in M(\Omega)$. It is easy to see that $S_{(\cdot)}$ is an isometric algebra isomorphism of $B(\Omega)$ into $\mathscr{B}(M(\Omega))$, that S_f is hermitian if and only if f is real-valued, and that for each $g \in C(\Omega)$, $S_g = (L_g)'$ (in the notation of (4.2)). In terms of the foregoing notation (4.3) has the following corollary.

(4.5) COROLLARY. *$\mathscr{H}(M(\Omega))$ is the closure in the strong operator topology of $\{S_f: f \in C(\Omega)$ and f is real-valued$\}$.*

Proof. By (4.2) and (4.3).

(4.6) Remarks. If $\mathscr{A}$ is a C^*-algebra, then it follows from (4.1) that each $T \in \mathscr{H}(\mathscr{A}'')$ is weak*-continuous on $\mathscr{A}''$, and hence is the adjoint of a (necessarily hermitian) operator on $\mathscr{A}'$. Thus the map which assigns Q' to Q is one-to-one from $\mathscr{H}(\mathscr{A}')$ onto $\mathscr{H}(\mathscr{A}'')$. It follows from this remark and (4.2) that $\mathscr{H}(M(\Omega))$ is a commutative subring of $\mathscr{B}(M(\Omega))$. This last fact is also clear from (4.5).

EXAMPLE. We show that $\mathscr{H}(M([0,1]))$ is strictly larger than $\{S_g: g \in B([0,1]),\ g$ real-valued$\}$. Indeed, the cardinal number of the latter set is c, the power of the continuum. We shall demonstrate that the set of idempotent elements in $\mathscr{H}(M([0,1]))$ has cardinal number at least 2^c. By virtue of (4.2) and the first part of (4.6) this amounts to showing that the maximal ideal space Ω_0 of $C[0,1]''$ has at least 2^c open-closed sets. Identify $C[0,1]''$ with $C(\Omega_0)$. For each $x \in [0,1]$, let h_x be the homomorphism of $C[0,1]$ onto the complex field given by $h_x(f) = f(x)$. Since $U(C[0,1])$ is weak*-dense in $C(\Omega_0)$, it is easy to see that evaluation at h_x is a weak*-continuous homomorphism of $C(\Omega_0)$ onto the complex field. Thus there is a one-to-one map $x \mapsto p_x$ of $[0,1]$ into Ω_0 such that unit mass at p_x is a normal measure on Ω_0 [5, Corollary, p. 171]. Thus by [5, Proposition 3] the singleton set $\{p_x\}$ is open-closed. For each subset a of $[0,1]$ let $\Gamma(a)$ be the closure in Ω_0 of $\{p_x: x \in a\}$. It is easy to see, since Ω_0 is stonian, that $\Gamma(a)$ is open-closed in Ω_0. Also, it is now easy to see that $\Gamma(\cdot)$ is one-to-one.

Remark. In [2, (3.3)] there was defined for an arbitrary Banach space X a notion of orthogonality (relative to a suitable family of idempotent elements of $\mathscr{H}(X)$). Let Ω be, as above, a compact Hausdorff space, and for each Borel set γ in Ω, let k_γ be the characteristic function of γ, and put $E(\gamma) = S_{k_\gamma}$. Let $\mathscr{F}$ be $\{E(\gamma): \gamma$ is a Borel set in $\Omega\}$. Then (in the notation of [2, § 3]) it is straightforward to see that for any two measures μ, ν in $M(\Omega)$, μ and ν are mutually singular if and only if $\mu \perp_{\mathscr{F}} \nu$ in $M(\Omega)$. We omit the details.

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