STUDIA MATHEMATICA, T. LII. (1974)

On a problem of Pełczyński: Milutin spaces, Dugundji spaces and AE(0-dim)

by

RICHARD HAYDON* (Oxford, England)

Abstract. It is shown that the class of Dugundji spaces, introduced by Pełczyński in [5], coincides with the class of absolute extensors for compact zero-dimensional spaces. It follows from this that every Dugundji space is a Milutin space.

1. Introduction. In [5], Pełczyński introduced the notions of Milutin space and Dugundji space and posed the problem ([5], Problem 15): Does the class of all Milutin spaces coincide with the class of all Dugundji spaces? S. Z. Ditor (unpublished Ph. D. thesis, University of California, Berkeley, 1968) and the present author (unpublished essay, University of Cambridge, 1971) proved that a Milutin space of topological weight at most ω_1 is necessarily a Dugundji space. It is the object of this article to show that every Dugundji space is a Milutin space, without restriction on topological weight.

The method of proof used here is to show that every Dugundji space is an absolute extensor for compact zero-dimensional spaces (AE(0-dim)). Such spaces were first introduced in this context by Ditor, who proved that every AE(0-dim) is a Milutin space ([1], Corollary 2). Theorem 1 in the present article includes a rather simpler proof of this result as well as a proof that every AE(0-dim) is Dugundji.

All the topological spaces considered in what follows will be compact (and Hausdorff); $\boldsymbol{I}$ will denote the unit interval $[0,1] \subset \boldsymbol{R}$ and $\boldsymbol{D}$ the two-point space $\{0, 1\}$. The identity function on S is written ι_S. If $\varphi: S \to T$, $\varrho: T \to S$ are continuous mappings with $\varrho \circ \varphi = \iota_S$, we say that ϱ is a retraction and φ a coretraction.

I follow the convention of [6] in identifying a cardinal number with the corresponding initial ordinal (and thus avoid the 'aleph' notation). The topological weight of the space S is the smallest cardinal τ such that there exists a base $\mathscr{B}$ for the topology of S with card $\mathscr{B} = \tau$; we can embed S in the product $\boldsymbol{I}^A$, with card $A = \tau$.

* The research was carried out while the author was supported by a grant from the Science Research Council of Great Britain.

$C(S)$ denotes the space of all continuous real-valued functions on S, equipped with the supremum norm. 1_S is the function that is identically 1 on S. The dual $C(S)'$ is identified, as usual, with the space $M(S)$ of all regular signed Borel measures on S. $P(S)$ is the compact convex set of probability measures on S, $\{\mu\in M(S)\colon \|\mu\| = \langle\mu, 1_S\rangle = 1\}$, equipped with the induced $\sigma\big(M(S), C(S)\big)$ topology. δ_S is the canonical embedding $S\to P(S)$.

If φ is a continuous mapping from S into T, $\tilde\varphi$ will denote the induced mapping $P(S)\to P(T)$, defined by $\langle\tilde\varphi\mu, f\rangle = \langle\mu, f\circ\varphi\rangle$ $(f\in C(T))$. In the case that T is a compact *convex* space (by which I mean "a compact convex subset of a locally convex space"), $\bar\varphi$ will denote the composition of $\tilde\varphi$ with the centroid mapping $P(T)\to T$ ([6], 23.4.2). We can characterize $\bar\varphi$ as the unique continuous affine mapping $P(S)\to T$ which satisfies $\varphi = \bar\varphi\circ\delta_S$.

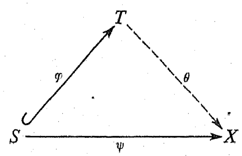
Recall that an operator $u\colon C(S)\to C(T)$ is called *regular* if u is positive and $u(1_S) = 1_T$. If $\varphi\colon S\to T$ is a homeomorphic embedding (resp. a continuous surjection), a regular operator $u\colon C(S)\to C(T)$ is called *a regular extension operator*, or r. e. o. (resp. a *regular averaging operator*, or r. a. o.) if $u(f)\circ\varphi = f$ for all $f\in C(S)$ (resp. if $u(g\circ\varphi) = g$ for all $g\in C(T)$).

By the integral representation of 4.1 of [5], an r. e. o. u for the embedding $\varphi\colon S\to T$ is determined by a continuous mapping $\sigma\colon T\to P(S)$ which satisfies $\sigma\circ\varphi = \delta_S$. We find that $\bar\sigma\colon P(T)\to P(S)$ is a retraction for the embedding $\tilde\varphi\colon P(S)\to P(T)$. Similarly, an r. a. o. for the surjection $\varphi\colon S\to T$ corresponds to a continuous mapping $\lambda\colon T\to P(S)$ with $\tilde\varphi\circ\lambda = \delta_T$. This just says that, for each $t\in T$, $\lambda(t)$ is a probability measure supported by $\varphi^{-1}(t)$. In this case $\bar\lambda$ is a coretraction for the surjection $\tilde\varphi\colon P(S)\to P(T)$.

A space S is called a *Dugundji space* if every embedding $\varphi\colon S\to T$ admits an r. e. o. It is enough that, for some index set A, and some embedding $\varphi\colon S\to \boldsymbol{I}^A$, φ should admit an r. e. o. S is called a *Milutin space* if there is a continuous surjection from some $\boldsymbol{D}^A$ onto S that admits an r. a. o. Every product of compact metrizable spaces is both a Milutin space and a Dugundji space ([5], 5.6 and 6.6).

2. Absolute retracts, Dugundji spaces and AE(0-dim). Before getting down to the main theorems, I devote a little space to an alternative approach to Dugundji spaces and indicate how similar considerations lead us naturally to the AE(0-dim). Let us consider the general extension

Diagram 1

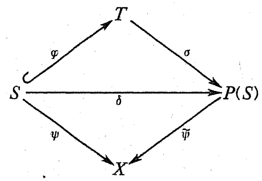


problem where φ is an embedding, ψ is a continuous mapping and we are trying to find θ with $\psi = \theta\circ\varphi$. It is well-known that the absolute retracts (AR's) are characterized either as those spaces X such that the above problem can be solved for arbitrary S, T, φ and ψ, or as those S such that it can be solved for arbitrary T, X, φ and ψ. (See, for instance, Theorems 3.1 and 3.2 of [3].)

The extension problem in its most general form seems to be quite intractable, and the problem of characterizing AR's to be very difficult. Certainly, anything resembling a complete description would probably require techniques of algebraic topology as well as of analysis. In order to obtain a problem allowing an analytical treatment, two approaches suggest themselves, the starting point in each case being an extension theorem involving metrizability.

In the first place, we can restrict X to be a compact *convex* space and characterize a class of spaces S by demanding that there should always be a solution to the extension problem. The Borsuk–Dugundji theorem (see, for instance, [6], Theorem 21.1.4) tells us that every compact metrizable space is such an S. In fact, it is easy to see that these spaces are exactly the Dugundji spaces. For if such a space S is embedded in T by φ, we need only take $X = P(S)$, $\psi = \delta_S$ to obtain a map $\theta\colon T\to P(S)$ which will determine an r. e. o. for φ. If, on the other hand, S is a Dugundji space, embedded in T by φ, and ψ is a continuous map from S into the compact convex space X, there is $\sigma\colon T\to P(S)$ with $\sigma\circ\varphi = \delta_S$ and we can take $\theta = \bar\psi\circ\sigma$. It is a triviality now to see that S is AR if and only if S is both a Dugundji space and a retract of some compact convex space.

Diagram 2



The second approach involves imposing conditions on T and seeing what class of spaces X we have characterized. Motivation for the choice of condition is provided by the theorem of E. Michael ([4], Theorem 2) that the extension problem can be solved for all metrizable X, provided we restrict T to be zero-dimensional. The spaces X that we obtain are called absolute extensors for zero-dimensional spaces, or AE(0-dim). The result to be proved, namely that the class of all Dugundji spaces coincides with the class of all AE(0-dim), shows that these two ways of generalizing from the good extension properties of metrizable spaces give exactly the same answer. The first step in the proof is the following theorem.

THEOREM 1. *If* X *is an* AE(0-dim), *then* X *is both a Milutin space and a Dugundji space.*

Proof. Embed X in some $Z = I^A$. As in the proof of 5.6 of [5], there is a continuous surjection $\varphi\colon \boldsymbol{D}^B \to Z$ (for suitable B) which allows a regular averaging operator. So there is $\lambda\colon Z \to P(\boldsymbol{D}^B)$ with $\tilde{\varphi}\circ\lambda = \delta_Z$. Since X is an AE(0-dim) and $\boldsymbol{D}^B$ is zero-dimensional, there is a map $\psi\colon \boldsymbol{D}^B \to X$ with $\psi|\varphi^{-1}[X] = \varphi|\varphi^{-1}[X]$. It follows that $\tilde{\psi}|P(\varphi^{-1}[X]) = \tilde{\varphi}|P(\varphi^{-1}[X])$ and thus that $(\tilde{\psi}\circ\lambda)|X = \delta_X$ (since $\lambda(x)$ is supported by $\varphi^{-1}[X]$ for each $x \in X$).

We now see that the surjection $\psi\colon \boldsymbol{D}^B \to X$ allows a regular averaging operator determined by $\lambda|X$, so that X is a Milutin space. If we define $\sigma\colon Z \to P(X)$ by $\sigma = \tilde{\psi}\circ\lambda$, then $\sigma|X = \delta_X$, σ determines a regular extension operator $C(X) \to C(Z)$, and X is a Dugundji space.

3. A sufficient condition for X to be AE(0-dim). The proof of Theorem 2 will require a selection theorem due to E. Michael. Let us recall that a set-valued function Φ from X into the set $\mathscr{P}Y$ of all subsets of Y is called *lower semicontinuous* if $\{x \in X\colon \Phi(x) \cap U \neq \emptyset\}$ is an open subset of X whenever U is an open subset of Y. A mapping $\varphi\colon X \to Y$ is called a *selection* for Φ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Theorem 2 of [4] states that if X is paracompact and zero-dimensional and Y is a complete metric space then every lower semicontinuous function from X into the closed, non-empty subsets of Y admits a continuous selection.

I introduce one new piece of terminology. Let us say that a continuous map $\varphi\colon S \to T$ has a *metrizable kernel* if there is a compact metrizable space K and an embedding $k\colon S \to T \times K$ such that $\varphi = \Pi_1 \circ k$, where Π_1 is the projection $T \times K \to T$.

THEOREM 2. *Let* X *be a compact space that can be represented as the inverse limit*

$$\varprojlim (X_\alpha, p_{\alpha,\beta})_{\alpha\leqslant\beta<\tau}$$

of a well-ordered inverse system, indexed by the ordinals less than some τ, *and satisfying*:

(a) X_0 *is metrizable*;

(b) *for all limit ordinals* $\gamma < \tau$, *the natural mapping from* X_γ *to* $\varprojlim (X_{\alpha'}, p_{\alpha\beta})_{\alpha\leqslant\beta<\gamma}$ *is a homeomorphism*;

(c) *for all* $\beta < \tau$, $p_{\beta,\beta+1}$ *is an open mapping with a metrizable kernel.*

Then X *is an* AE(0-dim).

Proof. Let $\varphi\colon S \hookrightarrow T$ be an embedding with T zero-dimensional and let ψ be a continuous map $S \to X$. Let us define, by transfinite induction, a family $(\theta_\alpha)_{\alpha<\tau}$ of continuous maps $T \to X_\alpha$ satisfying

$$p_{\alpha\beta}\circ\theta_\beta = \theta_\alpha \qquad (\alpha \leqslant \beta < \tau)$$

and

$$\theta_\alpha\circ\varphi = p_\alpha\circ\psi \qquad (\alpha < \tau),$$

where p_α denotes the canonical map $X \to X_\alpha$.

X_0 is metrizable, hence AE(0-dim), and so we can find $\theta_0\colon T \to X_0$ with $\theta_0\circ\varphi = p_0\circ\psi$.

If, for some limit ordinal $\gamma < \tau$, the θ_α have already been defined for $\alpha < \gamma$ and form a consistent system, then θ_γ is already determined, because of (b).

So let us suppose that θ_α has been defined for each $\alpha \leqslant$ some β. We need to define

$$\theta_{\beta+1}\colon T \to X_{\beta+1}$$

with

$$\theta_{\beta+1}\circ\varphi = p_{\beta+1}\circ\psi$$

and

$$p_{\beta,\beta+1}\circ\theta_{\beta+1} = \theta_\beta.$$

Consider the set-valued mapping

$$\Phi\colon T \to \mathscr{P}X_{\beta+1}$$

defined by

$$\Phi(t) = \{p_{\beta+1}\psi(s)\} \qquad (t = \varphi(s) \in \varphi[S]),$$

$$\Phi(t) = p_{\beta,\beta+1}^{-1}\theta_\beta(t) \qquad (t \in T\setminus\varphi[S]).$$

Certainly, each $\Phi(t)$ is a non-empty closed subset of $X_{\beta+1}$. Moreover, Φ is lower semicontinuous. For if U is open in $X_{\beta+1}$,

$$\{t \in T\colon \Phi(t)\cap U \neq \emptyset\} = \varphi\big[(p_{\beta+1}\circ\psi)^{-1}[U]\big] \cup (\theta_\beta^{-1}p_{\beta,\beta+1}[U])\setminus\varphi[S]$$
$$= (\theta_\beta^{-1}p_{\beta,\beta+1}[U])\setminus\varphi\big[S\setminus(p_{\beta+1}\circ\psi)^{-1}[U]\big].$$

Since $p_{\beta,\beta+1}$ is an open mapping and the other mappings are continuous, this is an open subset of T.

By hypothesis, there is a compact metrizable space K and an embedding $k\colon X_{\beta+1} \to X_\beta \times K$ such that $p_{\beta,\beta+1} = \Pi_1\circ k$. The set-valued mapping $\Psi\colon T \to \mathscr{P}K$, defined by $\Psi(t) = \Pi_2 k[\Phi(t)]$ is lower semicontinuous and each $\Psi(t)$ is a closed non-empty subset of K. So, by the theorem of Michael quoted earlier, there is a continuous selection λ for Ψ. We now need only to define $\theta_{\beta+1}$ by $\theta_{\beta+1}(t) = k^{-1}\big(\theta_\beta(t), \lambda(t)\big)$.

4. Some preliminary results. The aim from now on will be to show that Milutin and Dugundji spaces allow representations of the kind considered in Theorem 2. We shall thus see, in passing, that the apparently artificial sufficient condition of that theorem is in fact also a necessary condition. This section is devoted to some preliminary results that will be needed. First let us note the following fact about regular operators, which I refer to as a "module property". Corollary 1 is an easy case of a theorem of Tomiyama that is valid for arbitrary C^*-algebras (Theorem 3.1 of [7]).

Proposition 1. *Let $\varphi\colon S\to Z$ and $\psi\colon T\to Z$ be continuous mappings and suppose that u is a regular operator $C(S)\to C(T)$ that satisfies $u(f\circ\varphi) = (f\circ\psi)$ $(f\in C(Z))$. Then $u\big((f\circ\varphi)\cdot g\big) = (f\circ\psi)\cdot u(g)$ $(f\in C(Z),\ g\in C(S))$; i.e. u is a morphism of $C(Z)$-modules.*

Proof. Let $f\in C(Z)$, $g\in C(S)$ and suppose $t_0\in T$ with $(f\circ\psi)(t_0) = 0$. Then

$$-\|g\|(|f|\circ\varphi)\leqslant(f\circ\varphi)\cdot g\leqslant\|g\|(|f|\circ\varphi)$$

so that

$$-\|g\|\,(|f|\circ\psi)\leqslant u\big((f\circ\varphi)\cdot g\big)\leqslant\|g\|(\,|f|\circ\psi)$$

and we deduce that $u\big((f\circ\varphi)\cdot g\big)(t_0) = 0$.

If t_0 is now an arbitrary point of T, we may replace f by $f' = f-(f\circ\psi)(t_0)1_Z$ and apply the above argument to obtain $u\big((f'\circ\varphi)\cdot g\big)(t_0) = 0$ or

$$u\big((f\circ\varphi)\cdot g\big)(t_0) = (f\circ\psi)(t_0)\cdot u(g)(t_0).$$

Corollary 1. *Let u be a regular averaging operator for the continuous surjection $\varphi\colon S\to T$. Then*

$$u\big((f\circ\varphi)\cdot g\big) = f\cdot u(g)\qquad (f\in C(T),\ g\in C(S)).$$

Corollary 2. *Let u be a regular extension operator for the embedding $\theta\colon S\to T$ and let $\psi\colon T\to Z$ be such that*

$$u(f\circ\psi\circ\theta) = f\circ\psi\qquad (f\in C(Z)).$$

Then

$$u\big((f\circ\psi\circ\theta)\cdot g\big) = (f\circ\psi)\cdot u(g)\qquad (f\in C(Z),\ g\in C(S)).$$

We shall need some lemmas on open mappings.

Lemma 1. *Suppose that the diagram*

$$\begin{array}{ccccc} W & \overset{\varphi}{\hookrightarrow} & Z & \overset{\varrho}{\twoheadrightarrow} & W\\ {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle p}\downarrow\\ W_1 & \overset{\varphi_1}{\hookrightarrow} & Z_1 & \overset{\varrho_1}{\twoheadrightarrow} & W_1\end{array}$$

commutes and that $\varrho\circ\varphi = \iota_W$, $\varrho_1\circ\varphi_1 = \iota_{W_1}$. Then if π is an open mapping, so is p.

Proof. Let U be open in W. Then $\pi\varrho^{-1}[U]$ is open in Z_1 and so $\varphi_1^{-1}\pi\varrho^{-1}[U]$ is open in W_1. But

$$p[U]\subseteq\varphi_1^{-1}\pi\varrho^{-1}[U]\subseteq\varrho_1\pi\varrho^{-1}[U] = p\varrho\varrho^{-1}[U] = p[U].$$

Lemma 2. *Let $\varphi\colon S\hookrightarrow T$ (resp. $\varphi_1\colon S_1\hookrightarrow T_1$) be an embedding that admits a regular extension operator u (resp. u_1) and let π, p be continuous surjections such that the diagram*

$$\begin{array}{ccc} S & \overset{\varphi}{\hookrightarrow} & T\\ {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow\\ S_1 & \overset{\varphi_1}{\hookrightarrow} & T_1\end{array}$$

commutes. Suppose further that $u(f\circ p) = \big(u_1(f)\big)\circ\pi$ for all $f\in C(S_1)$. Then if π is an open mapping, so is p.

Proof. Let ϱ be the retraction of $P(T)$ onto $P(S)$ determined by u and define ϱ_1 similarly. Then the assumptions on u and u_1 imply that the diagram

Diagram 3

$$\begin{array}{ccccc} P(S) & \overset{\tilde\varphi}{\hookrightarrow} & P(T) & \overset{\varrho}{\twoheadrightarrow} & P(S)\\ {\scriptstyle \tilde p}\downarrow & & {\scriptstyle \tilde\pi}\downarrow & & {\scriptstyle \tilde p}\downarrow\\ P(S_1) & \overset{\tilde\varphi_1}{\hookrightarrow} & P(T_1) & \overset{\varrho_1}{\twoheadrightarrow} & P(S_1)\end{array}$$

commutes. A result of Ditor and Eifler ([2], § 4) states that a continuous surjection $\theta\colon X\to Y$ is open if and only if $\tilde\theta\colon P(X)\to P(Y)$ is open. Combining this with Lemma 1, we see that $\tilde p$, and hence also p, is an open mapping.

5. The main theorem.

Theorem 3. *If X is a Dugundji space, then X is also an AE(0-dim).*

Proof. Let τ be the topological weight of X. Since any metrizable space is AE(0-dim), we may assume that τ is uncountable. Take a set A of cardinality τ and embed X as a subspace of I^A. There is a regular extension operator $u\colon C(X)\to C(I^A)$. For $B\subset A$ denote by π_B the projection $I^A\to I^B$. I shall produce an increasing family $A(\alpha)$ of subsets of A, indexed by the ordinals $\alpha<\tau$, and define X_α to be the subspace $\pi_{A(\alpha)}[X]$ of $I^{A(\alpha)}$. $p_\alpha\colon X\to X_\alpha$ will be $\pi_{A(\alpha)}|X$ and $p_{\alpha,\beta}\colon X_\beta\to X_\alpha$ will be the restriction of the projection $I^{A(\beta)}\to I^{A(\alpha)}$, whenever $\alpha\leqslant\beta<\tau$.

The system will have the properties:

(i) $A(0)$ is countable;

(ii) when γ is a limit ordinal $<\tau$, $A(\gamma) = \bigcup_{\alpha<\gamma} A(\alpha)$;

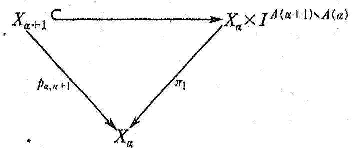
(iii) for all α, $A(\alpha+1)\setminus A(\alpha)$ is countable;

(iv) for all α and all $f \in C(X_\alpha)$, $u(f \circ p_\alpha)$ agrees with $(f \circ \pi_{A(\alpha)})$ on the subspace $X_\alpha \times I^{A \setminus A(\alpha)}$ of I^A;

(v) for all α and all $f \in C(X_{\alpha+1})$, $u(f \circ p_{\alpha+1})|(X \times I^{A \setminus A(\alpha)})$ factors through $\pi_{A(\alpha+1)}$.

I assert that if these conditions hold so do (a), (b) and (c) of Theorem 2. (a) is evident from (i); (b) from (ii). Now consider Diagram 4.

Diagram 4



It is immediate from (iii) that $p_{\alpha,\alpha+1}$ has a metrizable kernel. From (v) it follows that a regular extension operator

$$u_{\alpha+1}: C(X_{\alpha+1}) \to C(X_\alpha \times I^{A(\alpha+1) \setminus A(\alpha)})$$

is defined by requiring that $(u_{\alpha+1}(f)) \circ \pi_{A(\alpha+1)} = u(f \circ p_{\alpha+1})$ on $X_\alpha \times I^{A \setminus A(\alpha)}$ for all $f \in C(X_{\alpha+1})$. We deduce from (iv) that $u_{\alpha+1}(f \circ p_{\alpha,\alpha+1}) = f \circ \pi$ $(f \in C(X_\alpha))$ and, since π is trivially open it follows from Lemma 2 that $p_{\alpha,\alpha+1}$ is open.

The construction of the sets $A(\alpha)$ will be by transfinite induction. As a start, let us set $A(0) = \emptyset$, so that X_0 is a one-point space. Choose a family $(f_\xi)_{\xi<\tau}$ in $C(X)$ which separates the points of X.

Suppose that the $A(\alpha)$ have been defined for all α not greater than some $\beta < \tau$ and that (ii), (iii), (iv), (v) are satisfied so far. Let ξ be the first ordinal for which f_ξ does not factor through p_β. Let us recall (cf. 7.3.13 of [6]) that if f is a continuous real-valued function on I^A there is a countable subset C of A such that f factors through π_C. So there is a countable $B(0) \subset A$ such that f_ξ factors through $\pi_{B(0)}|X$. For the same reason, we can define $B(n)$ inductively by requiring $B(n+1)$ to be a countable subset of A, containing $B(n)$, and such that $u(f)$ factors through $\pi_{B(n+1)}$ whenever $f \in C(X)$ and f factors through $\pi_{B(n)}|X$. Let us now put $B = \bigcup B(n)$.

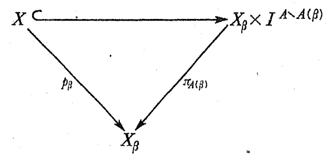
Then I assert that $u(f)$ factors through π_B whenever f is in $C(X)$ and f factors through $\pi_B|X$. For, given any such f and $\varepsilon > 0$, there is, by the Stone–Weierstrass theorem, a finite $C \subseteq B$ and g that factors through $\pi_C|X$ such that $\|f-g\| \leqslant \varepsilon$. For suitable n, $C \subseteq B(n)$ so that $u(g)$ factors through $\pi_{B(n+1)}$ while $\|u(f)-u(g)\| \leqslant \varepsilon$. Thus $u(f)$ can be approximated with functions that factor through π_B and so does itself factor in the same way.

Now define $A(\beta+1) = A(\beta) \cup B$ and consider Diagram 5.

A regular extension operator

$$v: C(X) \to C(X_\beta \times I^{A \setminus A(\beta)})$$

Diagram 5



is defined by $v(f) = u(f)|(X_\beta \times I^{A \setminus A(\beta)})$, and, because of (iv) (with $\alpha = \beta$), we have $v(f \circ p_\beta) = f \circ \pi_{A(\beta)}$ $(f \in C(X_\beta))$. Applying Corollary 2, we see that

$$v((f \circ p_\beta) \cdot g) = (f \circ \pi_{A(\beta)}) \cdot v(g)$$

for all $f \in C(X_\beta)$ and $g \in C(X)$. Thus if $f \in C(X_\beta)$ and g factors through $\pi_B|X$, $v((f \circ p_\beta) \cdot g)$ factors through $\pi_{A(\beta+1)}$. But this is enough to show that $v(h \circ p_{\beta+1})$ factors through $\pi_{A(\beta+1)}$ for all $h \in C(X_{\beta+1})$, which is (v) for $\alpha = \beta$. This also shows that (iv) holds for $\alpha = \beta+1$ since $u(f \circ p_{\beta+1})|(X_{\beta+1} \times I^{A \setminus A(\beta+1)})$ factors through $\pi_{A(\beta+1)}$ and can do so only as $f \circ \pi_{A(\beta+1)}$.

To finish the definition we must consider a limit ordinal $\gamma < \tau$ and suppose that the $A(\alpha)$ have been constructed for $\alpha < \gamma$. We put $A(\gamma) = \bigcup_{\alpha<\gamma} A(\alpha)$ and only have to verify (iv). Any $f \in C(X_\gamma)$ is a uniform limit of functions $g \circ p_{\alpha,\gamma}$ with $g \in C(X_\alpha)$ and $\alpha < \gamma$. For each of these, $u(g \circ p_{\alpha,\gamma} \circ p_\gamma)$ agrees with $g \circ p_{\alpha,\gamma} \circ \pi_{A(\gamma)}$ on $X_\alpha \times I^{A \setminus A(\alpha)}$, and hence on the subset $X_\gamma \times I^{A \setminus A(\gamma)}$. So $u(f \circ p_\gamma)$ agrees with $f \circ \pi_{A(\gamma)}$ on $X \times I^{A \setminus A(\gamma)}$.

To complete the proof of Theorem 2, we only have to check that the mapping $p: X \to \lim(X_\alpha, p_{\alpha,\beta})$ defined by the family (p_α) is injective (for it will then be a homeomorphism). This is immediate since each f_ξ factors through some p_α, hence through p, and the f_ξ were chosen so as to form a separating family of functions.

References

[1] S. Z. Ditor, *On a lemma of Milutin concerning averaging operators in continuous function spaces*, Trans. Amer. Math. Soc. 149 (1970), pp. 443–452.
[2] — and L. Q. Eifler, *Some open mapping theorems for measures*, Trans. Amer. Math. Soc. 164 (1972), pp. 287–293.
[3] S. T. Hu, *Theory of retracts*, Detroit 1965.
[4] E. Michael, *Selected selection theorems*, Am. Math. Monthly 63 (1956), pp. 233–238.
[5] A. Pełczyński, *Linear extensions, linear averagings, and their application to linear topological classification of spaces of continuous functions*, Dissertationes Math. 58, Warszawa 1968.
[6] Z. Semadeni, *Banach spaces of continuous functions*, Warszawa 1971.
[7] Y. Tomiyama, *Tensor products and projections of norm one in von Neumann algebras*, Seminar notes of the Mathematical Institute of the University of Copenhagen, 1970.

MATHEMATICAL INSTITUTE
OXFORD, ENGLAND.

Received April 19, 1973 (675)