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Addition to the paper "Translation invariant subspaces of $L^p(G)$" Studia Mathematica 48 (1973), pp. 245–250

by

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The purpose of this note is to prove that the condition $1 \leqslant p < \frac{4}{3}$ in Theorem 1.1 of [1] can be replaced by the condition $1 \leqslant p < 2$. That is, we prove:

THEOREM. *If G is a locally compact abelian group which is not compact and $1 \leqslant p < 2$, then $L^p(G)$ contains a closed translation invariant subspace which is not the closed span of translates of a single function.*

In what follows we keep all the definitions and notations of [1]. It follows from Proposition 3.1 of [1] that the Theorem will be proved once we show that the bilinear functional L — which is defined by formula (3.1) of [1]—which we proved there to be bounded with respect to the $\mathscr{F}_p(\Gamma)$ norm for $1 \leqslant p < \frac{4}{3}$, is in fact bounded for every $1 \leqslant p < 2$. Consequently, in virtue of formula (3.2) of [1], the theorem will follow from the following:

PROPOSITION. *If Γ is a locally compact abelian group which is not discrete, and $1 \leqslant p < 2$, then there exist in $A(\Gamma)$ real valued functions f_1, f_2 and a positive function φ, all with compact support, such that for every function g in $A(\Gamma)$ with compact support*

$$\Big| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^2 \langle g\bar{g}, \varphi \operatorname{Exp} [i(uf_1+vf_2)]\rangle \, du\, dv \Big| \leqslant C_p \|g\|^2_{\mathscr{F}_p(\Gamma)}$$

where C_p is a constant which depends only on p.

For the proof of the Proposition, the following analog of Lemma 2.1 in [1] is needed:

LEMMA. *If Γ is a locally compact abelian group which is not discrete, then there exist in $A(\Gamma)$ real valued functions f_1, f_2, and a positive function φ, all with compact support, such that for some absolute constant K,*

$$\|\varphi \operatorname{Exp}[(uf_1+vf_2)]\|_{PM(\Gamma)} \leqslant K \exp[-(|u|^{1/2}+|v|^{1/2})] \tag{1}$$

for all real u, v.

This Lemma is a version of a lemma of Malliavin [2] and is proved in Rudin [3], p. 181 for Γ compact with $\varphi \equiv 1$. The general case is deduced by standard arguments using the structure theorem of groups (see the remarks which follow the proof of Lemma 2.1 in [1]).

Proof of the Proposition. Let S denote the subspace of all functions in $A(\Gamma)$ with compact support. Notice that S is contained in $\mathscr{F}_r(\Gamma)$ and also in $PM_r(\Gamma)$ for every $1 \leqslant r \leqslant 2$. For every $(u, v) \in \boldsymbol{R}^2$ we consider the linear operator $T(u, v)$ on S defined by: $T(u, v)h = h\varphi \operatorname{Exp}[i(uf_1+vf_2)]$, $h \in S$; where φ, f_1, f_2 are as in the Lemma. For every $1 \leqslant r \leqslant 2$ let $\|T(u, v)\|_r$ denote the operator norm of $T(u, v)$ regarded as a transformation from S equipped with the $\mathscr{F}_r(\Gamma)$ norm to S equipped with the $PM_r(\Gamma)$ norm. It is easy to see that

$$\|T(u, v)\|_1 = \|\varphi \operatorname{Exp}[i(uf_1+vf_2)]\|_{PM_1}, \tag{2}$$

$$\|T(u, v)\|_2 = \|\varphi \operatorname{Exp}[i(uf_1+vf_2)]\|_\infty = \|\varphi\|_\infty. \tag{3}$$

On the other hand, since for every $1 \leqslant p \leqslant 2$ $PM_p(\Gamma)$ is isometrically isomorphic to $L^q(G)$ where $1/p+1/q = 1$, it follows from the Riesz–Thorin theorem that for every $1 \leqslant p \leqslant 2$

$$\|T(u, v)\|_p \leqslant \|T(u, v)\|_1^t \|T(u, v)\|_2^{1-t}$$

where t satisfies $1/p = t+(1-t)/2$. Therefore, taking into account (1), (2) and (3), we get that for $1 \leqslant p < 2$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 \|T(u, v)\|_p \, du\, dv = C_p < \infty.$$

Consequently, for every $g \in S$ and $1 \leqslant p < 2$:

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 \langle g\bar{g}, \varphi \operatorname{Exp}[i(uf_1+vf_2)]\rangle \, du\, dv\right| = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle g, T(u, v)\bar{g}\rangle \, du\, dv\right|$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|g\|_{\mathscr{F}_p(\Gamma)} \|T(u, v)\bar{g}\|_{PM_p(\Gamma)} \, du\, dv \leqslant C_p \|g\|^2_{\mathscr{F}_p(\Gamma)}.$$

This completes the proof of the Proposition.

References

[1] A. Atzmon, *Translation invariant subspaces of $L^p(G)$*, Studia Math. 48 (1973), pp. 245–250.

[2] P. Malliavin, *Calcul symbolique et sous-algèbres de $L^1(G)$*, Bull. Soc. Math. France 87 (1959), pp. 181–190.

[3] W. Rudin, *Fourier analysis on groups*, Interscience, 1962.

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