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Localization techniques in L^p spaces

by

A. PEŁCZYŃSKI (Warszawa), H. P. ROSENTHAL (Berkeley and Columbus)*

Abstract. Localization refers to obtaining quantitative finite-dimensional formulations of infinite-dimensional results or of unquantified finite-dimensional results. It is proved that for every $\varepsilon > 0$ and $1 < p < \infty$, every finite-dimensional subspace E of L^p is contained in a subspace F of L^p of dimension n which is $(1+\varepsilon)$-complemented a $(1+\varepsilon)$-isomorph of l_n^p, where n depends only on ε and the dimension of E. For $p > 2$, the infinite-dimensional result of Kadec and the first named author that every infinite-dimensional subspace of L^p is either isomorphic to l^2 and complemented, or contains a complemented isomorph of l^p, is localized as follows: there is a positive function g_p so that every subspace E of L^p with $\lambda = d(E, l^2_{\dim E}) < \infty$ is $g_p(\lambda)$-complemented; given n and ε, there is a k so that if $\lambda > k$, then E contains a $(1+\varepsilon)$-complemented $(1+\varepsilon)$ isomorph of l_n^p. One of the consequences of theses results is that for such p, all amost Euclidean subspaces of L^p of the same dimension are in the same position in the space. Other consequences and localization techniques are also obtained.

1. Introduction. In the present paper we are mainly concerned with the question of how finite-dimensional spaces are situated in L^p. In particular we are interested in the positions of "almost" Euclidean subspaces, ($= \lambda$-isomorphs of l_n^2 for λ small with respect to n). To state our results, in addition to the standard definition we introduce the following: Let X, Y, Z be Banach spaces with Y a subspace of X, let $\lambda \geqslant 1$. Y is *λ-complemented* in X if there exists a projection from X onto Y of norm less than or equal to λ. Y is a *λ-isomorph* of Z if there exists an invertible operator T from Y onto Z with $\|T\|\;\|T^{-1}\| \leqslant \lambda$. The g. l. b. of those λ that Y is a λ-isomorph of Z is denoted by $d(Y, Z)$.

The main result of Section 2 is

THEOREM A. *Given a positive integer k and $\varepsilon > 0$, there exists a positive integer $M = M(k, \varepsilon)$ so that if $1 \leqslant p \leqslant \infty$ and E is a subspace of L^p with $\dim E = k$, there exists a subspace F of L^p with $F \supset E$, so that $\dim F \leqslant M$ and F is a $(1+\varepsilon)$-complemented $(1+\varepsilon)$-isomorph of l_n^p.*

Roughly speaking, this result says that L^p has a very strong form of a uniform approximation property. This concept and its relatives are discussed in the begining of Section 2 and in the remark after the proof of Corollary 2.1. Theorem A yields easily that the analogous result holds for

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$\mathscr{L}_{p,\lambda}$ spaces as defined in [24]. The method of the proof of Theorem A is used to construct in L^p very many unconditional bases (with uniformly bounded unconditional constants) in the sense that every finite dimensional subspace of L^p has a "large" subspace spanned by disjoint blocks with respect to such a basis.

The main result of Section 3 "localizes" the following fact established in [22]. Every subspace of L^p $(2 \leqslant p < \infty)$ is either isomorphic to a Hibert space and complemented in L^p or contains a complemented isomorph of l^p. Our result is

THEOREM B. *Let $2 \leqslant p < \infty$. Then there exists a non decreasing positive function $g_p(\lambda)$ and a positive function $k_p(N, \varepsilon)$ of a positive real variable λ, and positive integer variable N and a positive real variable ε respectively, so that if E is a subspace of L^p and $\lambda = d(E, l^2_{\dim E}) < \infty$, then E is $g_p(\lambda)$-complemented, while if $\lambda > k_p(N, \varepsilon)$ then there exists a $(1+\varepsilon)$-complemented $(1+\varepsilon)$-isomorph of l^p_N contained in E.*

In the course of proving Theorem B, we also obtain that $d(E, l^2_{\dim E})$ can be functionally related to the ratio of the L^p-norm and a certain weighted L^2-norm on E; our technique and results of [37] also yield some applications to subspaces of L^p for $p < 2$. Theorem A and Theorem B together imply that the function g_p of Theorem B depends essentially on p (or more precisely, on the largeness of p).

A simple consequence of Theorem B is that in L^p $(p > 2)$ all "almost" Euclidean subspaces of the same dimension are in the same position in the space. Another corollary says that as $p \to \infty$ or $p \to 1$ all Euclidean subspaces of L^p of large dimension are badly complemented. Finally we study complemented almost Euclidean subspaces of L^p for $1 < p < 2$. We give a rather simple proof of a result announced by Milman [30] that if H is a subspace of L^p $(1 < p < 2)$ which is isomorphic to l^2, then H contains an infinite dimensional subspace which is complemented in L^p. We also prove a "local" analogue of this result.

We recall some standard definitions and notation. "Operator" stands for a bounded linear operator. If $(S, \mathscr{T}, \mu)$ is a measure space and $1 \leqslant p < \infty$ then $L^p(\mu)$ denotes the space of all μ-equivalence classes of p-absolutely integrable μ-measurable scalar valued functions on S with the norm

$$\|f\|_p = \Big(\int_S |f|^p \, d\mu\Big)^{1/p}.$$

If μ is the Lebesgue measure on the unit interval $[0;1]$ we write L^p instead of $L^p(\mu)$. By l^p_k we denote the space of all k-tuples of scalars with the norm

$$\|(x(j))\| = \Big(\sum_{j=1}^{k} |x(j)|^p\Big)^{1/p},$$

and by l^p the corresponding space of infinite sequences. If $(x_j)_{1\leqslant j\leqslant k}$ (k finite or infinite) is a sequence of elements of a Banach space X then $[x_j]_{1\leqslant j\leqslant k}$ denotes the closed linear span of the x_j's. The *unconditional constant* of a basis (e_n) is the g.l.b. of those K such that for all scalars $c_1, \ldots, c_n$ and all signs $\varepsilon_1, \ldots, \varepsilon_n$ and for $n = 1, 2, \ldots$ the inequality

$$\Big\|\sum_{i=1}^{n} \varepsilon_i c_i e_i\Big\| \leqslant K \Big\|\sum_{i=1}^{n} c_i e_i\Big\|$$

holds.

2. The uniform approximation property of $\mathscr{L}_p$ spaces.

DEFINITION. Let $b > 1$. A Banach space X is said to have the *b-uniform approximation property* if given a positive integer k there exists an $N = N(k)$ such that for every k-dimensional subspace E of X there exists an operator $T: X \to X$ such that $Te = e$ for $e \in E$, $\dim T(X) \leqslant N$ and $\|T\| < b$. For later reference we shall call the function $k \to N(k)$ the *uniformity function* for X. If moreover T can be chosen to be a projection then X is said to have the *b-uniform projection property*.

As was observed by W. B. Johnson there are Banach spaces with bases hence with the bounded approximation property (cf. [20] for the definition) which fail to have the b-uniform approximation property for any $b > 1$. We present his example.

Take any separable Banach space, say Z, which fails to have the bounded approximation property (cf. [10], [7] and [13]). Let $Z_1 \subset Z_2 \subset \ldots$ be any increasing sequence of finite dimensional subspaces of Z such that the union $\bigcup_{j=1}^{\infty} Z_j$ is dense in Z. Let $Y = (Z_1 \times Z_2 \times \ldots)_{l^2}$. It is easily seen that if Y had the b-uniform approximation property for some $b > 1$, then Z would have the bounded approximation property with a bound b. So Y fails to have the b-uniform approximation property for any $b > 1$ and the same is clearly true for any Banach space containing an isomorph of Y as a complemented subspace. Clearly Y has the bounded approximation property being an l^2-product of finite dimensional subspaces. Thus (cf. [20], [34]) there exists a Banach space, say X, with a basis which contains a complemented isomorph of Y. Clearly X provides the desired example.

In this section we shall show that for $1 \leqslant p \leqslant \infty$ all spaces $L_p(\mu)$ and more generally all $\mathscr{L}_{p,\lambda}$ spaces $(1 \leqslant \lambda < \infty)$ have the b-uniform projection property for every $b > 1$. Moreover the function $k \to N(k)$ can be chosen so that it depends only on b and λ but not on p and the particular $\mathscr{L}_{p,\lambda}$-space. This is an obvious consequence of Theorem A stated in the introduction and Corollary 2.1 below.

We proceed now to the proof of Theorem A. Since for all N and $1 \leqslant p \leqslant \infty$ any subspace of L^p isometric to l^p_N is the range of a contractive projection (cf. e. g. [1] and [23]), Theorem A follows immediately from the next result and a standard perturbation argument (see e. g. [20]; Lemma 24).

THEOREM 2.1. *Given k and $1>\varepsilon>0$, there exists an $N=N(k,\varepsilon)$ so that for any measure space $(T,\mathcal{S},\mu)$, $1\leqslant p\leqslant\infty$ and subspace E of $L^p(\mu)$ with $\dim E=k$, there exists a subspace F of $L^p(\mu)$ isometric to $l^p_{N'}$ with $N'\leqslant N$ so that for all $e\in E$ with $\|e\|=1$ there is an $x\in F$ with $\|x-e\|\leqslant\varepsilon$.*

Proof. We first need the following

Fact: Let Y be a k-dimensional subspace of a Banach space B. By a result of Auerbach (cf. Taylor [39]), there exist elements $y_1,y_2,\ldots,y_k$ in Y so that for all i, $\|y_i\|=1$ and $\|\sum a_jy_j\|\geqslant|a_i|$ for any scalars $a_1,a_2,\ldots,a_k$. Let $\eta>0$ and suppose $x_1,x_2,\ldots,x_k\in B$ are such that $\|x_i-y_i\|\leqslant\eta$ for all i. Then, given $y=\sum a_jy_j\in Y$ with $\|y\|=1$, there is an $x\in[x_1,x_2,\ldots,x_k]$ with $\|x-y\|\leqslant k\eta$, for example $x=\sum a_jx_j$.

The proof of Theorem 2.1 for the case $p=\infty$ follows easily from this Fact. Choose $e_1,e_2,\ldots,e_k$ in E all of norm one so that $\|\sum a_je_j\|\geqslant|a_i|$ for all i and scalars a_j. It is evident that for each j, we may choose $m=\left[\frac{2k}{\varepsilon}\right]$ disjoint measurable sets $G_1^j,G_2^j,\ldots,G_m^j$ so that e_j may be approximated to within ε/k by a linear combination of the characteristic functions $\chi_{G_1^j},\chi_{G_2^j},\ldots,\chi_{G_m^j}$. If we let $\mathfrak{A}$ be the algebra of sets generated by $\{G_i^j\colon 1\leqslant j\leqslant k,\ 1\leqslant i\leqslant m\}$, then it is evident that the linear span of the characteristic functions of elements of $\mathfrak{A}$ is isometric to $l^\infty_{N'}$, where $N'\leqslant N=\left[\frac{2k}{\varepsilon}\right]^k$.

Now we assume $1\leqslant p<\infty$, $p\neq2$ (the $p=2$ case is trivial). We shall prove Theorem 2.1 by induction on k. The first step is to show that $N(2,\varepsilon)$ exists for all $\varepsilon>0$. In fact, we shall prove that $N(2,\varepsilon)$ may be taken equal to $2\left[\frac{4}{\varepsilon^2}\right]+4$.

Let E be 2-dimensional and choose $f,g\in E$ so that $\|f\|=\|g\|=1$ and $\|af+bg\|\geqslant\max(|a|,|b|)$ for all scalars a and b. By the Fact it suffices to show that there exists an $F\subset L^p(\mu)$ which is isometric to $l^p_{N'}$, with $N'\leqslant2\left[\frac{4}{\varepsilon^2}\right]+4$, so that there are elements f_1 and g_1 in F with $\max(\|f-f_1\|,\|g-g_1\|)\leqslant\varepsilon/2$.

We accomplish the construction of F, essentially, by approximating f by linear combinations of characteristic functions of the measure $g\cdot d\mu$.

Put $n=\left[\frac{4}{\varepsilon^2}\right]+1$. Let $G=\left\{t\colon|f(t)|\leqslant\frac{2}{\varepsilon}|g(t)|\right\}$. Let $G^\sim=T\setminus G$.

Define G_j for $-n\leqslant j\leqslant n$ by

$$G_0=\{t\colon f(t)=0\},$$

$$G_j=\left\{t\colon\frac{2(j-1)}{\varepsilon n}g(t)<f(t)\leqslant\frac{2j}{\varepsilon n}g(t)\right\}\quad\text{for}\quad j=1,2,\ldots,n,$$

$$G_j=\left\{t\colon\frac{2j}{n\varepsilon}g(t)\leqslant f(t)<\frac{2(j+1)}{n\varepsilon}g(t)\right\}\quad\text{for}\quad j=-1,-2,\ldots,-n.$$

Evidently, $G=\bigcup_{j=-n}^{n}G_j$. We shall show that F may be chosen equal to the span of $f\chi_{G^\sim}$ and $\{g\chi_{G_j}\colon j=0,\pm1,\ldots,\pm n\}$. Evidently, $\dim F\leqslant2n+2=2\left[\frac{4}{\varepsilon^2}\right]+4$.

Let

$$f_1=f\cdot\chi_{G^\sim}+\sum_{j=-n}^{n}\frac{2j}{n\varepsilon}g\cdot\chi_{G_j},\qquad g_1=g\chi_G=\sum_{j=-n}^{n}g\cdot\chi_{G_j}.$$

Clearly for all j

$$\int_{G_j}\left|\frac{2j}{n\varepsilon}g(t)-f(t)\right|^p d\mu(t)\leqslant\frac{2}{n^p\varepsilon^p}\int_{G_j}|g(t)|^p d\mu(t).$$

Thus, remembering that $\|g\|_p=1$, we get, by the definition of n,

$$\|f-f_1\|_p=\Bigg(\sum_{\substack{j=-n\\ j\neq0}}^{n}\int_{G_j}\left|f(t)-\frac{2j}{n\varepsilon}g(t)\right|^p d\mu(t)\Bigg)^{1/p}$$

$$\leqslant\frac{2}{n\varepsilon}\Big(\int_{G}|g(t)|^p d\mu(t)\Big)^{1/p}\leqslant\frac{\varepsilon}{2}.$$

While

$$\|g-g_1\|_p=\Big(\int_{G^\sim}|g(t)|^p d\mu(t)\Big)^{1/p}\leqslant\frac{\varepsilon}{2}\Big(\int_{G^\sim}|f(t)|^p d\mu(t)\Big)^{1/p}\leqslant\frac{\varepsilon}{2},$$

because $\|f\|_p=1$ and $|g(t)|<\frac{\varepsilon}{2}|f(t)|$ for $t\in G^\sim$. This completes the proof of the existence of $N(2,\varepsilon)$ for all $\varepsilon>0$.

Now suppose that $N(k,\varepsilon)$ has been proved to exist for all $\varepsilon>0$. We shall show that

$$N(k+1,\varepsilon)=1+N\Big(k,\frac{\varepsilon}{(k+1)}\Big)N\left(2,\frac{\varepsilon}{(k+1)N\big(k,\frac{\varepsilon}{(k+1)}\big)}\right)$$

has the desired property. Let $\varepsilon>0$ and E a $(k+1)$-dimensional subspace of $L^p(\mu)$ be given, and choose $e_1, \ldots, e_{k+1}$ in E of norm one so that for all j and scalars $a_1, a_2, \ldots, a_{k+1}$, $\|\sum a_i e_i\| \geqslant |a_j|$.

By the Fact, it suffices to construct an F isometric to l_m^p where m depends only on $k+1$ and ε, so that each $e_1, e_2, \ldots, e_{k+1}$ can be approximated to within $\frac{\varepsilon}{k+1}$ by elements of F. We accomplish the construction of F as follows: we first choose a $Y \subset L^p(\mu)$ isometric to an l_n^p, so that the unit ball of $[e_1, e_2, \ldots, e_k]$ can be very closely approximated by elements of Y. Letting $(g_1, \ldots, g_n)$ be the natural basis for Y, the supports S_i's of the g_i's may be taken to be disjoint (cf. [1], [23]). We then apply the $k = 2$ part of the theorem to each of the (at most) two-dimensional spaces $[g_j, e_{k+1}|S_j]$ contained in $L^p(\mu|S_j)$. We thus choose X_j isomorphic to a suitable l_N^p, $X_j \subset L^p(\mu|S_j)$; the final space F is taken to be the linear span of $\bigcup_{j=1}^n X_j$ and $e_{k+1}\cdot\chi_{T\setminus\bigcup_{j=1}^n S_j}$. We pass to the details.

Let δ be defined by $3\delta(k+1) = \varepsilon$; choose $Y \subset L^p(\mu)$ isometric to l_n^p with $n \leqslant N(k, \delta)$, so that if $x \in [e_1, e_2, \ldots, e_k]$ and $\|x\| = 1$, then there is a $y \in Y$ with $\|y-x\| \leqslant \delta$. Since Y is isometric to l_n^p, there are $g_1, g_2, \ldots, g_n$ of norm one so that Y equals the span of $g_1, \ldots, g_n$ and so that putting $S_i = \{x\colon g_i(x) \neq 0\}$, then $S_i \cap S_j = \emptyset$ for $i \neq j$. Now for each $1 \leqslant j \leqslant n$, choose $X_j \subset L^p(\mu|S_j)$ with X_j isometric to $l_{N_j}^p$ where $N_j \leqslant N(2, \delta/n)$ so that each element of the span of $e_{k+1}|S_j$ and g_j, of norm one, can be approximated to within δ/n by an element of X_j. Now let F denote the linear span of $\bigcup_{i=1}^n X_i$ and $e_{k+1}\cdot\chi_{T\setminus\bigcup_{i=1}^n S_i}$. Evidently F is isometric to an l_N^p where $N \leqslant n\cdot N(2, \delta/n)+1 \leqslant N(k+1, \varepsilon)$.

For each $1 \leqslant j \leqslant n$, there is a $g_j' \in X_j$ with $\|g_j' - g_j\| \leqslant \delta/n$. Hence, by the Fact, any element in Y of norm one can be approximated to within $n\cdot\frac{\delta}{n} = \delta$ by some element of F. Thus, by the assumption on Y, each of $e_1, e_2, \ldots, e_k$ is at a distance of at most $2\delta+\delta^2$ from an element of F. Moreover for each $1 \leqslant j \leqslant n$, there is an $x_j \in X_j$, so that

$$\|x_j - e_{k+1}|S_j\|_p \leqslant \frac{\delta}{n}\|e_{k+1}|S_j\|_p.$$

Hence

$$\Big\|\sum_{j=1}^n x_j + e_{k+1}\chi_{T\setminus\bigcup_{i=1}^n S_i} - e_{k+1}\Big\|_p \leqslant \Big(\sum_{j=1}^n \frac{\delta^p}{n^p}\|e_{k+1}|S_j\|_p^p\Big)^{1/p} \leqslant \frac{\delta}{n} \leqslant \delta.$$

Thus by the Fact, since all $e_1, e_2, \ldots, e_{k+1}$ can be approximated to within $2\delta+\delta^2$ by elements of F, every element in E of norm one can be approximated to within $(2\delta+\delta^2)(k+1) < 3\delta(k+1) = \varepsilon$ by elements of F. Q.E.D.

After this paper had been submitted for publication, S. Kwapień communicated to the authors the following elegant proof of Theorem 2.1.

Let $\varepsilon > 0$. Let E be a subspace of L^p with $\dim E = k$. Let $y_1, y_2, \ldots, y_k$ be a normalized basis for E such that $\|\sum_{j=1}^k a_j y_j\| \geqslant |a_i|$ for $i = 1, 2, \ldots, k$ and for all scalars $a_1, a_2, \ldots, a_k$. Let

$$f = \sum_{j=1}^k |y_j| + \chi_A \quad \text{where} \quad A = \Big\{t \in [0;1]\colon \sum_{j=1}^k |y_j(t)| = 0\Big\}.$$

Then there exists an order preserving isometric isomorphism $T\colon L^p \xrightarrow[\text{onto}]{} L^p$ such that $T(f) = 1\cdot\|f\|_p$ (cf. S. Banach, *Théorie des Opérations Linéaires*, p. 178, chap. XI, § 3). Clearly if $y = \sum_{j=1}^k a_j y_j \in E$, with $\|y\|_p = 1$, then

$$|y(t)| < \sum_{j=1}^k |a_j|\,|y_j(t)| < f(t) \quad \text{for} \quad t \in [0;1].$$

Hence

$$|T(y)(t)| \leqslant T(f)(t) = \|f\|_p \leqslant k+1 \quad \text{for all } t \in [0;1].$$

Now the proof of Theorem 2.1 for $p = \infty$ yields the existence of an algebra $\mathfrak{A}$ of measurable subsets of $[0;1]$ consisting of at most $N = [2k(k+1)\varepsilon^{-1}]^k$ sets and such that given $z \in T(Y)$ with $\|z\|_\infty \leqslant k+1$, there exists a $w \in W$ such that $\|z-w\|_p \leqslant \|z-w\|_\infty \leqslant \varepsilon$ where W denotes the linear space consisting of all linear combinations of the characteristic functions of members of $\mathfrak{A}$. Clearly $\dim W \leqslant N$ and W regarded as a subspace of L^p is isometrically isomorphic to $l_{\dim W}^p$. Let $F = T^{-1}(W)$. Then F is the desired subspace of L^p. Q.E.D.

We recall that a Banach space X is called an $\mathscr{L}_{p,\lambda}$ space if for each finite-dimensional subspace E of X, there exists an N and a λ-isomorph F of l_N^p with $E \subset F \subset X$. We call X an $\mathscr{L}_p$ space if X is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda \in [1, \infty)$.

The next result extends Theorem A to the case of general $\mathscr{L}_p$ spaces.

COROLLARY 2.1. *Let $\lambda \geqslant 1$ be given. Then there exists a $\beta = \beta(\lambda)$, so that if k is a positive integer, there exists an $N = N(k)$ so that if $1 \leqslant p \leqslant \infty$, X is an infinite dimensional $\mathscr{L}_{p,\lambda}$ space, and E is a k-dimensional subspace of X, there exists an N'-dimensional subspace F of X containing E, so that F is a β-complemented β-isomorph of $l_{N'}^p$ and $N' \leqslant N$.*

Proof. The proof for $p = 2$ is trivial since every $\mathscr{L}_{2,\lambda}$ space is a λ-isomorph of a Hilbert space, so assume $p \neq 2$. By the results of [24] there exist an $L^p(\mu)$ space and a λ-complemented subspace Y of $L^p(\mu)$ with

Y a λ-isomorph of X^{**}. Now simply let $N = N(k, 2)$ as defined in Theorem A. We may then choose an $N' \leqslant N$ and a 2-complemented 2-isomorph of $l^p_{N'}$ containing E, contained in $L^p(\mu)$. The inspection of an argument in [26] (cf. the proofs of Theorems 2.2, 3.2 and 3.3 of [26]) yields that there is a K depending only on λ, so that E is contained in a K-complemented K isomorph of $l^p_{N'}$, contained in Y itself. Thus the result is proved for X^{**}, with $\beta = \lambda K$. But then the result follows for X in virtue of the local reflexivity principle (see e.g. Theorem 3.3 of [20]). Q.E.D.

Remark. The uniform approximation property may be used to obtain the localization of the concept of approximation property to finite dimensional spaces. The idea is to require that the identity operator can be approximated by low-rank operators. Precisely let $\mathscr{F}$ be a family of Banach spaces, B a given Banach space, and b and λ positive real numbers. We say that $\mathscr{F}$ satisfies the b-uniform approximation property if there exists a function $k \to N(k)$ with domain and range in the positive integers, so that for each $X \in \mathscr{F}$, X satisfies the b-uniform approximation property with $N_X(k) \leqslant N(k)$, for all k, where N_X denotes the uniformity function for X. Put another way, the family satisfies the b-uniform approximation property with a uniform uniformity function. Now suppose that all the members of $\mathscr{F}$ are finite dimensional. Say that $\mathscr{F}$ *λ-paves* B if every finite dimensional subspace E of B is contained in a λ-isomorph of a member of $\mathscr{F}$, i.e. there exists a $Y \subset B$ and $F \in \mathscr{F}$ such that $E \subset Y$ and $d(Y, F) \leqslant \lambda$. A compactness argument may then be used to prove the following

PROPOSITION. *If $\mathscr{F}$ satisfies the b-uniform approximation property and $\mathscr{F}$ λ-paves B, then B satisfies the λb-uniform approximation property.*

We do not know if a converse to this Proposition is true; that is, if B satisfies the uniform approximation property, is B paved by a family $\mathscr{F}$ of finite dimensional spaces such that $\mathscr{F}$ satisfies the uniform approximation property? The answer is easily seen to be yes if B in addition satisfies the (not necessarily uniform) bounded projection approximation property. Now our proof of Theorem A shows that if $\mathscr{F} = \{l^p_n: 1 \leqslant p \leqslant \infty, n = 1, 2, \ldots\}$ then $\mathscr{F}$ satisfies the b-uniform approximation property for all $b > 1$. By definition, if X is an $\mathscr{L}_{p,\lambda}$-space, then X is λ-paved by $\mathscr{F}$. Hence the above Proposition may be applied to show that X satisfies the $\lambda+\varepsilon$-uniform approximation property for all $\varepsilon > 0$. We note finally that this particular family only paves $\mathscr{L}_p$-spaces, i.e. if X is paved by $\mathscr{F}$ as above, then X is already an $\mathscr{L}_p$-space for some $1 \leqslant p \leqslant \infty$.

Our next result shows that for all k, there exists a Banach space X_k all of whose k-dimensional subspaces are such "arbitrarily close" to l^2_k and are the ranges of almost contractive projections, yet X_k is not isomorphic to a Hilbert space. It is a very simple consequence of Theorem A.

COROLLARY 2.2. *Let $\varepsilon > 0$ and k be a positive integer. Then there is a $\delta > 0$, so that if $|p-2| < \delta$, then every k-dimensional subspace of L^p is $(1+\varepsilon)$-complemented and a $(1+\varepsilon)$-isomorph of l^2_k.*

Proof. Choose $\eta > 0$, so that $(1+\eta)^2 < 1+\varepsilon/2$. Let $n = M(k, \eta)$ as defined in Theorem A, and choose $\delta > 0$, so that $|p-2| < \delta$ implies that

$$\left(1+\frac{\varepsilon}{2}\right) n^{\left|\frac{1}{p}-\frac{1}{2}\right|} < 1+\varepsilon.$$

If E is a k-dimensional subspace of L^p (for $|p-2| < \delta$), we may choose, by Theorem A, a subspace F of L^p with $F \supset E$, so that F is $(1+\eta)$-complemented $(1+\eta)$-isomorph of l^p_n. Since $d(l^p_n, l^2_n) \leqslant n^{\left|\frac{1}{p}-\frac{1}{2}\right|}$, it follows that

$$d(F, l^2_n) \leqslant (1+\eta) n^{\left|\frac{1}{p}-\frac{1}{2}\right|} \leqslant 1+\varepsilon,$$

hence F is a $(1+\varepsilon)$-isomorph of l^2_k. Moreover E is $(1+\eta) n^{\left|\frac{1}{p}-\frac{1}{2}\right|}$-complemented in F and F is $(1+\eta)$-complemented in L^p, hence E is $(1+\varepsilon)$-complemented.

A similar argument yields

COROLLARY 2.3. *Given an increasing sequence (k_n) with $k_1 > 1$ and $\lim_n k_n = \infty$, there exists a Banach space X non-isomorphic to a Hilbert space such that every n-dimensional subspace of X is a k_n-complemented k_n-isomorph of l^2_n. Precisely there are sequences $p_n \to 2$ and $m_n \to \infty$ as $n \to \infty$ such that the space $X = (l^{p_1}_{m_1} \times l^{p_2}_{m_2} \times \ldots)_{l^2}$ has the desired property.*

It is interesting to compare Corollary 2.3 with a result of Lindenstrauss–Tzafriri [27] which says that if X is a Banach space non-isomorphic to a Hilbert space then $h_n(X) \to \infty$ and $p_n(X) \to \infty$ as $n \to \infty$ where

$$h_n(X) = \sup\{d(E, l^2_n): E \subset X, \dim E = n\},$$

$$p_n(X) = \sup\{\inf\{\|P\|: P \text{ projection of } X \text{ onto } E\}: E \subset X, \dim E = n\}.$$

Our next result, Theorem 2.2, is a localization in L^p of a well-known fact [2] that if E is an infinite dimensional subspace of a Banach space with an unconditional basis then there exists a sequence of elements of E which is equivalent to a block basic sequence of the basis. We begin with a lemma on "controlled extension" of some basic sequences in L^p. To state the lemma we recall some facts about martingale-type basic sequences.

DEFINITION. Let $p \neq 2$. A *martingale-type basic sequence in L^p* is a monotone basic sequence $(b_j)_{1 \leqslant j < n+1}$, where n is either finite or infinite, such that the subspace $[b_j]_{1 \leqslant j < n+1}$ is the range of a contractive projection from L^p.

It follows from the definition that for each $k = 1, 2, \ldots, n$ the subspace $[b_j]_{1\leqslant j\leqslant k}$ is a range of a contractive projection from L^p which annihilates all b_j for $j > k$. Every finite dimensional contractive projection in L^p for $\infty > p \neq 2 > 1$ is of the form

$$(+)\qquad Qf = \sum_{j=1}^{n} (f, \psi_j) g_j \text{ where}$$

$$1 = \|\psi_j\|_{\frac{p}{p-1}} = \|g_j\|_p = (g_j, \psi_j) = \int g_j \psi_j \, dt$$

and if $i \neq j$, then $g_i g_j = g_i \cdot \psi_j = 0$ for $i, j = 1, 2, \ldots, n$.

As demonstrated in [40] this allows to show that every martingale-type basic sequence in L^p can be by an appropriate positive isometry of L^p onto itself transformed into the sequence of martingale differences. Now the deep result of Burkholder and Gundi [6] (cf. also [5] and [40]) yields that if $1 < p < \infty$ then any sequence of martingale differences is an unconditional basic sequence with an unconditional constant K_p where K_p depends only on p. Hence the same is true for any martingale-type basic sequence in L^p. In particular (as pointed out in [40]) we have

PROPOSITION 2.1. *Every monotone basis in L^p and more generally every monotone Schauder decomposition of L^p is unconditional.*

Now we pass to the lemma on extension of martingale-type basic sequences.

LEMMA 2.1. *Let $1 < p \neq 2 < \infty$, $\varepsilon > 0$ and n a positive integer. Then there exists a positive integer $M = M(p, n, \varepsilon)$ such that given a martingale-type basic sequence $(b_j)_{1\leqslant j\leqslant n}$ and an $e \in L^p$ with $\|e\| = 1$, there exists a martingale-type basic sequence $(a_\nu)_{1\leqslant \nu\leqslant M}$ such that there exist an $f \in [a_\nu]_{1\leqslant \nu\leqslant M}$ and an isometric embedding*

$$U\colon [b_j]_{1\leqslant j\leqslant n} \to [a_\nu]_{1\leqslant \nu\leqslant M}$$

such that

$$\|e - f\| < \varepsilon, \qquad Ub_j = a_j \quad \text{for } j = 1, 2, \ldots, n;$$

$$\|Uh - h\| \leqslant \varepsilon \|h\| \quad \text{for } h \in [b_j]_{1\leqslant j\leqslant n}.$$

Moreover if Q and Q_1 denote the contractive projections onto $[b_j]_{1\leqslant j\leqslant n}$ and $[Ub_j]_{1\leqslant j\leqslant n}$ respectively then $\|Q - Q_1\| \leqslant \varepsilon$.

Proof. Let $A_j = \{t\colon g_j(t) \neq 0\}$ where the functions $g_1, g_2, \ldots, g_n$ are defined by $(+)$ for the contractive projection Q from L^p onto $[b_j]_{1\leqslant j\leqslant n}$. Clearly if $i \neq j$ then $A_i \cap A_j = \emptyset$. Now fix $\eta > 0$ sufficiently small. How small η should be we shall determine later. Let us put $m = 2[4/\eta^2] + 3$. Then an inspection of the proof of Theorem 2.1 yields the existence of mutually disjoint sets $(B_{j,k})_{0\leqslant k\leqslant m}$ and numbers $c_{j,k}$ for $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ such that $B_{j,k} \subset A_j$ for all k and j and if

$$\tilde{A}_j = \bigcup_{k=1}^{m} B_{j,k}; \qquad e_j = e \cdot \chi_{A_j}; \qquad B_{n+1,0} = [0, 1] \setminus \bigcup_{j=1}^{n} A_j;$$

$$\tilde{e}_j = e \cdot \chi_{B_{j,0}} + \sum_{k=1}^{m} c_{j,k} g_j \chi_{B_{j,k}}; \qquad \tilde{g}_j = \|g_j \cdot \chi_{\tilde{A}_j}\|^{-1} \cdot g_j \cdot \chi_{\tilde{A}_j},$$

then

$$\|e_j - \tilde{e}_j\| \leqslant \eta \quad \text{and} \quad \|g_j - \tilde{g}_j\| \leqslant \eta \quad \text{for } j = 1, 2, \ldots, n.$$

Define $U\colon [g_j]_{1\leqslant j\leqslant n} \to [\tilde{g}_j]_{1\leqslant j\leqslant n}$ by

$$U\Big(\sum_{j=1}^{n} c_j g_j\Big) = \sum_{j=1}^{n} c_j \tilde{g}_j \quad \text{for all scalars } c_1, c_2, \ldots, c_n.$$

Clearly U is an isometry and for each $h \in [g_j]_{1\leqslant j\leqslant n} = [b_j]_{1\leqslant j\leqslant n}$ we have $\|Uh - h\| \leqslant n\eta \|h\|$ (because (g_j) is an Auerbach basis). Since $[\tilde{g}_j]_{1\leqslant j\leqslant n}$ is isometric to l_n^p, there exists a contractive projection, say Q_1, from L^p onto $[\tilde{g}_j]_{1\leqslant j\leqslant n}$ (cf. [1], [23]). Hence there exist $\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_n$ in $L^{\frac{p}{p-1}}$ so that

$$\|\tilde{\psi}_j\|_{\frac{p}{p-1}} = (\tilde{g}_j, \tilde{\psi}_j) = 1 \quad \text{for } j = 1, 2, \ldots, n$$

and

$$Q_1 h = \sum_{j=1}^{n} (h, \tilde{\psi}_j) \tilde{g}_j \quad \text{for all } h \in L^p.$$

Thus for each $h \in L^p$ we have

$$(Q - Q_1)h = \sum_{j=1}^{n} (h, (\psi_j - \tilde{\psi}_j)) g_j + \sum_{j=1}^{n} (h, \tilde{\psi}_j)(g_j - \tilde{g}_j)$$

where ψ_j $(j = 1, 2, \ldots, n)$ are defined for Q by $(+)$. It follows from the uniform convexity of $L^{\frac{p}{p-1}}$ (cf. [18]) that there is a positive constant C_p such that $\|g_j - \tilde{g}_j\|_p \leqslant \eta$ implies

$$\|\psi_j - \tilde{\psi}_j\|_{\frac{p}{p-1}} \leqslant C_p \eta^{\alpha_p} \quad \text{for } j = 1, 2, \ldots, n$$

where $\alpha_p = \min(\frac{1}{2}, 1 - 1/p)$. Hence

$$\begin{aligned}\|(Q - Q_1)(h)\|_p &\leqslant \Big\| \sum_j (h, (\psi_j - \tilde{\psi}_j)) g_j \Big\|_p + \Big\| \sum_j (h, \tilde{\psi}_j)(g_j - \tilde{g}_j) \Big\|_p \\ &\leqslant \sum_j \|h\|_p \|\psi_j - \tilde{\psi}_j\|_{\frac{p}{p-1}} + \sum_j \|h\|_p \|g_j - \tilde{g}_j\|_p \\ &\leqslant n(C_p \eta^{\alpha_p} + \eta) \|h\|_p.\end{aligned}$$

Thus $\|Q - Q_1\| \leqslant n(C_p \eta^{\alpha_p} + \eta)$.

Next we shall construct the basic sequence $(a_\nu)_{1\leqslant\nu\leqslant M}$. To this end let us set

$$F_0 = [\tilde{g}_j]_{1\leqslant j\leqslant n};$$
$$F_r = \operatorname{span}(F_{r-1}, g_j\cdot\chi_{B_{j,k}}) \quad \text{for } r = (j-1)(m-1)+k$$
$$(j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m-1);$$
$$F_{n(m-1)+1} = \operatorname{span}(F_{n(m-1)}, e\cdot\chi_{\bigcup_{j=1}^{n+1} B_{j,0}}).$$

It is easily seen that F_r is isometric to l^p_{r+n} and $F_{r-1}\subset F_r$ $(r = 0, 1, \ldots \ldots, n(m-1)+1)$. Hence each F_r is the range of a contractive projection, say Q_{r+1}, from L^p. Now we put $a_j = Ub_j$ for $j = 1, 2, \ldots, n$ and let a_{n+r} be any element of norm one belonging to the intersection $F_r\cap\ker Q_r$ $(r = 1, 2, \ldots, n(m-1)+1)$. It can easily be checked (cf. e.g. [29]) that the sequence $(a_\nu)_{1\leqslant\nu\leqslant nm+1}$ is a monotone basic sequence and therefore a martingale basic sequence (because $[a_\nu]_{1\leqslant\nu\leqslant nm+1} = F_{n(m-1)+1}$ is the range of the contractive projection $Q_{n(m-1)+2}$).

Next define $f\in F_{n(m-1)+1}$ by

$$f = e\cdot\chi_{\bigcup_{j=1}^{n+1} B_{j,0}} + \sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}g_j\cdot\chi_{B_{j,k}} = \sum_{j=1}^{n}\tilde{e}_j.$$

Then we have

$$\|e-f\| = \Big\|\sum_{j=1}^{n} e_j - \sum_{j=1}^{n}\tilde{e}_j\Big\| \leqslant \sum_{j=1}^{n}\|e_j-\tilde{e}_j\| \leqslant n\eta.$$

Now to guarantee that the sequence $(a_\nu)_{1\leqslant\nu\leqslant nm+1}$ has the properties required in Lemma 2.1 it suffices to choose $\eta > 0$, so that $n(\eta + C_p\eta^{a_p}) < \varepsilon$ and put

$$M = M(p, k, \varepsilon) = \left(2\left[\frac{4}{\eta^2}\right]+3\right)n+1. \quad \text{Q.E.D.}$$

LEMMA 2.2. *Let k be a positive integer, $1 > \varepsilon > 0$, and $1 < p \neq 2 < \infty$. Then there exist functions $d = d(p, k, \varepsilon)$ and $n = n(p, k, \varepsilon)$ such that for every subspace $E\subset L^p$ with $\dim E\geqslant d$ there exists a martingale-type basic sequence $(b_j)_{1\leqslant j\leqslant n}$ and elements of norm one $e_1, e_2, \ldots, e_k$ in E, and $z_1, z_2, \ldots, z_k$ in $[b_j]_{1\leqslant j\leqslant n}$ which are disjoint blocks with respect to the basic sequence $(b_j)_{1\leqslant j\leqslant n}$ and $\|z_i-e_i\|\leqslant\varepsilon$ for $i = 1, 2, \ldots, k$.*

Proof. Let us set

$$n(p, 1, \varepsilon) = d(p, 1, \varepsilon) = 1 \quad \text{for all } \varepsilon;$$
$$n(p, k+1, \varepsilon) = M(p, n(p, k+1, \varepsilon/3), \varepsilon/3),$$
$$d(p, k+1, \varepsilon) = \max\big(n(p, k, \varepsilon/3), d(p, k, \varepsilon/3)+1\big)$$

for $k = 1, 2, \ldots$, where $M(\cdot, \cdot, \cdot)$ is that of Lemma 2.1.

Clearly $n(p, 1, \varepsilon)$ and $d(p, 1, \varepsilon)$ have for all ε the desired properties. Assume that for some $k\geqslant 1$ and all $1 > \varepsilon > 0$ the integers $n(p, k, \varepsilon)$ and $d(p, k, \varepsilon)$ have the desired properties. Let E be a subspace of L^p with $\dim E\geqslant d(p, k+1, \varepsilon)$. In particular $\dim E > d(p, k, \varepsilon/3)$. Hence, by the inductive hypothesis, there exist a martingale-type basic sequence $(b'_j)_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$ and elements of norm one $e_1, e_2, \ldots, e_k$ in E and $z'_1, z'_2, \ldots, z'_k$ in $[b'_j]_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$ which are disjoint blocks with respect to the basic sequence $(b'_j)_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$ and $\|z'_i-e'_i\|\leqslant\varepsilon/3$ for $i = 1, 2, \ldots, k$. Let Q' denote the contractive projection from L^p onto $[b'_j]_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$. Since $\dim E > n(p, k, \varepsilon/3)$, there exists an e_{k+1} in E such that $\|e_{k+1}\| = 1$ and $Q'(e_{k+1}) = 0$. Now, by Lemma 2.1, there exists a martingale-type basic sequence $(b_\nu)_{1\leqslant\nu\leqslant n(p,k+1,\varepsilon/3)}$ and an isometry $U\colon [b'_j]_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}\to[b_\nu]_{1\leqslant\nu\leqslant n(p,k+1,\varepsilon/3)}$ and an $f\in[b_\nu]_{1\leqslant\nu\leqslant n(p,k+1,\varepsilon/3)}$ such that $\|e_{k+1}-f\|\leqslant\varepsilon/3$; $b_j = Ub'_j$ for $j = 1, 2, \ldots \ldots, n(p, k, \varepsilon/3)$; $\|Uh-h\|\leqslant\varepsilon/3\|h\|$ for $h\in[b'_j]_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$ and $\|Q'-Q\|\leqslant\varepsilon/3$ where Q denotes the contractive projection from L^p onto $[b_j]_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$.

Let us set $z_i = Uz'_i$ for $i = 1, 2, \ldots, k$ and $z_{k+1} = f-Qf$. Clearly $z_1, z_2, \ldots, z_{k+1}$ are disjoint blocks with respect to the basic sequence $(b_j)_{1\leqslant j\leqslant n(p,k+1,\varepsilon/3)}$ because $z'_1, \ldots, z'_k$ are disjoint blocks with respect to $(b'_j)_{1\leqslant j\leqslant n(p,k,\varepsilon/3)}$ and $Qz_i = z_i$ for $i = 1, 2, \ldots, k$ and $Qz_{k+1} = 0$. Next remembering that $\|e_i\| = 1$ and $\varepsilon < 1$ we have

$$\|e_i-z_i\|\leqslant\|e_i-z'_i\|+\|z'_i-Uz'_i\|\leqslant\varepsilon/3+\varepsilon\|z'_i\|/3 = \varepsilon(2+\varepsilon/3)/3\leqslant\varepsilon \quad \text{for } i = 1, 2, \ldots, k$$

and

$$\|e_{k+1}-z_{k+1}\| = \|e_{k+1}-f+Q(f-e_{k+1})+(Q-Q')(e_{k+1})\|\leqslant 2\|f-e_{k+1}\|+\varepsilon/3\leqslant\varepsilon.$$

This completes the induction and the proof of Lemma 2.2.

It follows immediately from Lemma 2.1 that every finite martingale-type basic sequence in L^p extends to a martingale-type basis for L^p. Hence, by Lemma 2.2, we get

THEOREM 2.2. *Let k be a positive integer, $\varepsilon > 0$ and $1 < p \neq 2 < \infty$. Then there exists a positive integer $\tilde{d} = \tilde{d}(p, k, \varepsilon)$ such that if E is a subspace of L^p with $\dim E\geqslant\tilde{d}$, then there exist k elements each of norm one, say $z_1, z_2, \ldots, z_k$, which are disjoint blocks with respect to a martingale-type basis for L^p and there exists a linear operator $U\colon [z_i]_{1\leqslant i\leqslant k}\to E$ such that*

$$\|z-Uz\|\leqslant\varepsilon\|z\| \quad \textit{for each } z\in[z_i]_{1\leqslant i\leqslant k}.$$

3. Almost Euclidean subspaces of $L^p(\mu)$ spaces. To prove Theorem B, we wish to obtain a suitable localized version of the result of [22] that every Hilbert space isomorph contained in L^p $(p > 2)$ has equivalent L^p and L^2 norms. Evidently in an appropriate finite dimensional analogue, the L^2 norm must somehow be modified, for there are obviously one-dimensional subspaces of L^p with the L^p and L^2 norms in arbitrarily large ratio. We accomplish this by considering weighted L^2-norms.

DEFINITION. Let $2 < p < \infty$ and let E be a (possibly finite-dimensional) subspace of L^p. Put

$$c_E = \inf_{\varphi} \sup_{f \in E} \|f\varphi^{\frac{p-2}{2p}}\|_2^{-1} \cdot \|f\|_p,$$

the infimum is taken over all functions φ, such that

(1) $\quad \varphi$ is measurable, $\quad \varphi \geqslant 0, \quad \int_0^1 \varphi(t)\,dt = 1, \quad$ and

$$[0,1] = \{x\colon \varphi(x) \neq 0\}.$$

Obviously $d(E, l^2_{\dim E}) \leqslant c_E$ and moreover, by the results of [22], c_E is finite if E is isomorphic to a Hilbert space.

We shall see below that there is a function

$$h_p\colon R^+ \to R^+, \quad \text{so that} \quad c_E \leqslant h_p\big(d(E, l^2_{\dim E})\big).$$

For the time being, we observe the simple

LEMMA 3.1. *If c_E is finite, E is c_E-complemented.*

Proof. Fix $\varepsilon > 0$ and pick φ satisfying (1), so that

$$\sup_{f \in E} \|f\|_p \cdot \|f\varphi^{\frac{p-2}{2p}}\|_2^{-1} \leqslant c_E + \varepsilon.$$

Let $\nu(A) = \int_A \varphi\,dx$ for all measurable $A \subset [0,1]$, and define T by $Tg = g\varphi^{-1/p}$ for all $g \in L^p$. Then T is an isometry of L^p onto $L^p(\nu)$ and moreover

$$\|Tg\|_{L^2(\nu)} = \Big(\int_0^1 g^2\varphi^{-2/p}\varphi\,dx\Big)^{1/2} = \Big\|g\varphi^{\frac{p-2}{2p}}\Big\|_2 \quad \text{for all } g.$$

It follows immediately that the orthogonal projection from $L^2(\nu)$ onto $T(E)$ yields a projection P from $L^2(\nu)$ onto $T(E)$ of norm at most

$$\sup_{f \in E} \|f\|_p \cdot \Big\|f\varphi^{\frac{p-2}{2p}}\Big\|_2 \leqslant c_E + \varepsilon.$$

Clearly $P_\varepsilon = T^{-1}PT$ is a projection from L^p onto E of norm at most $c_E + \varepsilon$. Letting ε tend to zero, an easy compactness argument using the reflexivity of L^p, now yields a projection onto E of norm at most c_E. Q.E.D.

Before proceeding to the main part of the argument of Theorem B, we have need of two additional lemmas. The proof of the first follows from the arguments of [22] and will be omitted.

LEMMA 3.2. *Let N, $\varepsilon > 0$, and $1 \leqslant p < \infty$ be given. Then there exists a $\delta = \delta(N, \varepsilon, p)$ such that if (Ω, μ) is a probability space and $f_1, \ldots, f_N$ are functions in $L^p(\mu)$ of norm one, and $A_1, \ldots, A_N$ are measurable subsets of Ω, so that for all $1 \leqslant j \leqslant N$ and all $1 \leqslant i < j$,*

$$\int_{A_j} |f_j|^p\,d\mu > 1 - \delta \quad \text{and} \quad \int_{A_j} |f_i|^p\,d\mu < \delta,$$

then

$$d([f_1, \ldots, f_N], l_N^p) < 1 + \varepsilon$$

and there is a projection from L^p onto $[f_1, \ldots, f_N]$ of norm at most $1 + \varepsilon$.

LEMMA 3.3. *Let ε, η, and $2 < p < \infty$ be given. Then there exists a $C = C(\varepsilon, \eta, p)$, so that if (Ω, μ) is a probability space and $f \in L^p(\mu)$ is such that if $\|f\|_p \cdot \|f\|_2^{-1} > C$, then there exists a measurable set A with $\mu(A) < \varepsilon$, so that*

$$\int_A |f|^p\,d\mu > (1-\eta)\|f\|_p^p.$$

Proof. Let $C = (\varepsilon^{\frac{p-2}{2p}}\eta^{1/2})^{-1}$. Suppose $\|f\|_p = 1$, $\|f\|_2^{-1} > C$. Let $A = \{x\colon |f(x)| > \varepsilon^{-1/p}\}$. Then evidently $\mu(A) < \varepsilon$, and

$$\int_{\Omega\setminus A} |f(x)|^p\,d\mu(x) \leqslant \int_{\Omega\setminus A} |f(x)|^2\,d\mu(x)\,\varepsilon^{-\frac{p-2}{p}}$$

$$\leqslant \|f\|_2^2\,\varepsilon^{-\frac{p-2}{p}} < \big(C^2\varepsilon^{\frac{p-2}{p}}\big)^{-1} = \eta.$$

Q.E.D.

Our next result and standard facts easily yield the proof of Theorem B. This result may be loosely stated as follows. If a subspace of L^p has the ratio of its L^p norm to all of its weighted L^2 norms quite large, then it has a subspace close to l_N^p for large N which is the range of an almost contractive projection. Precisely:

PROPOSITION 3.1. *Let N, $\varepsilon > 0$, and $2 < p < \infty$ be given. Then there exists a $k = k(N, \varepsilon, p)$, so that if E is a subspace of L^p with $c_E \geqslant k$, then there is a subspace $F \subset E$, such that F is a $(1+\varepsilon)$-isomorph of l_N^p and F is $(1+\varepsilon)$-complemented in L^p.*

Proof. Let $\delta = \delta(N, \varepsilon, p)$ be defined as in Lemma 3.2. Let $k = C(\delta/2N, \delta, p)$ be defined as in Lemma 3.3; and let E be a subspace of L^p with $c_E \geqslant k$. We shall construct elements $f_1, f_2, \ldots, f_N$ of E and measurable sets $A_1, A_2, \ldots, A_N$ in $[0,1]$ satisfying the hypothesis of Lemma 3.2.

Let f_1 be an arbitrary element of E of norm one and let $A_1 = \{x\colon f_1(x) \neq 0\}$. Let $1 \leqslant j < N$, and suppose that $f_1, f_2, \ldots, f_j$ have been chosen in E all of norm one, and $A_1, \ldots, A_j$ measurable subsets of $[0,1]$ have been chosen, satisfying together with the f_i's the hypotheses of Lemma 3.2. Let $S = \{x\colon |f_1|^p(x) + \ldots + |f_j|^p(x) \neq 0\}$ and put

$$\varphi = \frac{1}{2}\,\frac{|f_1|^p + \ldots + |f_j|^p}{j} + \frac{1}{2(1-m(S))}\,\chi_{[0,1]\setminus S}$$

if $m(S) \neq 1$, otherwise let

$$\varphi = \frac{|f_1|^p + \ldots + |f_j|^p}{j}$$

(m denotes Lebesgue measure). By the definition of c_E there exists an $f_{j+1} \in E$ of norm one, such that

$$\|f_{j+1} \cdot \varphi^{\frac{x-2}{2p}}\|_2^{-1} \geqslant k.$$

Letting $d\mu = \varphi\, dx$ and applying Lemma 3.3 to the function $f = f_{j+1}\varphi^{-1/p}$, there exists a measurable set A_{j+1} with

$$\int_{A_{j+1}} |f_{j+1}|^p dx = \int_{A_{j+1}} |f|^p d\mu \geqslant 1 - \delta \quad \text{and} \quad \int_{A_{j+1}} \varphi\, dx = \mu(A_{j+1}) < \delta/2N.$$

If $i < j+1$, then

$$\int_{A_{j+1}} |f_i|^p dx < 2j \int_{A_{j+1}} \varphi\, dt,$$

whence

$$\int_{A_{j+1}} |f_i|^p dx < \frac{2j\delta}{2N} \leqslant \delta.$$

This completes the construction of $f_1, \ldots, f_N$ and $A_1, \ldots, A_N$ by induction, and hence in virtue of Lemma 3.2, the proof of Proposition 3.1. Q.E.D.

Now we are ready to complete the proof of Theorem B.

Let p, E, and λ be as in the statement of Theorem B. Let N be such that $N^{1/2-1/p} > 2\lambda$. Now it is known that $d(l_N^2, l_N^p) = N^{1/2-1/p} > 2\lambda$ (see e.g. [17]). Thus if F is an N-dimensional subspace of E, $d(F, l_N^p) > 2$, for otherwise since $d(F, l_N^2) \leqslant \lambda$, $d(l_N^2, l_N^p) = N^{1/2-1/p} \leqslant 2\lambda$. Thus by Proposition 3.1, $c_E < k(N, 2, p)$. Thus since N depends only on λ and p, we obtain a function $h_p\colon R^+ \to R^+$, so that $c_E \leqslant h_p(\lambda)$. The existence of g_p follows immediately from the existence of h_p and Lemma 3.1, while the existence of $k_p(N, \varepsilon)$ of Theorem B follows immediately from Proposition 3.1 and from the fact that $c_E \geqslant \lambda$.

Remark 1. For $1 < p < 2$ and E a subspace of L^p define

$$c_E = \inf_{\varphi} \sup_{f \in E} \|f\|_p \cdot \|f \cdot \varphi^{p/(p-1)}\|_1^{-1},$$

the infimum is taken over all measurable φ satisfying (1). It follows easily from the results of [22] that if $c_E = \infty$, then for all $\varepsilon > 0$, there is a $(1+\varepsilon)$-isomorph of l^p contained in E which is $(1+\varepsilon)$-complemented in L^p. On the other hand if $c_E < \infty$, the results of [37] (see also [11] for a more elementary treatment) show that E contains no isomorph of l^p.

Now the proof of Proposition 3.1 yields that those results can be localized. An inspection of the proof shows that the statement of Proposition 3.1 holds for $1 < p < 2$ with the above definition of c_E. Moreover the results of [37] yield that:

Given $1 < p < 2$ and c, $k > 0$, there is an N so that for all $n > N$, if $E \subset L^p$ is a c-isomorph of l_n^p, then $c_E \geqslant k$.

Consequently fixing c and N, then c-isomorphs of l_k^p which are contained in L^p, for sufficiently large k, contain almost isometric copies of l_N^p. (For further complements and details see [37].)

Remark 2. Recently B. Maurey [28], [41] proved that if $\infty > p > 2$, then every operator from a subspace of L^p into a Hilbert space admits an extension to an operator from L^p. This implies the existence of the function g_p. In fact his method shows that for $E \subset L^p$, $c_E \leqslant K_p d(E, l^2_{\dim E})$ and hence $g_p(\lambda) \leqslant K_p\lambda$ with $K_p = O(\sqrt{p})$. Maurey's result suggests also the following

Problem. Let X be a Banach space with the property that there exists a function $g_X(\lambda)$ such that if $E \subset X$ is a λ-isomorph of $l^2_{\dim E}$, then E is $g_X(\lambda)$ complemented in X. Is it true that every operator from every subspace of X into a Hilbert space admits an extension to the whole space?

Next we observe that Proposition 3.1 together with an argument of Figiel yields the following variation of a result due to him (cf. [12], Proposition 3). Here and in the sequel we denote by $\varrho(X)$ the projection constant of a Banach space X (cf. [38] for the definition).

Corollary 3.1. *For all N, $2 < p < \infty$, and $\varepsilon > 0$ there is a $\tilde{k} = \tilde{k}(N, \varepsilon, p)$, so that for all n and subspaces E of l_n^p with $(\dim E)n^{-2/p} > \tilde{k}$, there is a $(1+\varepsilon)$-isomorph of l_N^p contained in E which is $(1+\varepsilon)$-complemented in l_n^p.*

Proof. Of course we may regard l_n^p as a subspace of L^p. Let $k = \dim E$, $\lambda = d(E, l_k^2)$. Following the argument of Figiel, we observe that there are absolute constants C_1 and C_2 with

$$\varrho(l_n^p) \leqslant C_1 n^{1/p} \quad \text{and} \quad \varrho(l_n^2) \geqslant C_2 n^{1/2} \quad \text{for all } n.$$

Thus letting g be the function of Theorem B, since E is $g_p(\lambda)$-complemented in l_n^p,

$$\varrho(E) \leqslant C_1 g_p(\lambda) n^{1/p}.$$

On the other hand,

$$\varrho(E) \geqslant \frac{C_2 k^{1/2}}{\lambda}.$$

Thus

(2) $$g_p(\lambda)\lambda > C_3 k^{1/2} n^{1/p},$$

where C_3 is an absolute constant.

Let h be the positive increasing function defined by $h(x) = \sup_{0<\lambda\leqslant x} g_p(\lambda)\lambda$ for all $x > 0$; let $k(N, \varepsilon, p)$ be the function defined in Proposition 3.1, and put $\bar{k} = k(N, \varepsilon, p)$. Now if $c_3 k^{1/2} n^{-1/p} \geqslant h(\bar{k})$, then $c_E \geqslant \lambda > \bar{k}$ by (2), and hence the conclusion follows by Proposition 3.1. Hence we may set $\tilde{k} = \big(h(\bar{k})^2 c_3\big)^{-2}$.

Roughly speaking, the next two corollaries say that in L^p $(p > 2)$ all Hilbert subspaces of the same dimension are in the same position.

Corollary 3.2. *Let $p > 2$, $\lambda > 0$. Then there exists a $b = b(p, \lambda)$ such that for any positive integer n there exists a positive integer $k = k(p, n)$ such that given $k' > k$ and n-dimensional subspaces H_1 and H_2 of $l^p_{k'}$, which are λ-isomorphs of l^2_n, there exists an isomorphism $T\colon l^p_{k'} \to l^p_{k'}$ such that $T(H_1) = H_2$ and $\|T\|\,\|T^{-1}\| \leqslant b(p, \lambda)$.*

Proof. Choose $n_1 = n_1(p, n)$, so that $l^p_{n_1}$ contains a 2-isomorph of l^2_n. Next, by Corollary 3.1, we choose $k = k(p, n)$, so that if $k' > k$ then every subspace of $l^p_{k'}$ of dimension $\geqslant k' - 2n$ contains a 2-complemented 2-isomorph of $l^p_{n_1}$. We shall show that k has the desired property. Let $k' > k$ and let $H_i \subset l^p_{k'}$, be λ-isomorphs of l^2_n $(i = 1, 2)$. By Theorem B, there exist projections P_i from $l^p_{k'}$ onto H_i with $\|P_i\| \leqslant g_p(\lambda)$ $(i = 1, 2)$. Let $E = \ker P_1 \cap \ker P_2$. Clearly $\dim E \geqslant k' - 2n$. Hence there is a projection $Q\colon l^p_{k'} \to l^p_{k'}$ with $\|Q\| \leqslant 2$ and $d(F, l^p_{n_1}) \leqslant 2$, where $F = Q(l^p_{k'}) \subset E$. Let H be a subspace of F which is a 2-isomorph of l^2_n. By Theorem B, there is a projection $R\colon F \xrightarrow[\text{onto}]{} H$ with $\|R\| \leqslant g_p(4)\cdot 2$. Cearly for $i = 1, 2$ there is an isomorphism A_i from $l^p_{k'}$ onto the product

$$Z_i = H_i \times (1_{l^p_{k'}} - P_i)(1_{l^p_{k'}} - Q)(l^p_{k'}) \times (1_F - R)(F) \times H$$

such that the norms $\|A_i\|$ and $\|A_i^{-1}\|$ depend only on λ and the norms of projections P_i, Q, R which depend only on p and λ (here by 1_X we denote the identity on X; Z_i is normed by $\|(x_1, x_2, x_3, x_4)\| = \max_{1\leqslant j\leqslant 4} \|x_j\|$). Since

$$\max_{i=1,2} d(H_i, H) \leqslant d(H, l^2_n) \max_{i=1,2} d(H_i, l^2_n) \leqslant 2\lambda,$$

there exist for $i = 1, 2$ isomorphisms

$$U_i\colon H_i \to H \quad \text{with } \|U_i\| = 1 \text{ and } \|U_i^{-1}\| \leqslant 4\lambda.$$

Let $B_i\colon Z_i \to Z_i$ be defined by $B_i(e^i, x, y, e) = (U_i^{-1}e, x, y\,U_i e^i)$. Clearly $\|B_i\|\,\|B_i^{-1}\| \leqslant 4\lambda$. Finally we put $T_i = A_i^{-1} B_i A_i$ for $i = 1, 2$ and $T = T_2^{-1} T_1$. One can easily check that $T(H_1) = H_2$ and the norms $\|T\|$ and $\|T^{-1}\|$ depend on p and λ only via the norms of A_i, A_i^{-1}, B_i. Q. E. D.

The above corollary immediately yields

Corollary 3.3. *Let $p > 2$ and let $\lambda > 0$. Then there exists a function $\tilde{b}(p, \lambda)$ such that if H_1 and H_2 are subspaces of L^p with $\max\big(d(H_i, l^2_n)\big) \leqslant \lambda$ for some $n = 1, 2, \ldots, \infty$, then there exists a surjective isomorphism $T\colon L^p \to L^p$ such that $T(H_1) = H_2$ and $\|T\|\,\|T^{-1}\| \leqslant \tilde{b}(p, \lambda)$.*

Our next corollary may be roughly stated as asserting that as $p \to 1$ or $p \to \infty$, all almost Euclidean subspaces of L^p are badly complemented. It follows that the function g_p necessarily depends on p (near infinity), and also gives some information concerning the following definition and question raised by Retherford and Stegall [35]: A space X is called *sufficiently Euclidean* if there exists a constant C satisfying

(3) for all positive integers, there exists a C-complemented C-isomorph of l^2_n contained in X.

Let us call the *sufficiently-Euclidean constant* of X, the infimum of the numbers C satisfying (3). The question is raised in [35] as to whether or not every reflexive space is sufficiently Euclidean. We do not know the answer; however our next result shows that the sufficiently-Euclidean constants of L^p tend to infinity as $p \to \infty$ or $p \to 1$; thus there is no universal C which satisfies (3) for all sufficiently-Euclidean spaces.

Corollary 3.4. *Given K and N, there exists an $\varepsilon > 0$ and k, so that if $p - 1 < \varepsilon$ or $1/p < \varepsilon$, $1 < p < \infty$, then if E is a subspace of L^p with $\dim E = k$ and $d(E, l^2_k) \leqslant K$, any projection from L^p onto E has norm at least as big as N.*

Proof. We first consider the case of large p. It is known that $\varrho(l^2_k) \geqslant \sqrt{k}/\sqrt{2\pi}$ for all k (cf. [38]). Now choose k, so that $\sqrt{k}/\sqrt{2\pi}\cdot 1/4K \geqslant N$ and let $n = M(k, 2)$ be defined as in Theorem A. Choose ε, so that $1/p < \varepsilon$ implies $n^{1/p} \leqslant 2$. Suppose that $E \subset L^p$ and $1/p < \varepsilon$ and $d(E, l^2_k) \leqslant K$. By Theorem A, we may choose a subspace F of L^p with $d(F, l^p_n) \leqslant 2$, with $F \supset E$. Since $d(l^p_n, l^\infty_n) \leqslant n^{1/p} \leqslant 2$, it follows that there is a subspace $\tilde{E}$ of l^∞_n with $d(E, \tilde{E}) \leqslant 2$. Hence $d(\tilde{E}, l^2_k) \leqslant 2K$, whence

$$\varrho(\tilde{E}) \geqslant \frac{\sqrt{k}}{\sqrt{2\pi}\cdot 2K}$$

and consequently if P is any projection from F onto E, then

$$\|P\| \geqslant \frac{\sqrt{k}}{\sqrt{2\pi}4K} \geqslant N.$$

The proof for the case of p close to one is almost identical. By a Theorem of Grothendieck [16] (cf. also [24]), it follows that there is an absolute constant C', so that if $\tilde{E}$ is any k-dimensional subspace of L^1, then if P is a projection from L^1 onto $\tilde{E}$,

$$\|P\| \geqslant \frac{C'\sqrt{k}}{d(\tilde{E}, l^k_2)}.$$

Choose k, so that $C'\sqrt{k}/4K \geqslant N$ and put $n = M(k, 2)$. Next choose $\varepsilon > 0$, so that $p-1 < \varepsilon$ implies $n^{1-1/p} \leqslant 2$. Then the same argument as above yields that any projection from L^p onto E has norm at least N, if $p-1 < \varepsilon$ (one needs, of course, that $d(l_n^p, l_n^1) \leqslant n^{1-1/p}$).

Remark 3. It follows from a recent result due to Gordon, Lewis and Retherford (cf. [14], Section 5) that there exists an absolute constant $C > 0$ independent of p, such that if Q is a projection from L^p onto a λ isomorph of l^2 then

$$\|Q\| \geqslant C\lambda^{-1}\left(\max\left(p, \frac{p}{p-1}\right)\right)^{1/2} \quad \text{for } \lambda > 1 \quad \text{and} \quad 1 < p < \infty.$$

If $1 < p < \frac{4}{3}$, then there are uncomplemented subspaces of L^p, which are isomorphic to a Hilbert space (cf. [36]). Hence Theorem B fails for $1 < p < \frac{4}{3}$ and probably for all $1 < p < 2$. However the next two results show that for all $1 < p < \infty$ there are in L^p very many nicely complemented almost Euclidean subspaces. A slightly weaker result that our Theorem 3.1 was announced by Milman [30].

THEOREM 3.1. *Let $1 < p < \infty$. There exists an absolute constant K_p, so that if $E \subset L^p$ is isomorphic to l^2 then E contains a K_p-complemented in L^p a K_p-isomorph of l^2.*

Proof. Let (b_n) be an unconditional basis for L^p, with unconditional-basis-constant U_p. It follows from the result of [2] that it suffices to prove the assertion of Theorem 3.1 for spaces E spanned by a block basis (e_j) of (b_n). Now first let $2 < p$. Following James [19] (cf. also [11] for a fixed ε with $0 < \varepsilon < 1$ we define an increasing sequence (k_ν) of the indices and a sequence of scalars (a_j) so that if $g_\nu = \sum_{j=k_{\nu-1}+1}^{k_\nu} a_j e_j$ then

$$\alpha(1-\varepsilon) \leqslant \|g_\nu\| \leqslant \alpha(1+\varepsilon)$$

and

$$\Big\|\sum_{j=k_0+1}^{\infty} c_j e_j\Big\| \geqslant \alpha(1-\varepsilon)\Big(\sum_{j=k_0+1}^{\infty} |c_j|^2\Big)^{1/2} \quad \text{for all } c_{k_0+1}, c_{k_0+2}, \ldots$$

where

$$\alpha = \liminf_m \Big\{\Big\|\sum_{j=m}^{\infty} c_j e_j\Big\|: \sum_{j=m}^{\infty} |c_j|^2 = 1\Big\}. \tag{4}$$

It follows by an argument of James [19] that if we put $y_\nu = g_\nu/\|g_\nu\|$ for all ν then

$$\Big(\sum |a_\nu|^2\Big)^{1/2} \leqslant \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\Big\|\sum a_\nu y_\nu\Big\| \quad \text{for all scalars } a_\nu.$$

It is a general fact about L^p (cf. [22]) that

(*) there is a constant B_p depending only on p, so that if (z_n) is an unconditional normalized basic sequence in L^p with unconditional constant u, then $\|\sum a_\nu z_\nu\| \leqslant B_p u(\sum |a_\nu|^2)^{1/2}$.

Hence since (y_ν) is a block basis of (e_j) and hence of (b_n),

$$d([y_\nu], l^2) \leqslant \frac{1+\varepsilon}{1-\varepsilon} B_p u_p$$

and so the desired conclusion follows from Theorem B.

For $1 < p \leqslant 2$, we again observe that by James' argument [19] (replacing in (4) "inf" by "sup"), for given $0 < \varepsilon < 1$ there exists a normalized block basis (y_ν) of (e_j) so that

$$\Big\|\sum a_\nu y_\nu\Big\| \leqslant \frac{1+\varepsilon}{1-\varepsilon}\Big(\sum |a_\nu|^2\Big)^{1/2}$$

for all scalars a_ν.

Now it is a general fact about L^p for $1 \leqslant p \leqslant 2$ that

(**) there exists an absolute constant B, so that if (z_ν) is a normalized unconditional basic sequence in the space with unconditional constant u, then $(\sum |a_\nu|^2)^{1/2} \leqslant Bu\|\sum a_\nu z_\nu\|$ for all scalars a_ν (cf. [8]).

Hence the space $[y_\nu]$ is a $\dfrac{1+\varepsilon}{1-\varepsilon}$ Bu-isomorph of l^2.

Now let $B_1, B_2, \ldots$ be disjoint finite subsets of the positive integers such that $y_\nu = \sum_{j\in B_\nu} c_j b_j$ for all ν and certain scalars c_j. For each ν, let P_ν be the projection of L^p onto $[b_j]_{j\in B_\nu}$ annihilating b_j for all $j \notin B_\nu$, and let Q_ν be a projection from $[b_j]_{j\in B_\nu}$ onto the one dimensional space spanned by the y_ν, with $\|Q_\nu\| = 1$. We shall show that $P = \sum Q_\nu P_\nu$ yields the desired projection. Let $x \in L^p$. Then $\sum_\nu P_\nu(x)$ converges unconditionally and moreover $(\sum \|P_\nu(x)\|^2)^{1/2} \leqslant u_p B$ (we apply (**) to the sequence $\left(\dfrac{P_\nu(x)}{\|P_\nu(x)\|}\right)$).

But since $Q_\nu P_\nu(x) = \lambda_\nu y_\nu$ for some scalar λ_ν, for all ν,

$$\Big(\sum |\lambda_\nu|^2\Big)^{1/2} \leqslant \Big(\sum \|P_\nu(x)\|^2\Big)^{1/2} \leqslant u_p B\|x\|.$$

Hence $\sum Q_\nu P_\nu(x) = \sum \lambda_\nu y_\nu$ converges and moreover

$$\Big\|\sum Q_\nu P_\nu(x)\Big\| \leqslant u_p B\|x\|,$$

hence $[y_\nu]$ is $u_p B$-complemented. Q.E.D.

Remark 4. The proof for $1 < p \leqslant 2$ shows that Theorem 3.1 generalizes as follows: its conclusion holds for any Banach space X with an unconditional basis, such that the space satisfies (**). In particular X satisfies (**) if it has the Orlicz property (cf. [8]), i.e.: There exists a constant C_X such that

$$\Big(\sum_i \|x_i\|^2\Big)^{1/2} \leqslant C_X \cdot \sup_{\varepsilon_i = \pm 1} \Big\|\sum_i \varepsilon_i x_i\Big\|$$

for all $x_1, x_2, \ldots, x_n$ in X and for $n = 1, 2, \ldots$

We note that the example of l^1 shows that this generalization is not localizable. Since L^1 has the Orlicz property (cf. [32]) and no infinite dimensional reflexive subspace of L^1 is complemented in L^1 (cf. [15], [33]), this yields yet another proof that L^1 has no unconditional basis.

We end this paper by discussing a local analogue of Theorem 3.1. We introduce the property of a space stronger than that of its being sufficiently-Euclidean.

Definition. Let X be an infinite dimensional Banach space. We say that X is *locally π-Euclidean* if there is a $\lambda > 0$ so that for all k and $\varepsilon > 0$, there is an $N = N_X(k, \varepsilon)$ such that every N dimensional subspace of X contains a $(1+\varepsilon)$-isomorph of l_k^2 which is λ-complemented in X.

Let us observe that in view of Dvoretzky's theorem on almost spherical sections [9], [31] a Banach space X is locally π-Euclidean if there exist a $\lambda > 0$ and a function $k \to N(k)$ such that every 2-isomorph of $l_{N(k)}^2$ in X contains a k-dimensional subspace which is λ-complemented in X.

Theorem 3.2. *If $1 < p < \infty$ then L^p is locally π-Euclidean.*

For $\infty > p \geqslant 2$ the above result follows immediately from Theorem B. For $2 > p > 1$ Theorem 3.2 can easily be deduced from Theorem 2.2 and from the final part of the argument of the proof of Theorem 3.1. However we present here yet another approach which admits some generalization.

Lemma 3.3. *Let $1 < p < \infty$. Then there is an absolute constant u_p (= the unconditional constant of the Haar basis) so that given $1 > \delta > 0$ and a positive integer k, there is an $n = n(p, k, \delta)$ such that: if $f_1, \ldots, f_n$ in L^p are elements of norm one with $\|\sum_{j=1}^n c_j f_j\| \geqslant \delta \max_{1\leqslant j\leqslant n} |c_j|$ for all scalars $c_1, c_2, \ldots, c_n$ then there are $1 \leqslant m_1 < m_2 < \ldots < m_{2k} \leqslant n$, so that putting $e_i = f_{m_{2i}} - f_{m_{2i-1}}$ for all i, then*

$$\Big\|\sum_{i=1}^k \pm c_i e_i\Big\| \leqslant 8u_p \Big\|\sum_{i=1}^k c_i e_i\Big\| \tag{4}$$

for all scalars $c_1, c_2, \ldots, c_k$ and all choices of $\pm$.

Proof. Assume to the contrary that there exist $\delta > 0$ and a positive integer k such that for each $n \geqslant k$ there are $f_1^n, \ldots, f_n^n$ which provide counter-examples. In the space F_0 of all eventually zero sequences we define a sequence of pseudonorms by $\|(c_i)\|_n = \|\sum_{i=1}^n c_i f_i^n\|$ for $(c_i) \in F_0$ and for $n = 1, 2, \ldots$ Then there exists a subsequence (n_j) such that for all $(c_j) \in F_0$, $\lim_j \|(c_i)\|_{n_j}$ exists. This follows by a standard compactness argument using the observation that for every $(c_j) \in F_0$ for almost all n we have

$$\delta \sup_i |c_i| \leqslant \|(c_i)\|_n \leqslant \sum_{i=1}^n |c_i|.$$

Now let E be the completion of F_0 with respect to the norm $\|\cdot\|$ defined by $\|(c_i)\| = \lim_j \|(c_i)\|_{n_j}$.

Since local isometric embeddibility in L^p implies global one (cf. [3], [24]), it follows that E is isometric to a subspace of L^p (because the spaces $(F_0, \|\cdot\|_n)$ have the same property).

Let $b_j = (0, \ldots, 0, \underset{j\text{-th place}}{1}, 0, \ldots)$ for $j = 1, 2, \ldots$ Clearly $\|b_j\| = 1$ and $\delta \leqslant \|b_i - b_j\|$ for $i \neq j$ $(i, j = 1, 2, \ldots)$. Hence the facts that E is isometric to a subspace of L^p and that L^p has an unconditional basis for $1 < p < \infty$ yield the existence of an increasing sequence of indices (m_ν) such that $(b_{m_{2\nu}} - b_{m_{2\nu-1}})$ is an unconditional basic sequence with unconditional constant $\leqslant 2u_p$ where u_p is the unconditional constant of the Haar basis in L^p (cf. [2] and [25]). Now pick $n > m_{2k}$ so large that for all scalars $c_1, c_2, \ldots, c_{m_{2k}}$,

$$2\Big\|\sum_{j=1}^{m_{2k}} c_j b_j\Big\|_n \geqslant \Big\|\sum_{j=1}^{m_{2k}} c_j b_j\Big\| \geqslant \tfrac{1}{2}\Big\|\sum_{j=1}^{m_{2k}} c_j b_j\Big\|_n.$$

(It follows from the definition of the norm $\|\cdot\|$ and a standard compactness argument that such an n exists.) Then for all $c_1, c_2, \ldots, c_k$ and for all choices of $\pm$ we have

$$\Big\|\sum_{i=1}^k \pm c_i (f_{m_{2i}}^n - f_{m_{2i-1}}^n)\Big\| = \Big\|\sum_{i=1}^k \pm c_i (b_{m_{2i}} - b_{m_{2i-1}})\Big\|_n$$
$$\leqslant 2\Big\|\sum_{i=1}^k \pm c_i (b_{m_{2i}} - b_{m_{2i-1}})\Big\| \leqslant 4u_p \Big\|\sum_{i=1}^k c_i (b_{m_{2i}} - b_{m_{2i-1}})\Big\|$$
$$\leqslant 8u_p \Big\|\sum_{i=1}^k c_i (b_{m_{2i}} - b_{m_{2i-1}})\Big\|_n = 8u_p \Big\|\sum_{i=1}^k c_i (f_{m_{2i}}^n - f_{m_{2i-1}}^n)\Big\|$$

which contradicts the choice of $f_1^n, \ldots, f_n^n$. Q.E.D.

Proof of Theorem 3.2 for $1<q<2$. Let E be a subspace of L^q of dimension $n=n\left(\frac{q}{q-1}, k, \frac{1}{2}\right)$ where n is that of Lemma 3.3. Assume that E is a 2-isomorph of l_n^2. Hence there are in E vectors $g_1, g_2, \ldots, g_n$ such that for all scalars $a_1, a_2, \ldots, a_n$

$$\sqrt{\sum_{j=1}^{n}|a_j|^2} \leqslant \Big\|\sum_{j=1}^{n} a_j g_j\Big\| \leqslant 2\sqrt{\sum_{j=1}^{n}|a_j|^2}. \tag{5}$$

In particular $|a_i| \leqslant \|\sum_{j=1}^{n} a_j g_j\|$ for all i. Hence, by the Hahn–Banach extension principle, there are $f_1, f_2, \ldots, f_n$ in L^p, with $p=\frac{q}{q-1}$, such that

$$\|f_j\|_p = 1 \quad \text{and} \quad a_j = \int_0^1 g f_j\,dt \quad \text{for } g=\sum_{i=1}^{n} a_i g_i \in E \quad (j=1,2,\ldots,n).$$

Clearly if $i \neq j$, then

$$2 \geqslant \|f_i - f_j\|_p \geqslant \|g_i - g_j\|^{-1}\Big|\int_0^1 (g_i-g_j)(f_i-f_j)\,dt\Big| \geqslant \frac{2}{2\sqrt{2}} > \frac{1}{2}.$$

Thus our specification of n and Lemma 3.3 yield the existence of indices $m_1 < m_2 < \ldots < m_{2k}$ such that the sequence $(e_i)_{1\leqslant i\leqslant k}$ satisfies (4) where $e_i = f_{m_{2i}} - f_{m_{2i-1}}$ $(i=1,2,\ldots,k)$. Let us set $F=[g_{m_{2i}}]_{1\leqslant i\leqslant k}$. Clearly F is a 2-isomorph of l_k^2. Define the projection $P\colon L^q \to F$ by

$$Pg = \sum_{i=1}^{k}\int_0^1 g e_i\,dt \cdot g_i \quad \text{for } g \in L^q.$$

For fixed $g\in L^q$ pick $\xi^* \in F^*$ so that $\|\xi^*\|=1$ and $\xi^*(Pg) = \|Pg\|$. Let $\xi^*(g_{m_{2i}}) = c_i$ for $i=1,2,\ldots,k$. It follows from (5) that

$$\Big(\sum_{i=1}^{k}|c_i|^2\Big)^{1/2} \leqslant 2\|\xi^*\| = 2.$$

Next recall that if $p>2$ then there exists an absolute constant K_p such that for all $\varphi_1, \varphi_2, \ldots, \varphi_m$ in L^p and $m=1,2,\ldots$

$$\Big\|\sum_{j=1}^{m}\varepsilon_j\varphi_j\Big\|_p \leqslant K_p\Big(\sum_{j=1}^{m}\|\varphi_j\|_p^2\Big)^{1/2}$$

for some choice of sequence $(\varepsilon_j)_{1\leqslant j\leqslant m}$ of signs (cf. [21], [22]). Combining the above two inequalities with (4) we get for an auxiliary choice of

signs $(\varepsilon_j)_{1\leqslant j\leqslant k}$

$$\begin{aligned}\|Pg\| &= \xi^*(Pg) = \Big|\int_0^1\Big(\sum_{i=1}^{k} a_i e_i\Big) g\,dt\Big| \\ &\leqslant \|g\|_q\Big\|\sum_{i=1}^{k} a_i e_i\Big\|_p \leqslant 8u_p\|g\|_q\Big\|\sum_{i=1}^{k}\varepsilon_i a_i e_i\Big\|_p \\ &\leqslant 8K_p u_p\|g\|_q\Big(\sum_{i=1}^{k}|a_i|^2\|e_i\|_p^2\Big)^{1/2} \leqslant 32K_p u_p\|g\|_q.\end{aligned}$$

Hence $\|P\| \leqslant 32K_p u_p$. Q.E.D.

Remarks. 1. The above proof shows in fact that every subspace of a quotient of L^q $(1<q<2)$ is locally π-Euclidean. For this note that the dual of a quotient of L^q is a subspace of L^p for some $p>2$.

2. Applying the Brunel–Sucheston [4] technique which uses the Ramsey theorem, one can prove an analogue of Lemma 3.3 for arbitrary Banach spaces. This allows one to generalize the part $1<p<2$ of Theorem 3.3 as follows

Let X be an infinite dimensional Banach space whose dual X^* has the following property: there exists a constant K_{X^*} such that for any x_1^*, $x_2^*, \ldots, x_m^*$ in X^* and $m=1,2,\ldots,$

$$\Big\|\sum_{j=1}^{m}\varepsilon_j x_j^*\Big\| \leqslant K_{X^*}\Big(\sum_{j=1}^{m}\|x_j^*\|^2\Big)^{1/2}$$

for some choice of a sequence $(\varepsilon_j)_{1\leqslant j\leqslant m}$ of signs. Then X is locally π-Euclidean.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY
THE OHIO STATE UNIVERSITY

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