that $\varphi_{j+1}^* - \varphi_j = (\alpha \oplus \bar{\partial})\chi$ in a neighbourhood of K_j (moreover, one may suppose that χ is defined on G). Now we can define $\varphi_{j+1} = \varphi_{j+1}^* - (\alpha \oplus \bar{\partial})\chi$. This completes our inductive argument.

With this sequence at hand, it is clear that $\varphi = \lim_{j\to\infty} \varphi_j$ exists and that $\psi = (\alpha \oplus \bar{\partial})\varphi$, so that Theorem 1 is proved.

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UNIVERSITY OF JASSY

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A multiplier theorem for Jacobi expansions

by

WILLIAM C. CONNETT and ALAN L. SCHWARTZ* (St. Louis, Mo.)

Abstract. Multiplier operators on Jacobi expansions of functions in L^p, $1 \leqslant p < \infty$, are studied by realizing these operators as a sequence of kernels of singular integral type. It then follows from the Calderón–Zygmund Theory that such operators must be of strong type (p, p) for $1 < p < \infty$ and weak type $(1, 1)$.

1. Introduction. In this paper we utilize a theory developed in an earlier paper [4] to prove new and interesting multiplier theorems for Jacobi expansions. The basic idea is to represent the multiplier transformation M as a limit of convolution operators with kernels that have the properties of singular integral kernels. It then follows from the Calderón–Zygmund Theory that the operator M must be of strong type (p, p) and weak type (1,1). This is a particular application of the idea of "spaces of homogeneous type" devised by Professors Coifman and Weiss in [3]. An exact statement of the theorem is given in § 3.

The key to the representation of M is finding an approximate identity with the desired properties. Here, as in the earlier paper, we use the Poisson kernel. There are many technical difficulties in these calculations, and many of the lemmas look quite different. One reason for this is the lack of symmetry in the polynomial $P_n^{(\alpha,\beta)}(x)$ which introduces more cases that must be handled. Another reason is the complicated expression for the Poisson kernel.

It is well known that any multiplier theorem for Jacobi polynomials will have important consequences in group theory. When $\alpha = \beta = (n-1)/2$, we obtain a multiplier theorem for the zonal spherical harmonics on the unit sphere Σ_n. When $\alpha = (n-1)/2$, $\beta = 0$, a multiplier theorem follows for the zonal spherical functions on the complex n-dimensional projective space. There are theorems of this sort for all of the compact rank -1 symmetric spaces. See Muckenhoupt and Stein [6], p. 22, Bonami and Clerc [2], § 7.

We mention here two other applications of our multiplier theorem, both of which will be developed elsewhere.

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The first application is to Hankel multipliers, that is, operators which satisfy

$$(mf)\hat{}(y) = m(y)\hat{f}(y)$$

where

$$f \in L^p(x^{2\alpha+1}dx)$$

and

$$\hat{f}(y) = \int_0^\infty f(x)(xy)^{-\alpha} J_\alpha(xy) x^{2\alpha+1} dx.$$

A multiplier theorem for these spaces can be proved using our theorem and an asymptotic relation between Bessel functions and Jacobi polynomials. (See the recent paper of Igari, *On the multipliers of Hankel Transforms*, Tohôku Math. J. 24 (1972), pp. 201–206.)

The second application is to mean summability of Jacobi series. Our theorem together with complex interpolation leads to results of the following type. If $\delta > \alpha + \frac{1}{2}$, the (C, δ) means of $f \in L^p$ converges to f in p-norm, $1 < p < \infty$.

2. Jacobi polynomials and the convolution structure. Let $\alpha > -1$ and $\beta > -1$ and define $L^p_{(\alpha,\beta)} = L^p$ to be the collection of all f for which

$$\|f\|_p = \Big[\int_{-1}^{+1} |f|^p dm_{(\alpha,\beta)}\Big]^{1/p} < \infty$$

where

$$dm_{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta dx.$$

The Jacobi polynomials are the polynomials $P_n^{(\alpha,\beta)}$ orthogonal with respect to $dm_{(\alpha,\beta)}$ and normalized so that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

They are also defined by the recurrence formula

$$(2.1)\quad 2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x)$$
$$= (2n+\alpha+\beta-1)\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\}P_{n-1}^{(\alpha,\beta)}(x) -$$
$$-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x), \quad n = 2, 3, 4, \ldots,$$
$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \tfrac{1}{2}(\alpha+\beta+2)x + \tfrac{1}{2}(\alpha-\beta).$$

The indices α and β will be omitted when there is no danger of confusion.

A detailed treatment of the Jacobi polynomials can be found in Szegö's book [7]. In what follows, we generally adopt the notation of Gasper [5]. We set

$$R_n(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$$

and

$$h_n = \Big[\int_{-1}^{1} [R_n(x)]^2 dm(x)\Big]^{-1} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{2^{\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)},$$

so by Stirling's formula

$$h_n \simeq Cn^{2\alpha+1}.$$

If $f \in L^1$, we define the Jacobi series of f by

$$f(x) \sim \sum_{n=0}^\infty f\hat{}(n) h_n R_n(x)$$

where

$$f\hat{}(n) = \int_{-1}^1 f(x) R_n(x) dm(x).$$

The convolution structure is based on the work of Gasper [5]. Let α and β satisfy

$$\alpha \geqslant \beta > -1 \quad \text{and} \quad \alpha+\beta > -1;$$

then there is a function $K(x, y, z)$ such that

$$(2.2)\qquad R_n(x)R_n(y) = \int_{-1}^1 K(x, y, z) R_n(z) dm(z)$$

satisfying

$$(2.3)\qquad \int_{-1}^1 |K(x, y, z)|\, dm(z) \leqslant K$$

for a constant K which depends only on α and β. If $\beta \geqslant -1/2$ or $\alpha+\beta \geqslant 0$ then $K(x, y, z)$ is non-negative and K has the value of unity. An immediate consequence of (2.2) and (2.3) is that if f and g are in L^1 then we can define

$$f*g(z) = \int\int K(x, y, z) f(x) g(y)\, dm(x)\, dm(y)$$

from which it follows that

$$(f*g)\hat{}(n) = f\hat{}(n) g\hat{}(n).$$

Moreover, if $1 \leqslant p, q, r \leqslant \infty$ satisfy

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

then

$$\|f*g\|_r \leqslant K\|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$.

We will make frequent use of a simple change of variables which is valid because x, y, and z are to be confined to the interval $[-1, 1]$. We define φ, θ, and ψ in $[0, \pi/2]$ by setting $x = \cos 2\theta$, $y = \cos 2\varphi$, and $z = \cos 2\psi$. The measure corresponding to m is defined by

$$d\mu(\theta) = 2^{\alpha+\beta+2}\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta,$$

and we write

$$G(\theta, \varphi, \psi) = K(\cos 2\theta, \cos 2\varphi, \cos 2\psi),$$

and

$$a = \sin\varphi\sin\psi, \quad b = \cos\varphi\cos\psi, \quad c = \cos\theta.$$

Gasper showed that $G(\theta, \varphi, \psi) = 0$ if

$$c < b-a \quad \text{or} \quad c > a+b,$$

hence

$$G(\theta, \varphi, \psi) = 0 \quad \text{unless} \quad |\varphi-\psi| < \theta < \varphi+\psi.$$

We let δ_φ be the unit mass concentrated at φ and we define a generalized translation by writing

$$f*\delta_\varphi(\theta) = \int f(\psi)G(\theta, \varphi, \psi)\, d\mu(\psi)$$

$\int \ldots d\mu$ and $\int \ldots dm$ denote integration over the intervals $[0, \pi/2]$ and $[-1, 1]$ respectively. $\sum$ denotes summation as all repeated indices range over $N = \{0, 1, 2, \ldots\}$. C represents a constant not necessarily the same at each occurrence.

3. The main result. If $\{m_n\}$ is a sequence of constants, the multiplier transformation M is defined by

$$Mf = \sum_{n=0}^{\infty} m_n f\hat{\ }(n) h_n R_n$$

at least for polynomials. Our theorem gives conditions on the sequence so that M can be extended to a continuous operator on L^p $(1 < p < \infty)$ and to a weak type operator on L^1 so that

$$(Mf)\hat{\ }(n) = m_n f\hat{\ }(n)$$

when $f \in L^p$ $(1 < p < \infty)$ and in a distribution sense when $p = 1$.

We define $[x]$ to be the greatest integer not exceeding x and $\langle x\rangle = x-[x]$, the fractional part of x; using this notation, we state

THEOREM. *Assume*

$$\alpha \geqslant \beta > -1 \quad \text{and} \quad \alpha+\beta > -1$$

and let $k = [\alpha+1]+1$ *and* $\gamma = 1-\langle\alpha+1\rangle$, *and suppose*

(a) $m_n = O(1)$,

(b) $\sum_{A}^{2A} |\Delta^k m_n|^2 h_n = O(A^{-2\gamma})$,

then M *is of strong type* (p, p) *for* $1 < p < \infty$ *and weak type* $(1, 1)$.

An interesting special case is $m_n = n^{ia}$ with $a > 0$ which shows that $\lim m_n$ need not exist for $\{m_n\}$ to be a multiplier.

The proof of the theorem is long and technical, so we outline it in this section and defer the technical argument to the last two sections.

The main idea is to represent M as a limit of operators K_n where

$$K_n f = k_n * f \qquad (f \in L^1)$$

for a continuous function k_n which will be defined below. The K_n are chosen so that at least for polynomials, f, $K_n f$ converges uniformly to Mf and so that there are constants C_p $(1 \leqslant p < \infty)$ independent of n such that if $1 < p < \infty$

$$\|K_n f\|_p \leqslant C_p\|f\|_p \qquad (f \in L^p) \tag{3.1}$$

and

$$m\{\theta\colon |K_n f(\theta)| \geqslant s\} < C_1\|f\|_1 s^{-1} \qquad (f \in L^1). \tag{3.2}$$

The uniformity of the bounds makes it an elementary matter to show that (3.1) and (3.2) hold with M in place of K_n (the notion of convergence in measure is used in the case of (3.2)).

Equations (3.1) and (3.2) are established by first showing that $k_n(\theta, \varphi) = k_n * \delta_\varphi(\theta)$ satisfies a version of Hörmander's condition:

$$\int_E |k_n(\theta, \varphi) - k_n(\theta, \varphi_0)|\, d\mu(\theta) \leqslant C \tag{3.3}$$

where

$$E = \{\theta\colon |\theta-\varphi_0| > 2|\varphi-\varphi_0|\}$$

and C is independent of n. Once (3.3) is established, methods of Coifman and Weiss ([3], p. 88) can be used to prove (3.1) and (3.2). Thus we need only to define k_n and show that (3.3) holds under the assumption that $\{m_n\}$ satisfies (a) and (b).

The idea of the proof rests upon the use of an approximate identity, in this case, the Poisson kernel

$$W_r(\theta, \varphi) = \sum (1-r)^n h_n R_n(\cos 2\theta) R_n(\cos 2\varphi).$$

We let $W_r(\theta) = W_r(\theta, 0)$, then examination of the Jacobi series of the two functions shows that

$$W_r(\theta, \varphi) = W_r * \delta_\varphi(\theta),$$

and $W_r(\theta)$ is a non-negative approximate identity (see Proposition 4.1 *infra* and the discussion preceeding it).

Now

$$MW_r(\theta) = \sum m_n(1-r)^n h_n R_n(\cos 2\theta)$$

is a continuous function, so we define

$$a_i(\theta) = [M(W_{2^{-i-1}} - W_{2^{-i}})] * (W_{2^{-i-1}} + W_{2^{-i}})(\theta)$$

and

$$k_n(\theta) = \sum_{i=0}^{n} a_i(\theta) + m_0$$

then k_n is a continuous function, and for any trigonometric polynomial f

$$Mf(\theta) = \lim_{n\to\infty} k_n * f(\theta)$$

uniformly in θ.

It is shown in § 5 that $2(1-x)R_n(x)$ is very nearly equal to $\Delta^2 R_{n-1}(x) = R_{n+1}(x) - 2R_n(x) + R_{n-1}(x)$ (equation 5.3 *infra*) and thus if $c_n = f\hat{\ }(n)$ then (Proposition 5.1, *infra*)

$$\int (1-x)^k [f(x)]^2 dm(x) \leqslant K \sum (\Delta^k c_n)^2 h_n. \tag{3.4}$$

We now let $V_r = W_r - W_{r/2}$; then if m_n satisfies (a) and (b), (3.4) together with an appropriate version of Parseval's theorem, can be used to prove

$$\int (1-\cos\theta)^k |MV_r|^2 d\mu(\theta) = O(r^{2\gamma}) \tag{3.5}$$

(see Lemma 5.1 of [4]). Then (3.5) can be used (as in Lemma 5.2 of [4]) to show that if $0 < \eta < \gamma$ there is a C such that

$$\int |MV_r(\theta)| \left(\frac{\theta}{r}\right)^\eta d\mu(\theta) \leqslant C. \tag{3.6}$$

Now (3.6) together with the inequalities which state that if $\alpha \geqslant \beta \geqslant -1$, there are constant C_1 and C_η such that

$$\int |W_r(\theta,\varphi) - W_r(\theta,\varphi_0)|\, d\mu(\theta) \leqslant \frac{C_1}{r}|\varphi-\varphi_0|,$$

$$\int W_r(\theta,\varphi)\left(\frac{|\theta-\varphi|}{r}\right)^\eta d\mu(\theta) \leqslant C_\eta \qquad (0<\eta<1)$$

(Proposition 4.2, *infra*) can be used to show as in Lemma 5.3 of [4], that

$$\int_E |a_i * \delta_\varphi(\theta) - a_i * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant C \min\left\{\left[\frac{r}{|\varphi-\varphi_0|}\right]^\eta, \frac{|\varphi-\varphi_0|}{r}\right\}. \tag{3.7}$$

Finally (3.3) is proved by summing (3.7) as i takes the values $0, 1, 2, \ldots, n$.

4. The Poisson kernel. In this section the Poisson kernel for Jacobi polynomials is shown to be an approximate identity satisfying the same types of inequalities as are satisfied by the Poisson kernel for the ultraspherical polynomials. We assume throughout that $\alpha > \beta > -1$ and $\alpha + \beta > -1$.

The Poisson kernel can be expressed as a hypergeometric function (see [1], p. 102) and as a consequence is non-negative; its integral with respect to $d\mu$ is unity since $R_0 \equiv 1$; in particular, $\|W_r\|_1 = 1$. This is enough to prove the following

PROPOSITION 4.1.

$$\|W_r * f - f\|_p \to 0 \quad \text{as } r\to 0 \text{ for } 1 \leqslant p < \infty. \tag{4.1}$$

To prove the proposition, let $\varepsilon > 0$; then there is a polynomial Q such that $\|Q-f\|_p < \varepsilon/3$. Now $W_r * Q$ converges uniformly to Q so there is a number r_0 such that if $r < r_0$,

$$\|W_r * Q - Q\|_p < \varepsilon/3.$$

Finally if $r < r_0$,

$$\|W_r*f - f\|_p \leqslant \|W_r*f - W_r*Q\|_p + \|W_r*Q - Q\|_p + \|Q-f\|_p$$
$$\leqslant \|W_r\|_1 \|f-Q\|_p + \varepsilon/3 + \varepsilon/3 \leqslant \varepsilon.$$

We also need the following proposition analogous to the corresponding inequalities in [4].

PROPOSITION 4.2. *There are constants C_1 and C_η such that*

$$\int |W_r(\theta,\varphi) - W_r(\theta,\varphi_0)|\, d\mu(\theta) \leqslant \frac{C_1}{r}|\varphi-\varphi_0| \tag{a}$$

and

$$\int W_r(\theta,\varphi)\left(\frac{|\theta-\varphi|}{r}\right)^\eta d\mu(\theta) \leqslant C_\eta\,, \qquad 0<\eta<1. \tag{b}$$

The proof of the proposition is based on the following lemmas, and is almost identical to the corresponding argument in [4], so we delete it; however, the proofs of the lemmas are much more difficult than the corresponding proofs in [4], so we give them in some detail.

LEMMA 4.1. *There exists a function $W_r^*(\theta,\varphi)$ which is a bound for $W_r(\theta,\varphi)$ and which, in turn satisfies the following bounds*

$$W_r^*(\theta,\varphi) \leqslant \frac{Cr}{[r^2+(\theta-\varphi)^2]^{(2q+3)/2}}, \tag{4.2}$$

$$W_r^*(\theta,\varphi) \leqslant \frac{Cr}{(ab)^{q+\frac12}[r^2+(\theta-\varphi)^2]}, \tag{4.3}$$

where q can be taken to be α for $0 \leqslant \theta, \varphi \leqslant \pi/2$, and in addition q can be given the value β if both θ and φ exceed $\pi/3$. (Recall that $a = \sin\varphi\sin\theta$ and $b = \cos\varphi\cos\theta$.)

LEMMA 4.2.

$$\left|\frac{\partial}{\partial\varphi} W_r(\theta,\varphi)\right| \leqslant \frac{C}{r} W_r^*(\theta,\varphi). \tag{4.4}$$

Lemmas 4.1 and 4.2 are generalizations of results which were proved for ultraspherical polynomials in [6] and [4] respectively. In [4] W_r^* was not introduced because when $\alpha=\beta$ the integrand of (4.5) (*infra*) does not change sign.

Before proving the lemmas, we give the kernel in two closed forms, both due to Watson. That author gives an expression for the kernel which is equivalent to the following

$$(4.5)\quad W_r(\theta,\varphi) = \frac{-r}{4\pi(1-r)^{(\alpha+\beta+2)/2}}\int_0^{\pi/2}\frac{\partial}{\partial k}(k^{\alpha+\beta+1}A^{-\alpha}B^{-\beta}Z^{-1/2})\cos^{\alpha+\beta}\omega\cos(\alpha-\beta)\omega\,d\omega,$$

where

$$k=\tfrac12(t^{1/2}+t^{-1/2}),\qquad Z=[k^2-(a^2+b^2)\cos^2\omega]^2-4a^2b^2\cos^4\omega,$$

$$A=k^2-(b^2-a^2)\cos^2\omega+Z^{1/2},\qquad B=k^2-(a^2-b^2)\cos^2\omega+Z^{1/2}.$$

We assume without loss that $1/2\leqslant t\leqslant 1$ so that

$$1\leqslant k\leqslant 3/2\quad\text{and}\quad r^2/4\leqslant k^2-1\leqslant r^2/2.$$

Watson then performs a change of variables and transforms (4.5) into

$$(4.6)\quad W_r(\theta,\varphi) = \frac{-r}{\pi[4(1-r)]^{(\alpha+\beta+2)/2}}\int_0^{\pi}\frac{\partial}{\partial k}[\cos(\alpha-\beta)\omega E^{-(\alpha+\beta+1)/2}(F/G)^{(\alpha-\beta)/2}]\sin^{\alpha+\beta}\chi\,d\chi$$

where

$$E=k^2-(a^2+2ab\cos\chi+b^2),\qquad F=k^2-(a+b\cos\chi)^2,$$
$$G=k^2-(b+a\cos\chi)^2,$$

and ω is the acute angle, positive or negative satisfying

$$\cot\omega=\frac{P}{Q}$$

for

$$P=k\sin\chi E^{1/2}\quad\text{and}\quad Q=k^2\cos\chi-(a+b\cos\chi)(b+a\cos\chi).$$

The quantity estimated in Lemma 4.2 is actually $W_r^*(\theta,\varphi)$ defined by

$$W_r^*(\theta,\varphi) = \frac{r}{\pi[4(1-r)]^{(\alpha+\beta+2)/2}}\int_0^{\pi}\left|\frac{\partial}{\partial k}[\cos(\alpha-\beta)\omega E^{-(\alpha+\beta+1)/2}(F/G)^{(\alpha-\beta)/2}]\right|\sin^{\alpha+\beta}\chi\,d\chi$$

so that we obviously have

$$W_r(\theta,\varphi)\leqslant W_r^*(\theta,\varphi).$$

Proof of Lemma 4.1. We can write (4.6) as

$$(4.7)\quad W_r^*(\theta,\varphi) = \frac{r}{\pi[4(1-r)]^{(\alpha+\beta+2)/2}}\int_0^{\pi}E^{-(\alpha+\beta+1)/2}(F/G)^{(\alpha-\beta)/2}\times$$
$$\times\left|\frac{d}{dk}\cos(\alpha-\beta)\omega+k\cos(\alpha-\beta)\omega\left[-\frac{\alpha+\beta+1}{E}+\frac{\alpha-\beta}{F}+\right.\right.$$
$$\left.\left.+\frac{\beta-\alpha}{G}\right]\right|\sin^{\alpha+\beta}\chi\,d\chi.$$

The estimation of the integral requires us to bound each term inside the absolute value signs by C/E. For the first term we observe that

$$\frac{\partial}{\partial k}\cos(\alpha-\beta)\omega=(\beta-\alpha)\sin(\alpha-\beta)\omega\frac{\partial\omega}{\partial k},$$

$$\frac{\partial\omega}{\partial k}=\frac{Q\dfrac{\partial P}{\partial k}-P\dfrac{\partial Q}{\partial k}}{P^2+Q^2}=\frac{\sin\chi}{E^{1/2}}\,\frac{H}{P^2+Q^2},$$

where H is defined by

$$H=\{k^2\cos\chi-(a+b\cos\chi)(b+a\cos\chi)\}\{E+k^2\}-2k^2\cos\chi\cdot E$$
$$=(a+b\cos\chi)(b+a\cos\chi)[a^2+2ab\cos\chi+b^2]-k^2[2ab+(a^2+b^2)\cos\chi]$$
$$=(a^2+2ab\cos\chi+b^2)(-Q)+k^2[2ab(\cos^2\chi-1)]$$

and finally,

$$(4.8)\quad |H|\leqslant(a+b)^2Q+k^2(4ab\sin^2\chi)$$

since $-\pi/2<\omega<\pi/2$ we also have

$$|\sin(\alpha-\beta)\omega|\leqslant|\alpha-\beta||\omega|\leqslant\frac{\pi}{2}|\alpha-\beta|/\cot\omega=\frac{\pi}{2}|\alpha-\beta|Q/P$$

so we obtain the bound

$$\left|\frac{\partial}{\partial k}\cos(\alpha-\beta)\omega\right|\leqslant\frac{\sin\chi|H|\cdot|\alpha-\beta|\cdot Q}{E^{1/2}(P^2+Q)^2\cdot P}=\frac{|\alpha-\beta|Q|H|}{kE(P^2+Q^2)}$$

which by (4.8) is

$$\leqslant|\alpha-\beta|\left(\frac{(a+b)^2Q^2}{kEQ^2}+\frac{k^2ab\sin^2\chi Q}{kEPQ}\right).$$

The first term is clearly bounded by CE^{-1}. The bound for the second term follows from the observation that

$$P=k\sin\chi E^{1/2}\geqslant k\sin\chi[2ab(1-\cos\chi)]^{1/2}$$

and consequently

$$\frac{k\,ab\sin^2\chi}{EP} \leqslant \frac{ab\sin\chi}{2E(ab)^{1/2}(\sin\chi/2)} \leqslant \frac{\sqrt{ab}}{E} \leqslant \frac{1}{E}.$$

We now examine the remaining three terms in the square brackets on the right side of (4.7). The first term is clearly bounded by CE^{-1}. That the second two terms have the same bound, follows from the observation that

$$F \geqslant E \quad \text{and} \quad G \geqslant E.$$

Now to obtain the first estimate of the integral we introduce the following definition,

$$\Delta = r^2 + (\varphi - \theta)^2,$$

then

$$E \geqslant C[\Delta + ab\chi^2].$$

This estimate follows directly from the definition. For example, if $t > 1/2$

$$\begin{aligned}(4.9)\qquad E &= (k^2-1) + [1-(a^2+2ab\cos\chi+b^2)]\\ &= (k^2-1) + \big(1-(a+b)^2\big) + 2ab(1-\cos\chi)\\ &= (k^2-1) + \sin^2(\varphi-\theta) + 4ab\sin^2(\chi/2) \geqslant C(\Delta + ab\chi^2).\end{aligned}$$

Finally, we return to (4.7) and using the above estimates obtain that $W_r^*(\theta,\varphi)$ is less than or equal to

$$\begin{aligned}(4.10)\qquad & Cr\int_0^\pi \frac{1}{E^{(\alpha+\beta+1)/2}}\cdot\left(\frac{F}{G}\right)^{(\alpha-\beta)/2}\cdot\frac{1}{E}\cdot\sin^{\alpha+\beta}\chi\,d\chi\\ &\leqslant Cr\int_0^\pi \frac{1}{E^{(\alpha+\beta+1)/2}}\left(\frac{1}{E}\right)^{(\alpha-\beta)/2}\cdot\frac{1}{E}\cdot\sin^{\alpha+\beta}\chi\,d\chi\\ &= Cr\int_0^\pi \frac{\sin^{\alpha+\beta}\chi\,d\chi}{E^{\alpha+3/2}} \leqslant Cr\int_0^\pi \frac{\chi^{\alpha+\beta}\,d\chi}{(\Delta+ab\chi^2)^{\alpha+3/2}}.\end{aligned}$$

Now to obtain the two estimates we make the change of variables $u = (ab/\Delta)^{1/2}\chi$ and obtain

$$(4.11)\qquad Cr\frac{(\Delta/ab)^{(\alpha+\beta+1)/2}}{\Delta^{\alpha+3/2}}\int_0^{\pi(ab/\Delta)^{1/2}}\frac{u^{\alpha+\beta}}{(1+u^2)^{\alpha+3/2}}\,du.$$

This last integral is bounded by

$$\int_0^{\pi(ab/\Delta)^{1/2}} u^{\alpha+\beta}\,du = C(ab/\Delta)^{(\alpha+\beta+1)/2}.$$

If this is substituted in (4.11) we obtain the first estimate of the Lemma 4.1 with $q = \alpha$.

To obtain the second estimate we observe

$$\frac{F}{G} = 1 + \frac{(b^2-a^2)\sin^2\chi}{G},$$

and

$$\begin{aligned}(4.12)\qquad G &= (k^2-1) + [1-(a+b)^2] + 2ab(1-\cos\chi) + a^2(1-\cos^2\chi)\\ &\geqslant C(ab+a^2)\chi^2\end{aligned}$$

so

$$\frac{F}{G} \leqslant \frac{C}{ab}.$$

Inserting this in (4.10) we obtain, instead of (4.11)

$$Cr\frac{(\Delta/ab)^{(\alpha+\beta+1)/2}}{\Delta^{(\alpha+\beta+3)/2}(ab)^{(\alpha-\beta)/2}}\int_0^{\pi(ab/\Delta)^{1/2}}\frac{u^{\alpha+\beta}}{(1+u^2)^{(\alpha+\beta+3)/2}}\,du.$$

The integral is bounded irrespective of the upper limit, so the second estimate of the lemma follows with $q = \alpha$.

To complete the proof we assume φ and θ exceed $\pi/4$ so that $a \geqslant 1/2$. Under this assumption F/E and χ^2/G are bounded because of (4.9) and (4.12) respectively, so the integral in (4.10) is bounded by

$$\begin{aligned}& Cr\int_0^\pi\left(\frac{F}{E}\right)^{(\alpha-\beta)/2}\frac{1}{E^{(2\beta+3)/2}}\left(\frac{\chi^2}{G}\right)^{(\alpha-\beta)/2}\chi^{2\beta}\,d\chi \leqslant Cr\int_0^\pi\frac{\chi^{2\beta}}{E^{(2\beta+3)/2}}\,d\chi\\ &\leqslant Cr\frac{(\Delta/ab)^{(2\beta+1)/2}}{\Delta^{(2\beta+3)/2}}\int_0^{\pi(ab/\Delta)^{1/2}}\frac{u^{2\beta}}{(1+u^2)^{(2\beta+3)/2}}\,du.\end{aligned}$$

The integral is bounded uniformly by a constant and by $(ab/\Delta)^{(2\beta+1)/2}$. The two bounds can then be used to obtain the rest of Lemma 4.1.

Proof of Lemma 4.2. To prove this result, we begin with Watson's first expression (4.5) for the kernel and obtain

$$\frac{\partial}{\partial\varphi}W_r(\theta,\varphi)$$

$$= \frac{r}{4\pi(1-r)^{(\alpha+\beta+2)/2}}\int_0^{\pi/2} U\frac{\partial}{\partial k}\left(k^{\alpha+\beta+1}A^{-\alpha}B^{-\beta}Z^{-1/2}\right)\cos^{\alpha+\beta}\omega\cos(\alpha-\beta)\omega\,d\omega$$

where

$$U = S + \frac{\partial R/\partial\varphi}{R},$$

$$R = \frac{\alpha\partial A/\partial k}{A} + \frac{\beta\partial B/\partial k}{B} + \frac{\partial Z/\partial k}{2Z} - \frac{\alpha+\beta+1}{k}$$

and

$$-S = \frac{\alpha\partial A/\partial\varphi}{A} + \frac{\beta\partial B/\partial\varphi}{B} + \frac{\partial Z/\partial\varphi}{2Z}.$$

It will suffice to show

$$|U| \leqslant \frac{C}{r}$$

uniformly in all variables, for then we would have

$$\left|\frac{\partial}{\partial\varphi} W_r(\theta,\varphi)\right| \leqslant \frac{C}{r}\,\frac{r}{(1-r)^{(\alpha+\beta+2)/2}} \int_0^{\pi/2} \left|\frac{\partial}{\partial k}\left(k^{\alpha+\beta+1}A^{-\alpha}B^{-\beta}Z^{-1/2}\right)\cos(\alpha-\beta)\omega\right| \cos^{\alpha+\beta}\omega\, d\omega.$$

With Watson's change of variables, this last expression is bounded by

$$\frac{C}{r} W_r^*(\theta,\varphi)$$

which yields the desired result.

In order to proceed with the argument, we shall first obtain some bounds for the quantities A, B, and Z. Since

$$b \pm a = \cos(\varphi \mp \theta),$$

we can write

$$Z = Z_+ Z_-$$

where

(4.13) $$Z_\pm = k^2 - \cos^2(\varphi\pm\theta)\cos^2\omega$$

so

(4.14) $$Z_\pm = (k^2-1) + \sin^2(\varphi\pm\theta)\cos^2\omega + \sin^2\omega,$$

and

(4.15) $$Z_\pm \geqslant Cr|\sin(\varphi\pm\theta)|\cos\omega.$$

We also note that

$$A = k^2 - \cos(\varphi-\theta)\cos(\varphi+\theta)\cos^2\omega + Z^{1/2},$$

so

(4.16) $$A \geqslant (k^2-1) + \tfrac{1}{2}[\sin^2(\varphi+\theta)+\sin^2(\varphi-\theta)]\cos^2\omega + Z^{1/2},$$

and so

(4.17) $$A \geqslant r\sin(\varphi+\theta)\cos\omega + r|\sin(\varphi-\theta)|\cos\omega + Z^{1/2}.$$

The last two inequalities also hold with B in place of A.

We shall first estimate S. The last term of $-S$ is

$$\left|\frac{1}{2Z}\frac{\partial Z}{\partial\varphi}\right| = \left|\frac{1}{2Z_+}\frac{\partial Z_+}{\partial\varphi} + \frac{1}{2Z_-}\frac{\partial Z_-}{\partial\varphi}\right| < \left|\frac{1}{r}[\cos(\varphi+\theta)+\cos(\varphi-\theta)]\cos\omega\right|;$$

thus

(4.18) $$\left|\frac{1}{2Z}\frac{\partial Z}{\partial\varphi}\right| \leqslant \frac{2}{r}.$$

To deal with the first term of $-S$ we note

$$\left|\frac{1}{A}\frac{\partial A}{\partial\varphi}\right| = \left|\frac{\cos^2\omega}{A}[\sin(\varphi+\theta)\cos(\varphi-\theta)+\sin(\varphi-\theta)\cos(\varphi+\theta)] + \frac{1}{2AZ^{1/2}}\frac{\partial Z}{\partial\varphi}\right|$$
$$\leqslant \frac{2\cos\omega}{r}[\cos(\varphi-\theta)+|\cos(\varphi+\theta)|] + \left|\frac{1}{2Z}\frac{\partial Z}{\partial\varphi}\right|.$$

Thus

$$\left|\frac{1}{A}\frac{\partial A}{\partial\varphi}\right| \leqslant \frac{6}{r}.$$

The last inequality follows from (4.17) and (4.18). The second term of $-S$ can be dealt with in the same manner, so we finally obtain

$$|S| \leqslant \frac{C}{r}.$$

We now estimate the term $R^{-1}\partial R/\partial\varphi$. We write R as

(4.19) $$R = \alpha\left[\frac{\partial A/\partial k}{A} - \frac{1}{k}\right] + \beta\left[\frac{\partial B/\partial k}{B} - \frac{1}{k}\right] + \frac{1}{2}\left[\frac{\partial Z/\partial k}{Z} - \frac{2}{k}\right]$$
$$= \alpha R_A + \beta R_B + \frac{1}{2}R_Z,$$

so

(4.20) $$\frac{1}{R}\frac{\partial R}{\partial\varphi} = \frac{\alpha}{R}\frac{\partial R_A}{\partial\varphi} + \frac{\beta}{R}\frac{\partial R_B}{\partial\varphi} + \frac{1}{2R}\frac{\partial R_Z}{\partial\varphi}.$$

The quantities R_A, R_B and R_Z are non-negative. In fact

$$R_Z = \frac{2}{Z}[k^2+(b^2-a^2)\cos^2\omega][k^2-(b^2-a^2)\cos^2\omega].$$

Now a simple computation shows that

$$\tfrac{1}{2}R_Z - R_A = \frac{k[\cos(\varphi+\theta)-\cos(\varphi-\theta)]^2\cos^2\omega}{AZ}[k^2+\cos(\varphi+\theta)\cos(\varphi-\theta)\cos^2\omega] \geqslant 0;$$

similarly $\frac{1}{2}R_Z - R_B \geqslant 0$; thus, because $\alpha \geqslant \beta > -1$ and $\alpha+\beta > -1$ we see that for some constant C

$$R \geqslant CR_Z, \quad R \geqslant CR_A, \quad \text{and} \quad R \geqslant CR_B. \tag{4.21}$$

Now, of the two factors of R_Z at least one must exceed unity; exactly which one does depends on the sign of a^2-b^2; but, if we assume for the moment that $b^2-a^2 \geqslant 0$, then

$$R_Z \geqslant Z^{-1}[k^2-(b^2-a^2)\cos^2\omega] = Z^{-1}\{(k^2-1)+[1-(b^2-a^2)]\cos^2\omega+\sin^2\omega\},$$

so

$$R_Z \geqslant Z^{-1}\{(k^2-1)+\tfrac{1}{2}\sin^2(\varphi+\theta)\cos^2\omega+\tfrac{1}{2}\sin^2(\varphi-\theta)\cos^2\omega+\sin^2\omega\};$$

the same inequality holds if $b^2-a^2 \leqslant 0$.

Referring to (4.14) we see at once that

$$R_Z \geqslant \frac{1}{2}Z^{-1}[Z_+ + Z_-] \geqslant Z^{-1/2}. \tag{4.22}$$

We now estimate last term of (4.20). It is bounded by $R_Z^{-1}|\partial R_Z/\partial\varphi|$.

Differentiation of $R_Z = \dfrac{1}{Z}\dfrac{\partial z}{\partial k} - \dfrac{2}{k}$ yields

$$\frac{\partial R_Z}{\partial\varphi} = \frac{1}{Z^2}\left\{Z\frac{\partial^2 Z}{\partial\varphi\partial k} - \frac{\partial Z}{\partial\varphi}\frac{\partial Z}{\partial k}\right\}. \tag{4.23}$$

Since $Z = Z_+Z_-$,

$$\frac{\partial Z}{\partial k} = 2k[Z_+ + Z_-], \quad \frac{\partial Z}{\partial\varphi} = [\sin(2\varphi-2\theta)Z_+ + \sin(2\varphi+2\theta)Z_-]\cos^2\omega, \tag{4.24}$$

and

$$\frac{\partial^2 Z}{\partial\varphi\partial k} = 2k[\sin(2\varphi-2\theta)+\sin(2\varphi+2\theta)]\cos^2\omega. \tag{4.25}$$

A simple computation finally shows

$$\frac{\partial R_Z}{\partial\varphi} = \frac{-2k}{Z^2}[\sin(2\varphi-2\theta)Z_+^2+\sin(2\varphi+2\theta)Z_-^2]\cos^2\omega;$$

thus using (4.22) we obtain the following bound for $R_Z^{-1}|\partial R_Z/\partial\varphi|$:

$$\frac{4k\cos^2\omega}{[Z_+ + Z_-]Z}|\sin(2\varphi-2\theta)Z_+^2+\sin(2\varphi+2\theta)Z_-^2] \leqslant 8k\cos^2\omega\left[\frac{\sin(\varphi-\theta)}{Z_-}\cos(\varphi-\theta)+\frac{\sin(\varphi+\theta)}{Z_+}\cos(\varphi+\theta)\right],$$

so, from (4.15)

$$\frac{1}{R_Z}\left|\frac{\partial R_Z}{\partial\varphi}\right| \leqslant \frac{C}{r}. \tag{4.26}$$

We now consider the first term of (4.20); it is easy to see that

$$\frac{\partial R_A}{\partial\varphi} = \frac{1}{A^2}(T_1+T_2-T_3)$$

where

$$T_1 = \left[\frac{k^2-\cos(\varphi+\theta)\cos(\varphi-\theta)\cos^2\omega+2Z^{1/2}}{4Z^{3/2}}\right]\left[Z\frac{\partial^2 Z}{\partial\varphi\partial k} - \frac{\partial Z}{\partial\varphi}\frac{\partial Z}{\partial k}\right],$$

$$T_2 = [k^2-\cos(\varphi+\theta)\cos(\varphi-\theta)\cos^2\omega]\frac{1}{4Z^{1/2}}\frac{\partial^2 Z}{\partial\varphi\partial k},$$

and

$$T_3 = 2k\sin 2\varphi\cos^2\omega + kZ^{-1/2}\frac{\partial z}{\partial\varphi} + \frac{1}{2}(\sin 2\varphi\cos^2\omega)Z^{-1/2}\frac{\partial Z}{\partial k}.$$

The balance of the proof consists of estimates of the quantities

$$T_j/RA^2 \quad (j = 1, 2, 3).$$

We recall (4.19) where R is represented in terms of the quantities R_A, R_B, and R_Z. The R's are non-negative; in fact,

$$R_A = \frac{k^2+(b^2-a^2)}{kA} + \frac{Z^{1/2}R_Z}{2A} \geqslant \frac{k^2+(b^2-a^2)}{kA}. \tag{4.27}$$

By symmetry we obtain

$$R_B \geqslant \frac{k^2-(b^2-a^2)}{kB}.$$

Since at least one of $k^2 \pm (b^2-a^2)$ exceeds unity, it follows from (4.21) that

$$R^{-1} \leqslant C\min(A, B),$$

so

$$(RA^2)^{-1} \leqslant CA^{-1}. \tag{4.28}$$

From (4.21) and (4.23)

$$\frac{T_1}{RA^2} = Z^{1/2}\frac{[k^2-\cos(\varphi+\theta)\cos(\varphi-\theta)\cos^2\omega+2Z^{1/2}]}{4A^2}\,\frac{1}{R}\,\frac{\partial R_Z}{\partial\varphi}$$

$$\leqslant \frac{Z^{1/2}}{4A}\cdot\frac{A+Z^{1/2}}{A}\cdot\frac{1}{R_Z}\cdot\frac{\partial R_Z}{\partial\varphi}$$

so, for some constant C, (4.26) implies

$$\frac{T_1}{RA^2}\leqslant\frac{C}{R}.$$

From (4.28) and (4.25) it follows that

$$\frac{T_2}{RA^2}\leqslant\frac{C}{4Z^{1/2}}\frac{\partial^2 Z}{\partial\varphi\partial k}$$

$$= Ck\cos^2\omega\left[\frac{\sin(\varphi-\theta)}{Z_-^{1/2}}\,\frac{\cos(\varphi-\theta)}{Z_+^{1/2}}+\frac{\sin(\varphi+\theta)}{Z_+^{1/2}}\,\frac{\cos(\varphi+\theta)}{Z_-^{1/2}}\right],$$

whence from (4.14) and (4.15)

$$\frac{T_2}{RA^2}\leqslant\frac{C}{r}.$$

Finally, we deal individually with the three terms of $T_3/(RA^2)$. First, writing $2\varphi=(\varphi+\theta)+(\varphi-\theta)$ and applying (4.28), we find

$$(RA^2)^{-1}2k|\sin 2\varphi|\cos^2\omega\leqslant\frac{C}{A}\,|\sin(\varphi+\theta)\cos(\varphi-\theta)+\cos(\varphi+\theta)\sin(\varphi-\theta)|\cos^2\omega$$

$$\leqslant C\left|\frac{2\cos(\varphi-\theta)}{r}+\frac{2\cos(\varphi+\theta)}{r}\right|\cos\omega\leqslant\frac{C}{r}.$$

Next since $(RA^2)^{-1}\leqslant CA^{-1}\leqslant CZ^{-1/2}$,

$$\frac{1}{RA^2}\,\frac{k}{Z^{1/2}}\,\frac{\partial Z}{\partial\varphi}\leqslant\frac{C}{Z}\cdot\frac{\partial Z}{\partial\varphi}\leqslant\frac{C}{r}$$

by (4.18). And for the last term, we observe that by (4.24)

$$\frac{\partial Z}{\partial k}=2k[Z_++Z_-]\leqslant kA,$$

so using $(RA^2)^{-1}\leqslant CA^{-1}$

$$(RA^2)^{-1}\frac{1}{2}(\sin 2\varphi\cos^2\omega)Z^{-1/2}\partial Z/\partial k\leqslant C(\sin 2\varphi\cos^2\omega)Z^{-1/2}\leqslant\frac{C}{r}$$

by the first estimate.

We have thus shown that there is a constant C such that

$$\frac{1}{R}\,\frac{\partial R_A}{\partial\varphi}\leqslant\frac{C}{r};$$

a similar result holds for the second term of (4.20).

5. Difference calculations. From the recursion relation (2.1) we obtain the following formula:

(5.1) $$A_nP_{n+1}^{(\alpha,\beta)}=(B_n+C_nx)P_n^{(\alpha,\beta)}-D_nP_{n-1}^{(\alpha,\beta)},$$

where

$$A_n=2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta),\qquad B_n=(2n+\alpha+\beta+1)(\alpha^2-\beta^2),$$

$$C_n=(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)(2n+\alpha+\beta),$$

$$D_n=2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2).$$

Now if we multiply equation (5.1) by $h_n/P_n^{(\alpha,\beta)}(1)$ we obtain

(5.2) $$A_n\left(\frac{h_n}{P_n^{(\alpha,\beta)}(1)}\,\frac{P_{n+1}^{(\alpha,\beta)}(1)}{h_{n+1}}\right)h_{n+1}R_{n+1}(x)$$

$$=(B_n+C_nx)h_nR_n(x)-D_n\left(\frac{h_n}{P_n^{(\alpha,\beta)}(1)}\,\frac{P_{n-1}^{(\alpha,\beta)}(1)}{h_{n-1}}\right)h_{n-1}R_{n-1}(x).$$

Notice that

$$\frac{h_n}{P_n^{(\alpha,\beta)}(1)}\,\frac{P_{n+1}^{(\alpha,\beta)}(1)}{h_{n+1}}=\frac{(2n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(n+\alpha+\beta+1)}=Q_n$$

and

$$\frac{h_n}{P_n^{(\alpha,\beta)}(1)}\,\frac{P_{n-1}^{(\alpha,\beta)}(1)}{h_{n-1}}=\frac{1}{Q_{n-1}}$$

so

$$xh_nR_n(x)=\frac{A_nQ_n}{C_n}h_{n+1}R_{n+1}(x)-\frac{B_n}{C_n}h_nR_n(x)+\frac{D_n}{C_nQ_{n-1}}R_{n-1}(x).$$

This allows us to compute the following formula:

(5.3) $$2(x-1)h_nR_n(x)=a_nh_{n+1}R_{n+1}(x)-b_nh_nR_n(x)+c_nh_{n-1}R_{n-1}(x)$$

with

$$a_n=\frac{2A_nQ_n}{C_n}\simeq 1,\qquad b_n=\frac{2B_n}{C_n}-2\simeq -2,\qquad c_n=\frac{2D_n}{Q_{n-1}C_n}\simeq 1,$$

which is clearly asymptotic to a second difference formula, $\Delta^2(h_n R_n(x))$. In particular, in order to obtain this difference we assume that $f(x)$ is a polynomial and

$$f(x) \sim \sum d_n h_n R_n(x)$$

and calculate

$$(5.4)\qquad 2(x-1)f(x) \sim \sum d_n(a_n h_{n+1} R_{n+1} - b_n h_n R_n + c_n h_{n-1} R_{n-1}) = \sum (d_{n-1}a_{n-1} - d_n b_n + d_{n+1}c_{n+1}) h_n R_n$$

with

$$d_{n-1}a_{n-1} - d_n b_n + d_{n+1}c_{n+1} = \Delta^2 d_n + (a_{n-1}-1)d_{n-1} + (2-b_n)d_n + (c_{n+1}-1)d_{n+1}.$$

It is another simple calculation to show that

$$a_{n-1}-1 = \frac{-(1+2\alpha)}{(2n+\alpha+\beta+1)} + \frac{(\alpha^2-\beta^2)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},$$

$$c_{n+1}-1 = \frac{(1+2\alpha)}{(2n+\alpha+\beta+1)} + \frac{(\alpha^2-\beta^2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}.$$

Therefore the coefficients in (5.4) become equal to

$$(5.5)\qquad \Delta^2 d_n + \frac{1+2\alpha}{(2n+\alpha+\beta+1)}[d_{n+1}-d_{n-1}] + (\alpha^2-\beta^2)\left\{\frac{d_{n+1}}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} - \frac{d_n}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)} + \frac{d_{n-1}}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\right\}.$$

We observe that each of these terms has the form of a difference divided by a polynomial in n. We estimate these terms using the same techniques as in the paper [4]. The above calculation can be used to obtain

PROPOSITION 5.1. *Suppose k is an integer, $f(x)\in L^2$, and $f \sim \sum c_n h_n R_n$; then there is a constant C, independent of f such that*

$$\int (1-x)^k [f(x)]^2 dm(x) \leqslant C \sum (\Delta^k c_n)^2 h_n.$$

The proof of this fact is so similar to the proof of the corresponding fact in [4], that we omit it.

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