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Another note on Kalton's theorems

by

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Abstract. The purpose of this paper is to give new proofs for the two Orlicz–Pettis type theorems, due to Kalton ([3], [4]), which are formulated below.

Let G be an abelian group, $N = \{1, 2, \ldots\}$, $\mathscr{P} = \mathscr{P}(N) = \{A: A \subset N\}$ and $\boldsymbol{m}: \mathscr{P} \to G$ an additive set function.

If $\boldsymbol{m}$ is countably additive (resp. exhaustive) when G is endowed with a Hausdorff group topology τ, we shall say that $\boldsymbol{m}$ is a *τ-measure* (resp. *τ-exhaustive*, i.e. $\boldsymbol{m}A_n \xrightarrow{\tau} 0$ for every infinite sequence of disjoint sets $A_n \subset N$). A subset A of (G, τ) will be called *transseparable* (trans for translation) if for every neighbourhood V of 0 in (G, τ) there exists a countable subset B of G such that $A \subset B + V$. (It can be assumed that $B \subset A$.) It can easily be shown that (G, τ) is transseparable iff it is isomorphic with a subgroup of the product $\prod_{i \in I} G_i$ of metrizable separable groups G_i $(i \in I)$. Thus, in particular, if X is any locally convex topological vector space, then $(X, \sigma(X, X'))$ is transseparable.

Throughout α and β denote two Hausdorff group topologies on G. We shall write $\alpha \vdash \beta$ (resp. $\alpha \vdash_s \beta$) if β has a base of α-closed (resp. sequentially α-closed) neighbourhoods of 0. Though $\alpha \vdash \beta$ does not imply $\alpha \subset \beta$, α can always be replaced by a Hausdorff group topology γ such that $\gamma \subset \alpha$, $\gamma \subset \beta$ and $\gamma \vdash \beta$ (e.g., the group topology $\gamma = \inf\{\alpha, \beta\}$).

THEOREM 1. *Suppose $\alpha \vdash_s \beta$ and $\boldsymbol{m}$ is an α-measure. Then if the range $\boldsymbol{m}[\mathscr{P}]$ of $\boldsymbol{m}$ is β-transseparable, $\boldsymbol{m}$ is also a β-measure.* (The converse is trivial.)

THEOREM 2. *Suppose $\alpha \vdash \beta$ and $\boldsymbol{m}$ is α-exhaustive. Then if $\boldsymbol{m}[\mathscr{P}]$ is β-transseparable, $\boldsymbol{m}$ is also β-exhaustive.*

(The above formulations slightly differ from the original results of Kalton.) A short proof of Theorem 1, based on the Baire category theorem, can be found in [2]. Below we give a still more direct proof. Our proof of Theorem 2 reduces it to Theorem 1; the method we use was inspired in part by Labuda's paper [5].

It is not clear to the author if exhaustivity of $\boldsymbol{m}$ implies transseparability of its range or if $\alpha \vdash \beta$ in Theorem 2 may be replaced by $\alpha \vdash_s \beta$.

Proof of Theorem 1. It suffices to show that $\boldsymbol{m}$ is β-exhaustive. (For, this implies that if a sequence $(A_n) \subset \mathscr{P}$ is disjoint then the series $\sum \boldsymbol{m}A_n$ is β-Cauchy, and since it α-converges to $\boldsymbol{m}(\bigcup A_n)$, it has the same limit in β.)

Suppose $\boldsymbol{m}$ is not β-exhaustive. Then we can find a disjoint sequence (E_n) and a symmetric sequentially α-closed β-neighbourhood V of 0 in G such that $\boldsymbol{m}E_n \notin V$ for $n \in \boldsymbol{N}$. Without loss of generality we can assume that $E_n = \{n\}$. Thus

(1) $$\boldsymbol{m}\{n\} \notin V, \quad n \in \boldsymbol{N}.$$

We claim that for each n there exists $k(n) > n$ such that

(2) $$\boldsymbol{m}\{n\} + \boldsymbol{m}A - \boldsymbol{m}B \notin V \quad \text{if } A, B \subset \boldsymbol{N} \text{ and } \inf(A \cup B) \geqslant k(n).$$

In fact, otherwise for some n_0 there exist sequences (A_i), (B_i) such that

$$\inf(A_i \cup B_i) \to \infty \quad \text{and} \quad \boldsymbol{m}\{n_0\} + \boldsymbol{m}A_i - \boldsymbol{m}B_i \in V, \quad i \in \boldsymbol{N}.$$

But $\boldsymbol{m}A_i \xrightarrow{\alpha} 0$, $\boldsymbol{m}B_i \xrightarrow{\alpha} 0$, so $\boldsymbol{m}\{n_0\} \in V$, for V is sequentially α-closed. This is a contradiction with (1).

Now define $q_1 = 1$, $q_{i+1} = k(q_i)$ for $i \in \boldsymbol{N}$, and set $Q = \{q_1, q_2, \ldots\}$. We claim that

(3) $$\text{if } A, B \subset Q \text{ and } A \neq B, \text{ then } \boldsymbol{m}A - \boldsymbol{m}B \notin V.$$

In fact, let $q = \inf((A \setminus B) \cup (B \setminus A))$ and consider, for instance, the case when $q \in B \setminus A$. Then

$$\boldsymbol{m}A - \boldsymbol{m}B = -[\boldsymbol{m}\{q\} + \boldsymbol{m}((B \setminus A) \setminus \{q\}) - \boldsymbol{m}(A \setminus B)] \notin -V = V$$

by (2), for

$$\inf[(A \setminus B) \cup ((B \setminus A) \setminus \{q\})] \geqslant k(q).$$

Now, as $\mathscr{P}(Q)$ is uncountable, (3) implies that $\boldsymbol{m}[\mathscr{P}]$ is not β-transseparable. This is a contradiction with the assumption.

We shall need the following

Lemma. *If $\alpha \vdash \beta$ and (G, β) is metrizable and separable, then there exists a metrizable group topology γ on G such that $\gamma \subset \alpha$ and $\gamma \vdash \beta$.*

Proof. Let $(V_n)_{n \in \boldsymbol{N}}$ be a base at 0 in (G, β) consisting of α-closed sets such that

$$V_{n+1} - V_{n+1} \subset V_n, \quad n \in \boldsymbol{N}$$

and let $\mathscr{A}$ be a base at 0 in (G, α).

For each $n \in \boldsymbol{N}$ and $U \in \mathscr{A}$ set

$$G(n, U) = \{x \in G \colon (x + U) \cap V_n = \emptyset\}.$$

Then

(4) $$G(n, U) \subset \operatorname{Int}_\beta G(n+1, U).$$

In fact, if $x \in G(n, U)$ then $x + V_{n+1} \subset G(n+1, U)$.

As V_n is α-closed,

$$G \setminus V_n = \bigcup_{U \in \mathscr{A}} G(n, U)$$

and therefore from (4) and the Lindelöf property of (G, β) we deduce the existence of a sequence $(U_k^n)_{k \in \boldsymbol{N}} \subset \mathscr{A}$ such that

(5) $$G \setminus V_n \subset \bigcup_{k=1}^{\infty} G(n+1, U_k^n).$$

Now we can easily define a countable family $\mathscr{U} \subset \mathscr{A}$ which is a base at 0 for some group topology γ on G and contains all the sets U_k^n $(k, n \in \boldsymbol{N})$. Evidently $\gamma \subset \alpha$.

If $x \notin V_n$, then we deduce from (5) the existence of $U \in \mathscr{U}$ such that $(x + U) \cap V_{n+1} = \emptyset$. It follows that $V_{n+1} \subset \mathrm{cl}_\gamma V_{n+1} \subset V_n$ and that γ is Hausdorff. Hence γ is metrizable and $\gamma \vdash \beta$.

Proof of Theorem 2. We can assume that (G, β) is transseparable.

First consider the case when β is metrizable; then (G, β) is separable. Let γ be a topology whose existence is asserted in our lemma. Since $\boldsymbol{m}$ is α-exhaustive and $\gamma \subset \alpha$, $\boldsymbol{m}$ is also γ-exhaustive. Let (E_n) be a disjoint sequence in $\mathscr{P}$. Then, by Proposition 1 in [1], we can find an infinite subsequence (F_n) of (E_n) such that $\boldsymbol{m}$ restricted to the σ-ring $\mathscr{F}$ generated by (F_n) is a γ-measure. Now $\gamma \vdash \beta$ and Theorem 1 imply that $\boldsymbol{m}|\mathscr{F}$ is a β-measure, so $\boldsymbol{m}F_n \xrightarrow{\beta} 0$. It follows easily that $\boldsymbol{m}E_n \xrightarrow{\beta} 0$. Thus $\boldsymbol{m}$ is β-exhaustive.

In the general case we proceed as follows. Let V_0 be any α-closed β-neighbourhood of 0. Then we can define inductively a sequence $(V_n)_{n \in \boldsymbol{N}}$ of symmetric α-closed β-neighbourhoods of 0 such that

$$V_n + V_n \subset V_{n-1}, \quad n \in \boldsymbol{N}.$$

Let β' be the group topology on G for which (V_n) is a base at 0.

The subgroup $H = \bigcap_n V_n$ of G is α- and β'-closed, so the quotient topologies $\hat{\alpha}$ and $\hat{\beta}$ on $\hat{G} = G/H$ corresponding to α and β' are Hausdorff. Evidently $(\hat{G}, \hat{\beta})$ is metrizable and separable. Let h be the quotient mapping of G onto $\hat{G}$. Then the sets $\hat{V}_n = h(V_n)$ $(n \in N)$ form a base at 0 for $\hat{\beta}$. We claim that $\hat{a} \vdash \hat{\beta}$.

It suffices to verify that

$$\mathrm{cl}_{\hat{a}} \hat{V}_n \subset \hat{V}_{n-1}, \quad n \in N. \tag{6}$$

Suppose $\hat{x} = h(x) \notin \hat{V}_{n-1}$. Then $x \notin V_{n-1}$, and since V_{n-1} is α-closed, there is an α-neighbourhood U of x such that $U \cap V_{n-1} = \emptyset$. Then $\hat{U} = h(U)$ is an $\hat{a}$-neighbourhood of $\hat{x}$ and $\hat{U} \cap \hat{V}_n = \emptyset$. (Otherwise we can find $u \in U$ and $v \in V_n$ such that $u - v \in H$, and then $u = (u-v) + v \in H + V_n \subset V_n + V_n \subset V_{n-1}$, so that $U \cap V_{n-1} \neq \emptyset$.)

Since $h: (G, \alpha) \to (\hat{G}, \hat{a})$ is continuous, the additive set function $\hat{m} = h \circ m: \mathscr{P} \to \hat{G}$ is $\hat{a}$-exhaustive. By the first part of the proof $\hat{m}$ is also $\hat{\beta}$-exhaustive. Thus if a sequence $(E_n) \subset \mathscr{P}$ is disjoint, there is n_0 such that $\hat{m}E_n \in \hat{V}_1$ for $n \geqslant n_0$. It follows that $mE_n \in H + V_1 \subset V_0$ for $n \geqslant n_0$, and we conclude that m is β-exhaustive.

Corollary. *If $\alpha \subset \beta$ and (G, β) is separable and metrizable by a complete metric, then*

(a) *if m is an α-measure, it is also a β-measure,*

(b *if m is α-exhaustive, it is also β-exhaustive.*

Part (a) of this corollary belongs to Kalton [3]; part (b) has been proved by Kalton [4] and independently by Labuda [5]. Part (a) was shown in [2] to be a simple consequence of Theorem 1 and a closed graph theorem for groups, and in exactly the same way part (b) can be derived from Theorem 2. Labuda [5] proved in a very clever manner that (a) implies (b). Let us note that when (b) is proved, the metrizable case in the proof of Theorem 2 can be reduced to (b) (assume $\alpha \subset \beta$, then complete (G, β) and (G, α), and apply (b)); then the lemma we have established before proving Theorem 2 is not needed. This lemma may be however of some independent interest.

Remark. From that proof of Theorem 1, which is given in [2], it is immediately seen that instead of $\alpha \vdash_s \beta$ one can suppose that β has a base at 0 consisting of sets V such that V is the countable union of sequentially α-closed sets.

References

[1] L. Drewnowski, *Equivalence of Brooks–Jewett, Vitali–Hahn–Saks and Nikodym theorems*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 725–731.

[2] — *On the Orlicz–Pettis type theorems of Kalton*, ibid. 21 (1973), pp. 515–518.

[3] N. J. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. 10 (1971), pp. 402–412.

[4] — *Topologies on Riesz groups and application to measure theory*, Proc. London Math. Soc. (3), 28 (1974), pp. 253–273.

[5] I. Labuda, *A generalization of Kalton's theorem*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 21 (1973), pp. 509–510.

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