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Isomorphisms of continuous function spaces

by

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Abstract. This paper is an outgrowth of the works of Bessaga and Pełczyński [2] and Semadeni [9] concerning the isomorphic classification of spaces of continuous functions defined on spaces of ordinal numbers.

For each ordinal number α, let C^α denote the supremum-normed Banach space of continuous complex-valued functions defined on the space of ordinal numbers not exceeding α. Let ω and Ω denote the first infinite ordinal number and the first uncountable ordinal number respectively. Bessaga and Pełczyński obtained a complete isomorphic classification of the spaces C^α for α less than Ω while Semadeni obtained the corresponding result for uncountable ordinal numbers α less than $\Omega\cdot\omega$. In this paper, the isomorphic classification of the spaces C^α is obtained for all α less than Ω^ω. In addition, some partial isomorphic classifications of the spaces C^α are obtained for arbitrarily large ordinal numbers of certain types.

0. Introduction. We begin by establishing some notation. For each ordinal number α, $\Gamma(\alpha)$ will denote the topological space of the non-zero ordinal numbers not exceeding α, equipped with the interval topology (cf. [3], page 57). The smallest ordinal number and the smallest uncountable ordinal number will be denoted by ω and Ω respectively. In order to facilitate the statement of results, Greek letters will always denote ordinal numbers throughout this paper unless otherwise specified. Any facts concerning ordinal numbers, not proved in this paper, can be found in Chapter XIV of [11]. Arithmetic properties of ordinal numbers obtained from this source will be used without specific reference.

Let X be a Banach space (real or complex). Following Bessaga and Pełczyński [2], we define X^α to be the Banach space of continuous X-valued functions defined on $\Gamma(\alpha)$, equipped with the supremum norm. X_0^α will denote the closed subspace of functions in X^α which vanish at α.

C will denote either the field of real numbers or the field of complex numbers. As customary, for any compact Hausdorff space S, $C(S)$ will denote the supremum-normed Banach space of continuous C-valued functions defined on S. Thus $C^\alpha = C(\Gamma(\alpha))$ for each ordinal number α.

If X and Y are Banach spaces, then X is said to be *isomorphic* (*isometric*) to Y, written $X \sim Y$, ($X \approx Y$) provided there is a one-one, bounded, (norm-preserving) linear operator from X onto Y.

The primary purpose of this paper is to find conditions on the ordinal numbers α and β which ensure that C^{α} is isomorphic to C^{β}. In [2], Bessaga and Pełczyński gave a complete solution to this problem for countable ordinal numbers with the following:

Suppose $\omega \leqslant \alpha \leqslant \beta < \Omega$. *Then* $C^{\alpha} \sim C^{\beta}$ *if and only if* $\beta < \alpha^{\omega}$.

In Section 1 of this paper, we apply some of the techniques of Bessaga and Pełczyński to uncountable ordinal numbers. Their methods are combined with some of the techniques arising in the proof of Miljutin's Theorem [6], which states that $C(S)$ is isomorphic to $C(T)$ for all uncountable compact metric spaces S and T (cf. [10], page 379 or [7], page 41). The following partial extension of the Bessaga–Pełczyński result to uncountable ordinal numbers is thereby obtained:

Suppose $\omega \leqslant \beta \leqslant \alpha$ *and* β *has the same cardinality as* α. *In addition, suppose* $\alpha\beta \leqslant \gamma$. *Then,* $C^{\gamma} \sim C^{\alpha\beta}$ *if and only if* $\gamma < \alpha^{\omega} = (\alpha\beta)^{\omega}$ (Theorem 1.10).

In Section 2, the results of Section 1 are combined with an isomorphism invariant obtained by Semadeni in [9] to yield the complete isomorphic classification of the spaces C^{α} for $\Omega \leqslant \alpha < \Omega^{\omega}$. This classification is due to Semadeni for $\Omega \leqslant \alpha < \Omega\cdot\omega$:

Suppose $1 \leqslant n < \omega$. *Then* $C^{\Omega\cdot n} \sim C^{\alpha}$ *if and only if* $\Omega\cdot n \leqslant \alpha < \Omega\cdot(n+1)$.

This should be compared with the following results which we obtain in the second section:

(1) $C^{\Omega\cdot\omega} \sim C^{\alpha}$ if and only if $\Omega\cdot\omega \leqslant \alpha < \Omega^{2}$ (Corollary 2.8).

(2) $C^{\Omega^{2}} \sim C^{\alpha}$ if and only if $\Omega^{2} \leqslant \alpha < \Omega^{\omega}$ (Corollary 2.9).

Theorems on isomorphic classification involving arbitrarily large ordinal numbers of certain types are also given in Section 2.

Before beginning Section 1, we need to introduce a few more conventions and definitions. We recall that a subspace Y of the Banach space X is said to be *complemented* if there is a bounded linear operator P from X onto Y satisfying $P^{2} = P$. A Banach space Z is said to be a *factor of* X provided Z is isomorphic to a complemented subspace of X.

For Banach spaces X and Y, $X\times Y$ will denote the Banach space obtained by defining operations coordinate-wise on the *Cartesian product* of X with Y, and normed by taking the maximum of the norms of the coordinates. We recall the following relation between factor spaces and Cartesian products of Banach spaces: If Y is a factor of X, then there is a Banach space Z such that X is isomorphic to $Y\times Z$.

For each ordinal number α, $\Delta(\alpha)$ will denote the set consisting of all non-zero numbers not exceeding α. For a Banach space X, $X^{\Delta(\alpha)}$ will denote the supremum-normed Banach space of X-valued functions f de-

fined on $\Delta(\alpha)$ and having the following property: for each positive real number ε, the set $\{\lambda\colon \lambda\in\Delta(\alpha) \text{ and } \|f(\lambda)\| \geqslant \varepsilon\}$ is finite; equivalently, if $\Delta(\alpha)$ is equipped with the discrete topology, $X^{\Delta(\alpha)}$ can be considered as the space of continuous X-valued functions on $\Delta(\alpha)$ which vanish at ∞. In particular, $X^{\Delta(\omega)}$ is isometric to X_0^{ω}.

Finally, in order to simplify statements of theorems, X and Y will always denote Banach spaces in the sequel.

1. We will need the following easily proved lemmas.

LEMMA 1.1. *If* $\alpha \geqslant \omega$, *then* $X_0^{\alpha} \sim X^{\alpha}$.

Proof. For a proof see page 55 of [2].

LEMMA 1.2. *Suppose* X *is a factor of* Y *and* Y *is a factor of* X. *If* $X^{\omega} \sim X$, *then* $X \sim Y$.

Proof. A proof can be obtained by a trivial adaptation of an argument appearing on page 41 of [7].

If $\alpha < \beta$, then X^{α} can be isometrically embedded as a complemented subspace of X^{β} by extending functions in X^{α} to be zero outside of $\Gamma(\alpha)$. Hence, the following corollary is an immediate consequence of the preceding lemma.

COROLLARY 1.3. *Suppose* $\alpha < \beta$ *and* $(X^{\alpha})^{\omega} \sim X^{\alpha} \sim X^{\beta}$. *If* $\alpha \leqslant \gamma \leqslant \beta$, *then* $X^{\alpha} \sim X^{\gamma}$.

The following lemma together with its proof is derived from an argument used by Bessaga and Pełczyński ([2], page 55) in the case of countable ordinal numbers.

LEMMA 1.4. *If* $\alpha\beta > 0$, *then* $X_0^{\alpha\beta} \sim X_0^{\beta} \times (X_0^{\alpha})^{\Delta(\beta)}$.

Proof. We first consider the case $\beta \geqslant \omega$. Set

$$Y = \{f \in X_0^{\alpha\beta}\colon f \text{ is constant on the interval } (\alpha\lambda, \alpha(\lambda+1)] \text{ for } 0 \leqslant \lambda < \beta\}$$

and

$$Z = \{f \in X_0^{\alpha\beta}\colon f(\alpha\lambda) = 0 \text{ for } 1 \leqslant \lambda \leqslant \beta\}.$$

Then $X_0^{\alpha\beta} \sim Y \times Z$ (cf. [2], page 55). Since Y is clearly isometric to X_0^{β}, it remains only to show that Z is isomorphic to $(X_0^{\alpha})^{\Delta(\beta)}$, for the case $\beta \geqslant \omega$. Now for each f belonging to Z and $\varepsilon > 0$, the set $\{\lambda\colon \sup_{\alpha\lambda<\gamma\leqslant\alpha(\lambda+1)} \|f(\gamma)\| \geqslant \varepsilon\}$ is easily seen to be finite (cf. [2], page 56). Consequently, Z is isomorphic to the space $\{g \in (X_0^{\alpha})^{\Delta(\beta)}\colon g(\beta) \equiv 0\}$ which in turn is isomorphic to $(X_0^{\alpha})^{\Delta(\beta)}$ since $\beta \geqslant \omega$.

It remains to consider the case $\beta < \omega$. If $\beta < \omega$ and $\alpha < \omega$, then $X_0^{\alpha\beta} \sim X^{\alpha\beta-1} \sim X^{\beta-1} \times (X^{\alpha-1})^{\beta} \sim X_0^{\beta} \times (X_0^{\alpha})^{\Delta(\beta)}$. If $\beta < \omega$ and $\alpha \geqslant \omega$, then

$$X_0^{\alpha\beta} \sim X^{\alpha\beta} \sim \underbrace{X^{\alpha}\times \ldots \times X^{\alpha}}_{\beta} \sim \underbrace{X\times \ldots \times X}_{\beta-1} \times \underbrace{X^{\alpha}\times \ldots \times X^{\alpha}}_{\beta} \sim X_0^{\beta} \times (X_0^{\alpha})^{\Delta(\beta)}$$

by Lemma 1.1.

The following trivial result will prove to be quite useful when used in conjunction with Lemma 1.2.

LEMMA 1.5. *If* $\alpha \geqslant \omega$, *then* $(X^{\Delta(\alpha)})^{\omega} \sim X^{\Delta(\alpha)}$.

Proof. By Lemma 1.1, it suffices to show $(X^{\Delta(\alpha)})^{\Delta(\omega)} \sim X^{\Delta(\alpha)}$. This is immediate since $\Delta(\alpha)$ can be written as a countably infinite union of disjoint sets A_n each having the same cardinality as $\Delta(\alpha)$.

We note that a trivial extension of the proof of the preceding lemma yields the following stronger result: If $\alpha \geqslant \omega$ and the cardinality of β is less than or equal to the cardinality of α, then $X^{\Delta(\alpha)} \sim (X^{\Delta(\alpha)})^{\Delta(\beta)}$.

COROLLARY 1.6. *If* $\omega \leqslant \beta \leqslant \alpha$, *then* $X^{\alpha\beta} \sim (X^{\alpha})^{\Delta(\beta)}$.

Proof. By Lemmas 1.1 and 1.4, we have $X^{\alpha\beta} \sim X_0^{\alpha\beta} \sim X_0^{\beta} \times (X_0^{\alpha})^{\Delta(\beta)}$. Set $Y = X_0^{\beta} \times (X_0^{\alpha})^{\Delta(\beta)}$. Then $X_0^{\alpha} \times (X_0^{\alpha})^{\Delta(\beta)}$ is a factor of Y, and Y is a factor of $X_0^{\alpha} \times (X_0^{\alpha})^{\Delta(\beta)}$ which is isomorphic to $(X_0^{\alpha})^{\Delta(\beta)}$ (since β is infinite). Applications of Lemmas 1.5 and 1.2 yield $Y \sim (X_0^{\alpha})^{\Delta(\beta)}$. Hence, $X^{\alpha\beta} \sim (X_0^{\alpha})^{\Delta(\beta)} \sim (X^{\alpha})^{\Delta(\beta)}$ by Lemma 1.1.

COROLLARY 1.7. *If* $\alpha \geqslant \omega$ *and* $\alpha^2 \leqslant \beta < \alpha^{\omega}$, *then* $X^{\alpha^2} \sim X^{\beta}$.

Proof. Since $\lim\limits_{n<\omega} \alpha^n = \alpha^{\omega}$, it suffices by Corollary 1.3 to show that $X^{\alpha^2} \sim (X^{\alpha^2})^{\omega}$ and $X^{\alpha^2} \sim X^{\alpha^n}$ for $2 \leqslant n < \omega$. Using Corollary 1.6 and Lemma 1.5, we have

$$X^{\alpha^2} \sim (X^{\alpha})^{\Delta(\alpha)} \sim [(X^{\alpha})^{\Delta(\alpha)}]^{\omega} \sim (X^{\alpha^2})^{\omega}$$

as desired. Now, suppose $X^{\alpha^2} \sim X^{\alpha^n}$ for some natural number $n \geqslant 2$. Then by Corollary 1.6 and the remark following Lemma 1.5, we have

$$X^{\alpha^{n+1}} = X^{\alpha^n\alpha} \sim (X^{\alpha^n})^{\Delta(\alpha)} \sim (X^{\alpha^2})^{\Delta(\alpha)} \sim [(X^{\alpha})^{\Delta(\alpha)}]^{\Delta(\alpha)} \sim (X^{\alpha})^{\Delta(\alpha)} \sim X^{\alpha^2}.$$

By induction, $X^{\alpha^n} \sim X^{\alpha^2}$ for each natural number $n \geqslant 2$.

THEOREM 1.8. *Suppose* $\omega \leqslant \beta \leqslant \alpha$ *and* β *has the same cardinality as* α. *If* $\alpha\beta \leqslant \gamma < \alpha^{\omega}$, *then* $X^{\alpha\beta} \sim X^{\gamma}$.

Proof. By Corollary 1.3, Lemma 1.5, and Corollary 1.7, it suffices to show that $X^{\alpha\beta} \sim X^{\alpha^2}$. This is an immediate consequence of Corollary 1.6 since α and β have the same cardinality:

$$X^{\alpha\beta} \sim (X^{\alpha})^{\Delta(\beta)} \approx (X^{\alpha})^{\Delta(\alpha)} \sim X^{\alpha^2}.$$

The following theorem due to Bessaga and Pełczyński ([2], page 59) can be used to obtain a partial converse of the preceding result in the case of $X = C$.

THEOREM 1.9. *If* $C^{\alpha} \sim C^{\beta}$, *then* $\beta < \alpha^{\omega}$.

THEOREM 1.10. *Suppose* $\omega \leqslant \beta \leqslant \alpha$ *and* β *has the same cardinality as* α. *Also, suppose* $\gamma \geqslant \alpha\beta$. *Then* $C^{\gamma} \sim C^{\alpha\beta}$ *if and only if* $\gamma < \alpha^{\omega}$.

Proof. The "only if" part of the statement follows from Theorem 1.9 since $C^{\gamma} \sim C^{\alpha\beta}$ thereby implies $\gamma < (\alpha\beta)^{\omega} \leqslant (\alpha^2)^{\omega} = \alpha^{2\cdot\omega} = \alpha^{\omega}$. The "if" part is Theorem 1.8.

A more general version of Theorem 1.10 can be obtained with the aid of the following definition.

DEFINITION. An infinite ordinal number α is called *decomposable* if and only if there exist ordinal numbers σ and λ such that $\lambda \leqslant \sigma$, λ and σ have the same cardinality, and $\sigma\lambda \leqslant \alpha < \sigma^{\omega}$.

The following result is an uncountable analog of Theorem 1 of [2] which dealt with countable ordinal numbers.

COROLLARY 1.11. *Suppose* α *is decomposable and* $\alpha \leqslant \beta$. *Then* $C^{\alpha} \sim C^{\beta}$ *if and only if* $\beta < \alpha^{\omega}$.

Proof. Suppose $\beta < \alpha^{\omega}$. Choose σ and λ satisfying the conditions of the above definition with respect to α. Since $\alpha < \sigma^{\omega}$ and $\lim\limits_{n<\omega} = \sigma^{\omega}$, there exists a natural number n satisfying $\alpha < \sigma^n$. So, $\alpha^{\omega} \leqslant (\sigma^n)^{\omega} = \sigma^{n\cdot\omega} = \sigma^{\omega} \leqslant \alpha^{\omega}$ and consequently $\beta < \sigma^{\omega}$. Hence $\sigma\lambda \leqslant \alpha \leqslant \beta < \sigma^{\omega}$ and $C^{\alpha} \sim C^{\sigma\lambda} \sim C^{\beta}$ by Theorem 1.10.

The converse is a specialization of Theorem 1.9.

2. We begin by introducing some notation and definitions. Let A be a subset of a topological space S. The *first derived set* of A is the set $A^{(1)}$ consisting of the accumulation points of A. For any ordinal number $\alpha > 1$, $A^{(\alpha)}$, the α-th *derived set* of A is defined as follows: $A^{(\alpha)} = (A^{(\beta)})^{(1)}$ if $\alpha = \beta + 1$; and $A^{(\alpha)} = \bigcap\limits_{\lambda<\alpha} A^{(\lambda)}$ if λ is a limit ordinal number. S is said to be *dispersed* provided $S^{(\alpha)}$ is empty for some α; equivalently S is dispersed if S has no non-empty perfect subsets (cf. [10], page 147 or [4], page 261).

Let S be a dispersed compact Hausdorff space. Then there is a unique ordinal number α such that $S^{(\alpha)}$ is non-empty and finite [1]. Suppose $S^{(\alpha)}$ contains exactly n elements. The ordered pair (α, n) is called the *characteristic system* of S and α is called the *characteristic* of S.

The following result due to Baker [1] gives a characterization of spaces of ordinal numbers among compact Hausdorff spaces.

THEOREM 2.1. *Let* S *be a compact Hausdorff space. Then* S *is homeomorphic to* $\Gamma(\alpha)$ *for some ordinal number* α *if and only if the following conditions are satisfied:*

(1) *S is dispersed.*

(2) *For each* s *in* S, *there is a decreasing sequence (possibly transfinite) of closed-open sets* $\langle U_{\lambda}\rangle_{\lambda<\gamma_s}$ *which form a neighborhood base at* s. *Moreover, for each limit ordinal number* $\beta < \gamma_s$, *the set* $\bigcap\limits_{\lambda<\beta} U_{\lambda} \setminus U_{\beta}$ *contains at most one point.*

If the compact Hausdorff space S satisfies conditions (1) *and* (2) *and has characteristic system* (σ, n), *then S is homeomorphic to* $\Gamma(\omega^\sigma \cdot n)$.

In what follows, the preceding theorem will be applied to closed subspaces S of $\Gamma(\alpha)$ for some α. It is a routine matter to verify that such spaces satisfy the conditions of the theorem.

We recall that an ordinal number α is said to have an *immediate predecessor* provided the equation $\alpha = \beta+1$ has a solution. If α is the limit of a sequence of distinct ordinal numbers, then α is said to be *co-final with* ω. These notions occur in the following result which was obtained for countable ordinal numbers by Bessaga and Pełczyński ([2], page 55).

LEMMA 2.2. *If either* (1) *α has an immediate predecessor, or* (2) *α is co-final with ω, then* $X^{\omega^\alpha} \sim (X^{\omega^\alpha})^\omega$ *for every Banach space X.*

Proof. By Lemma 1.1, it suffices to show $X_0^{\omega^\alpha} \approx (X_0^{\omega^\alpha})_0^\omega$. Set $S = \Gamma(\omega^\alpha)$. We will construct a countable sequence $\langle S_n \rangle$ of subsets of S having the following properties:

(i) $S = \bigcap_{1 \leqslant n < \omega} S_n$,

(ii) If $\beta_n \in S_n$ for $1 \leqslant n < \omega$, then $\lim_{n<\omega} \beta_n = \omega^\alpha$,

(iii) $S_m \cap S_n = \{\omega^\alpha\}$ for $m \neq n$,

(iv) S_n is homeomorphic to S for $1 \leqslant n < \omega$.

The homeomorphism from S_n onto S obtained from (iv) will necessarily fix ω^α for each n. Using this decomposition of S, it is then an easy matter to construct an isometry from $X_0^{\omega^\alpha}$ onto $(X_0^{\omega^\alpha})_0^\omega$ (cf. [2], page 55). In order to obtain the desired decomposition of S, we consider two cases separately depending on whether α has an immediate predecessor or α is co-final with ω.

Case I: α has an immediate predecessor. Let $\alpha = \beta+1$. Partition the set of non-negative integers into countably many infinite mutually disjoint sets $A_1, A_2, \ldots$ For each natural number n, set

$$S_n = \{\lambda:\ \omega^\beta \cdot i < \lambda \leqslant \omega^\beta \cdot (i+1) \text{ for some } i \in A_n\} \cup \{\omega^\alpha\}.$$

Each S_n is then the union of the singleton set $\{\omega^\alpha\}$ and the countable union of mutually disjoint closed-open subsets of S.

The sequence $\langle S_n \rangle$ satisfies (i) and (ii) since

$$\lim_{i<\omega} \omega^\beta \cdot i = \omega^\beta \cdot \omega = \omega^{\beta+1} = \omega^\alpha.$$

If $\gamma \neq \omega^\alpha$ and $\gamma \in S_m \cap S_n$ for some $m \neq n$, then there exist non-negative integers $j < k$ such that $\omega^\beta \cdot j < \gamma \leqslant \omega^\beta \cdot (j+1)$ and $\omega^\beta \cdot k < \gamma \leqslant \omega^\beta \cdot (k+1)$ by (i) and the disjointness of the A_n. Hence, $\omega^\beta \cdot k < \omega^\beta \cdot (j+1)$ which forces $k < j+1$, contradicting the assumption on j and k. Consequently, $S_m \cap S_n = \{\omega^\alpha\}$ for $m \neq n$ and (iii) is verified.

It remains to show that the sequence $\langle S_n \rangle$ satisfies (iv). Let m be a fixed natural number. Then $S \setminus S_m = \bigcup_{n \neq m} (S_n \setminus \{\omega^\alpha\})$ from (i) and (iii), and $S_n \setminus \{\omega^\alpha\}$ is open for each n. Hence, S_m is a closed subset of the compact Hausdorff dispersed space $S = \Gamma(\omega^\alpha)$ ([10], Corollary 8.6.7), and it is easy to verify that S_m satisfies all the conditions of Theorem 2.1 as was mentioned prior to the statement of this lemma. Consequently, to show that S_m is homeomorphic to S, we need only show that the characteristic system of S_m is $(\alpha, 1)$.

To determine the characteristic system of S_m we will need the following algebraic descriptions of the βth and αth derived sets of S:

$$S^{(\beta)} = \{\omega^\beta \cdot n:\ 0 < n \leqslant \omega\} \quad \text{and} \quad S^{(\alpha)} = \{\omega^\alpha\}$$

([10], Theorem 8.6.6).

We will first show that $S_m^{(\beta)}$ is infinite. It will then follow that $S^{(\alpha)} = S^{(\beta)(1)} \neq \emptyset$ since $S_m^{(\beta)}$ is compact. Suppose $j \in A_m$. Then $\omega^\beta \cdot (j+1)$ is contained in a closed-open set (relative to S) which is contained in S_m. Consequently, $\omega^\beta \cdot (j+1) \in S_m^{(\beta)}$ for each j belonging to the infinite set A_m. Hence, $S_m^{(\beta)}$ is infinite and

$$\emptyset \neq (S_m^{(\beta)})^{(1)} = S_m^{(\alpha)} \subseteq S^{(\alpha)} = \{\omega^\alpha\}.$$

Thus, $S_m^{(\alpha)} = \{\omega^\alpha\}$ and the characteristic system of S_m is $(\alpha, 1)$. Finally, we note that any homeomorphism from S_m onto S must carry ω^α into itself since $S_m^{(\alpha)} = \{\omega^\alpha\} = S^{(\alpha)}$ from above (cf. [8], Lemma 1).

Case II: α is co-final with ω. Let $\langle \alpha_n \rangle$ be an increasing sequence of non-zero ordinal numbers which converges to α. Set $\alpha_0 = 0$. Choose a sequence $\langle A_n \rangle$ of subsets of non-negative integers as in Case I. Assume without loss of generality that $0 \in A_1$. Define a sequence $\langle S_n \rangle$ of subsets of S as follows:

$$S_1 = \{\lambda:\ \omega^{\alpha_i} < \lambda \leqslant \omega^{\alpha_{i+1}} \text{ for some } i \in A_1\} \cup \{1, \omega^\alpha\}$$

and

$$S_n = \{\lambda:\ \omega^{\alpha_i} < \lambda \leqslant \omega^{\alpha_{i+1}} \text{ for some } i \in A_n\} \cup \{\omega^\alpha\} \quad \text{for} \quad 2 \leqslant n < \omega.$$

As in Case I, each S_n is the union of the singleton set $\{\omega^\alpha\}$ and the countable union of mutually disjoint closed-open subsets of S (except for S_1 which is only trivially different from the other S_n).

To verify that the sequence $\langle S_n \rangle$ satisfies the properties (i)–(iv), we proceed essentially as in Case I. First, $\langle S_n \rangle$ satisfies (i) and (ii) since $\lim_{i<\omega} \alpha_i = \alpha$ implies $\lim_{i<\omega} \omega^{\alpha_i} = \omega^\alpha$. Next, if $\gamma \neq \omega^\alpha$ and $\gamma \in S_m \cap S_n$ for some $m \neq n$, then there exist non-negative integers $j < k$ such that $\omega^{\alpha_j} < \gamma \leqslant \omega^{\alpha_{j+1}}$ and $\omega^{\alpha_k} < \gamma \leqslant \omega^{\alpha_{k+1}}$ from (i) and the disjointness of the A_n. Thus,

$\omega^{a_k} < \omega^{a_j+1}$ which forces $k < j+1$ since $\langle a_n \rangle$ is an increasing sequence. This contradiction establishes (iii).

To show that the sequence $\langle S_n \rangle$ satisfies (iv), we need only show that each S_n is a closed subset of S having characteristic $(\alpha, 1)$, as in Case I. Let m be a fixed natural number. To show that S_m is closed in S, simply use (ii) and (iii) as in Case I. Now, suppose $j \in A_m$. From Theorem 8.6.6 of [10] already cited in Case I, it follows that $\omega^{a_j+1} \in S^{(a_j+1)}$. Since ω^{a_j+1} is contained in a closed-open set (relative to S) which is contained in S_m, it is easy to show that $\omega^{a_j+1} \in S_m^{(a_j+1)}$. Hence, $S_m^{(a_j+1)} \neq \emptyset$ for each $j \in A_m$. Since $\sup\{a_{j+1}\colon j \in A_m\} = \alpha$ and $\langle S_m^{(\lambda)} \rangle_{\lambda < m}$ is a decreasing sequence (possibly transfinite) of sets, it follows that $S_m^{(\lambda)} \neq \emptyset$ for each $\lambda < \alpha$. Hence $S_m^{(\alpha)} = \bigcap_{\lambda<\alpha} S_m^{(\alpha)} \neq \emptyset$ since $S_m^{(\lambda)}$ is compact for each $\lambda < \alpha$. It then follows that $S^{(\alpha)} = \{\omega^\alpha\}$ since $S_m^{(\alpha)} \subseteq S^{(\alpha)} = \{\omega^\alpha\}$. Consequently, the characteristic system of S_m is $(\alpha, 1)$ as desired.

As an application of the preceding lemma, we will prove the "if" part of the following theorem due to Bessaga and Pełczyński [2]. Our proof appears to differ somewhat from their proof.

THEOREM 2.3 (Bessaga and Pełczyński) *Suppose* $\omega \leqslant \alpha \leqslant \beta < \Omega$. *Then,* $C^\alpha \sim C^\beta$ *if and only if* $\beta < \alpha^\omega$.

Proof. If $C^\alpha \sim C^\beta$, then $\beta < \alpha^\omega$ as has already been noted in Theorem 1.9. Conversely, suppose $\omega \leqslant \alpha \leqslant \beta < \alpha^\omega < \Omega$. Choose an ordinal number γ so that $\omega^\gamma \leqslant \alpha < \omega^{\gamma+1}$. Then, $\omega^{\gamma\cdot\omega} \leqslant \alpha^\omega \leqslant (\omega^{\gamma+1})^\omega \leqslant (\omega^{\gamma\cdot 2})^\omega = \omega^{\gamma(2\cdot\omega)} = \omega^{\gamma\cdot\omega}$ so that $\alpha^\omega = (\omega^\gamma)^\omega$. Consequently, $\omega^\gamma \leqslant \alpha \leqslant \beta < (\omega^\gamma)^\omega$. Hence, to show $C^\alpha \sim C^\beta$, it suffices by Corollary 1.3 and Theorem 1.8 to show $C^{\omega^\gamma} \sim C^{\omega^{\gamma\cdot\omega}}$. Since γ, being countable, either has an immediate predecessor or is co-final with ω, the existence of the desired isomorphism is an easy consequence of Lemma 2.2 together with Lemma 1.1 and Corollary 1.6:

$$C^{\omega^\gamma} \sim (C^{\omega^\gamma})^\omega \sim (C^{\omega^\gamma})^{\Delta(\omega)} \sim C^{\omega^{\gamma\cdot\omega}}.$$

The preceding result was used by Bessaga and Pełczyński to obtain an isomorphic classification of the spaces $C(S)$ for S a countable compact metric topological space. Since there appears to be a slight error [1] in this classification ([2], Theorem 2) we will give a corrected version. We first note that a countable compact metric space S is 0-dimensional ([4], page 286) and dispersed (as can be easily shown via an easy application of the Baire Category Theorem). Theorem 2.1 can then be used to conclude that S is homeomorphic to a space of ordinal numbers.

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(1) For a counter-example, consider the topological spaces $Q = \Gamma(\omega^\omega)$ and $Q_1 = (\omega^{\omega^2})$. Then $C(Q) \sim C(Q_1)$ according to Theorem 2 of [2]: however, $C(Q)$ cannot be isomorphic to $C(Q_1)$ by Theorem 2.3.

THEOREM 2.4. *Let S and T be countably infinite compact metric spaces with characteristics α and β respectively. Suppose $\alpha \leqslant \beta$. Then, $C(S) \sim C(T)$ if and only if $\beta < \alpha\cdot\omega$.*

Proof. Suppose S and T have characteristic systems (α, m) and (β, n) respectively. According to Theorem 2.1, S is homeomorphic to $\Gamma(\omega^\alpha\cdot m)$ and T is homeomorphic to $\Gamma(\omega^\beta\cdot n)$. Since $\omega^\alpha \leqslant \omega^\alpha\cdot m < (\omega^\alpha)^\omega$ and $\omega^\beta \leqslant \omega^\beta\cdot n < (\omega^\beta)^\omega$, it follows from Theorem 2.3 that $C(S) \sim C(T)$ if and only if $C^{\omega^\alpha} \sim C^{\omega^\beta}$. Using Theorem 2.3 again, this is equivalent to $\omega^\beta < (\omega^\alpha)^\omega = \omega^{\alpha\cdot\omega}$ which in turn is equivalent to $\beta < \alpha\cdot\omega$.

LEMMA 2.5. *Suppose α is an uncountable ordinal number such that either* (1) *α has an immediate predecessor or* (2) *α is co-final with ω. If $\omega^\alpha \leqslant \beta < \omega^\alpha\cdot\Omega$, then $X^{\omega^\alpha} \sim X^\beta$.*

Proof. Since $\lim_{\lambda<\Omega} \omega^\alpha\cdot\lambda = \omega^\alpha\cdot\Omega$, it suffices to show $X^{\omega^\alpha} \sim X^{\omega^\alpha\cdot\lambda}$ for $\omega \leqslant \lambda < \Omega$. by Lemma 2.2 and Corollary 1.3. So, suppose $\omega \leqslant \lambda < \Omega$. The existence of the desired isomorphism follows from Corollary 1.6, Lemma 1.1, and Lemma 2.2 which yield the following string of isomorphisms:

$$X^{\omega^\alpha\cdot\lambda} \sim (X^{\omega^\alpha})^{\Delta(\lambda)} \approx (X^{\omega^\alpha})^{\Delta(\omega)} \sim (X^{\omega^\alpha})^\omega \sim X^{\omega^\alpha}.$$

We will need the following result concerning an isomorphism invariant obtained by Semadeni in [9] for certain spaces of continuous functions.

THEOREM 2.6. *Suppose $S = \Gamma(\Omega\cdot\alpha)$ and $T = \Gamma(\Omega\cdot\beta)$. If $C(S) \sim C(T)$, then the cardinality of α is equal to the cardinality of β.*

The following result proved in [9] is an easy consequence of Theorem 2.1 and Theorem 2.6.

COROLLARY 2.7 (SEMADENI). *Suppose $1 \leqslant n < \omega$. Then, $C^\alpha \sim C^{\Omega\cdot n}$ if and only if $n \leqslant \alpha < \Omega\cdot(n+1)$.*

COROLLARY 2.8. *$C^\alpha \sim C^{\Omega\cdot\omega}$ if and only if $\Omega\cdot\omega \leqslant \alpha < \Omega^2$.*

Proof. Since $\omega^\Omega = \Omega$, it follows that $\Omega\cdot\omega = \omega^\Omega\cdot\omega = \omega^{\Omega+1}$. Thus, $C^\alpha \sim C^{\Omega\cdot\omega}$ whenever $\Omega\cdot\omega \leqslant \alpha < (\Omega\cdot\omega)\cdot\Omega = \Omega^2$ by Lemma 2.5.

For the converse, it suffices to show that C^{Ω^2} is not isomorphic to $C^{\Omega\cdot\omega}$, by Corollaries 1.3 and 2.7. By Theorem 2.6, $C^{\Omega^2} = C^{\Omega\cdot\Omega}$ cannot be isomorphic to $C^{\Omega\cdot\omega}$.

COROLLARY 2.9. *$C^{\Omega^2} \sim C^\alpha$ if and only if $\Omega^2 \leqslant \alpha < \Omega^\omega$.*

Proof. This is an immediate consequence of the preceding Corollaries and Theorem 1.10.

We do not have a complete isomorphic classification of any of the Banach spaces C^α for $\alpha \geqslant \Omega^\omega$ even though we have obtained several partial results. The first stumbling block appears to be finding the answer to the following question:

Is C^{Ω^ω} isomorphic to $C^{\Omega^{\omega+1}}$?

It will follow easily, from the proposition to be proved below, that one of the following statements must be true: Either (1) $C^{\Omega^\omega} \sim C^\alpha$ if and only if $\Omega^\omega \leqslant \alpha < \Omega^{\omega+1}$ or (2) $C^{\Omega^\omega} \sim C^\alpha$ if and only if $\Omega^\omega \leqslant \alpha < \Omega^2$.

Hence, an answer to the question raised above would yield the complete isomorphic classification of the space C^{Ω^ω}. Indeed, it seems almost certain that the ostensibly new techniques needed to settle this question would lead to the complete isomorphic classification of the spaces C^α for $\Omega^\omega \leqslant \alpha < \Omega^\Omega$. Evidence for this assertion is provided by the following somewhat complicated result.

PROPOSITION 2.10. *Suppose α has the same cardinality as Ω, and $\beta = \omega^{\omega^\alpha}$. Also, suppose α has an immediate predecessor or α is co-final with ω. Then, either the first statement below is true or the last two statements are both true:*

(1) $C^\beta \sim C^\gamma$ *if and only if* $\beta \leqslant \gamma < \beta^\omega$,
(2) $C^\beta \sim C^\gamma$ *if and only if* $\beta \leqslant \gamma < \beta \cdot \Omega$,
(3) $C^{\beta \cdot \Omega} \sim C^\gamma$ *if and only if* $\beta \cdot \Omega \leqslant \gamma < \beta^\omega$.

Proof. Since $\beta = \omega^{\omega^\alpha}$, it is not difficult using transfinite induction to show that $\lambda^\omega \leqslant \beta$ whenever $\lambda < \beta$ (cf. [5], page 33). Thus if $C^\beta \sim C^\gamma$, it follows that $\beta \leqslant \gamma$ from Theorem 1.9. To finish the proof, we consider two cases depending on whether or not C^β is isomorphic to $C^{\beta\cdot\Omega}$. If $C^\beta \sim C^{\beta\cdot\Omega}$, then (1) is true by Lemma 2.5, Theorem 1.10, and the comments at the beginning of this proof.

If C^β is not isomorphic to $C^{\beta\cdot\Omega}$, then (2) holds as a consequence of Corollary 1.3. Lemma 2.5, and the first part of this proof; statement (3) is true by Corollary 1.3 and Theorem 1.10.

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