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# Concrete subspaces of nuclear Fréchet spaces

by

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**Abstract.** Subspaces of arbitrary nuclear Fréchet spaces are studied. It is shown that every nuclear Fréchet space (other than $\omega$) contains a subspace which is isomorphic to a space of type $D_1$ and also a subspace which contains no subspaces isomorphic to power series spaces. It is also shown that any two nuclear Fréchet spaces (other than $\omega$) have a subspace in common. From this it follows that for any class of nuclear Fréchet spaces, either every nuclear Fréchet space (other than $\omega$) contains a subspace from the class or every nuclear Fréchet space has a subspace which contains no member of the class. For the class of all subspaces of a nuclear Fréchet space (other than $\omega$) the first alternative holds whereas the second alternative holds for any countable class.

The methods involve the construction of certain block basic sequences and also an application of the theory of $L_f(b, r)$ spaces.

In the study of nuclear Fréchet spaces, certain concrete examples appear to play a fundamental role. From the point of view of function spaces, analytic functions, differentiable functions, distributions and partial differential equations are standard objects in analysis that have generated interest in nuclear spaces. If we consider spaces with bases, then very deep results have been obtained by M. M. Dragilev, B. S. Mitiagin and others by restricting attention to certain classes of spaces such as power series spaces and spaces of type $D_1$ or $D_2$ (see, for example, [3], [4] and [7]).

It is natural to ask questions like: "What concrete objects does a general nuclear Fréchet space contain?" In the case of Banach spaces (still a rich source of questions, if not answers, for the study of nuclear Fréchet spaces) the question takes the form of the general conjecture (1) that every infinite dimensional Banach space contains a subspace isomorphic to $c_0$ or some $l_p$. (See [6] for some recent progress and bibliography on this problem.)

In this paper we investigate the possibility of finding a power series

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(1) The conjecture has recently been proved in the negative by Tzirel'son, Functional Analysis and its Applications 8.2 (1974), pp. 57-60 (Russian).

space, a space of type $D_1$ or a space of type $D_2$ as a subspace of an arbitrary nuclear Fréchet space. Contrary to the situation with Banach spaces, the solutions to such problems, while not completely trivial are possible.

To describe our results in intuitive terms we can say that $D_1$-spaces are rather common whereas $D_2$-spaces and power series spaces are rare, as subspaces of arbitrary nuclear Fréchet spaces. Moreover, it is found that any two nuclear Fréchet spaces (other than $\omega$) have a subspace in common. This result has several interesting consequences.

To present our results in detail it is necessary to recall some definitions. By a *nuclear Fréchet space* we shall mean inter alia an infinite dimensional locally convex topological (real or complex) vector space. The simplest example of such a space is the countable product of one dimensional spaces with the product topology. We shall call it $\omega$ and exclude it from most of our considerations. By a *subspace* we shall always mean an infinite dimensional closed subspace. When we say that $E$ *contains* $F$ we shall mean that $F$ is isomorphic to a subspace of $E$.

A sequence $(x_n)$ in a nuclear Fréchet space $E$ is a *basis* if each $x \in E$ has a representation $x = \sum t_n x_n$ where the scalars $t_n$ are uniquely determined by $x$. A sequence $(x_n)$ is a *basic sequence* if it is a basis for the subspace it generates. Corresponding to a basis $(x_n)$ in $E$ we have *representing matrices* $(a_n^k)$ obtained by choosing a fundamental sequence of seminorms $(\|\cdot\|_k)$ for $E$ and writing $a_n^k = \|x_n\|_k$. Because of the nuclearity, we can always choose the seminorms so that the matrix satisfies

$$0 \leqslant a_n^k \leqslant a_n^{k+1} \quad \text{for all } n, k \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n^k}{a_n^{k+1}} < \infty \quad \text{for all } k.$$

(The ratio in the sum is taken to be 0 if $a_n^{k+1} = 0$.) Conversely if we have a matrix $a = (a_n^k)$ satisfying these conditions it determines a nuclear Fréchet space $K(a) = \{\xi = (\xi_n)\colon p_k(\xi) = \|(\xi_n a_n^k)\|_p < \infty\}$ with topology determined by the seminorms $p_k$. Here $\|\cdot\|_p$ is the $l_p$-norm of the sequence and $p$ is any number with $1 \leqslant p \leqslant \infty$. It is not hard to see that $K(a)$ is isomorphic to $E$. The following condition is obviously sufficient for $K(a)$ to be isomorphic (indeed equal) to $K(b)$.

$$\forall k \exists j, l \ni a_n^k \leqslant b_n^j \leqslant a_n^l \quad \text{for all } n.$$

A representing matrix is *regular* if

$$\frac{a_n^k}{a_n^{k+1}} > \frac{a_{n+1}^k}{a_{n+1}^{k+1}} \quad \text{for all } n, k.$$

(We have used the strict inequality here for technical convenience. It is not hard to see that this is equivalent to the more standard definition [4].) A basis is *regular* if it has a regular representing matrix.



A nuclear Fréchet space with a basis is of *type* $D_1$ if the basis has a regular representing matrix $(a_n^k)$ which satisfies:

$$a_n^1 = 1 \text{ for all } n \text{ and } \forall k \exists j \ni (a_n^k)^2 \leqslant a_n^j \text{ for all } n.$$

It is of *type* $D_2$ if it has a regular representing matrix $(a_n^k)$ which satisfies:

$$\lim_k a_n^k = 1 \text{ for all } n \text{ and } \forall k \exists j \ni a_n^k \leqslant (a_n^j)^2 \text{ for all } n.$$

It is well known [4] that these two types are independent of the choice of basis and are mutually exclusive.

A nuclear Fréchet space with a basis is a *power series space of infinite (finite) type* if the basis has a representing matrix $a_n^k = k^{a_n}$ $\left(\left(\frac{k}{k+1}\right)^{a_n}\right)$ where $(a_n)$ is a nondecreasing sequence of nonnegative numbers satisfying

$$\sup_n \frac{\log n}{a_n} < \infty \quad \left(\lim_n \frac{\log n}{a_n} = 0\right).$$

Again the definitions are independent of the choice of basis. A power series space of infinite (finite) type is obviously a space of type $D_1$ ($D_2$). We shall write $\Lambda_\infty(a)$ ($\Lambda_1(a)$) to indicate a power series space of infinite (finite) type.

The symbol $\boldsymbol{N}$ will denote the sequence of positive integers. If $(s_j)$ is a sequence of numbers then the symbol

$$\prod_{j=1}^{k-1} s_j = s_1 s_2 \ldots s_{k-1},$$

is clear if $k \geqslant 1$. If $k = 1$ we interpret this symbol to be 1.

LEMMA. *Every nuclear Fréchet space which is not isomorphic to $\omega$ contains a subspace with a regular basis.*

Proof. Using results of C. Bessaga and A. Pełczyński ([1], [2]) we can find a subspace $E$ which has a basis $(x_n)$ and admits a continuous norm. Thus, by the nuclearity, we can find a sequence of norms $(\|\cdot\|_k)$ defining the topology of $E$ and satisfying

$$\lim_n \frac{\|x_n\|_k}{\|x_n\|_{k+1}} = 0, \quad k = 1, 2, \ldots$$

Using a diagonal procedure we can find a subsequence $(n_j)$ of indices such that

$$\frac{\|x_{n_j}\|_k}{\|x_{n_j}\|_{k+1}} > \frac{\|x_{n_{j+1}}\|_k}{\|x_{n_{j+1}}\|_{k+1}} \quad \text{for} \quad j \geqslant k, \ k = 1, 2, \ldots$$

If $E_0$ is the subspace generated by $(x_{n_j})$ and we define the norms $\|\cdot\|_k^0$ in $E_0$ inductively on $k$ by setting $\|x\|_k^0 = \|x_k\|$ for $k = 1, 2$ and for $k = 3, 4, \ldots$

$$\|x\|_k^0 = \frac{\|x_{n_{k-1}}\|_k}{\|x_{n_{k-1}}\|_{k-1}} \sum_{j=1}^{k-2} |\xi_j| \frac{\|x_{n_j}\|_{k-1}^0}{k-j} + \sum_{j=k-1}^{\infty} |\xi_j| \, \|x_{n_j}\|_{k-1}$$

$$\text{for} \quad x = \sum_{j=1}^{\infty} \xi_j x_{n_j},$$

then the sequence $(\|\cdot\|_k^0)$ also defines the topology of $E_0$ and satisfies the condition of regularity for $(x_{n_j})$.

Remark. The argument in the above lemma can be extended to show that any basis in a nuclear Fréchet space with a continuous norm can be decomposed into an at most countable set of disjoint subsequences and a fundamental sequence of norms for the whole space can be found which satisfies the condition of regularity for each subsequence (cf. [5]).

THEOREM 1. *Every nuclear Fréchet space which is not isomorphic to $\omega$ contains a subspace of type $D_1$.*

Proof. According to the lemma we may assume that our space has a regular basis $(\tilde{x}_n)$ and is represented by a matrix $(c_n^k)$ satisfying:

(1) $0 < c_n^k \leqslant c_n^{k+1}$ for all $k$, $n$ and $\left(\dfrac{c_n^k}{c_n^{k+1}}\right)$ converges to 0 strictly monotonically in $n$ for each $k$.

Let $(p_n)$ be any strictly increasing sequence of indices with $p_0 = 0$ and $\lim(p_n - p_{n-1}) = \infty$. Given a strictly increasing sequence of indices $(\nu_n)$, we will set $b_n^k = c_{\nu_n}^k$. We will show inductively that $(\nu_n)$ can be chosen so that

$$\frac{b_{p_{n-1}+k}^k}{b_{p_{n-1}+k}^{k+1}} \leqslant \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^j}{b_{p_{n-1}+j}^{j+1}}, \qquad k = 1, \ldots, p_n - p_{n-1};\ n = 1, 2, \ldots \tag{2}$$

If $n = 1$ we set $\nu_1 = 1$ and define $\nu_2, \ldots, \nu_{p_1}$ according to the following scheme given for $n > 1$. If $n > 1$ we assume that $\nu_1, \ldots, \nu_{p_{n-1}}$ have been defined so that (2) holds. We set $\nu_{p_{n-1}+1} = \nu_{p_{n-1}} + 1$ and observe that (2) follows from our convention and the fact that $c_n^k \leqslant c_n^{k+1}$. Once $\nu_{p_{n-1}+1}, \ldots, \nu_{p_{n-1}+k-1}$, $1 < k \leqslant p_n - p_{n-1}$ have been successfully defined we apply (1) to obtain $\nu_{p_{n-1}+k}$ as the smallest integer $i > \nu_{p_{n-1}+k-1}$ such that

$$\frac{c_i^k}{c_i^{k+1}} \leqslant \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^j}{b_{p_{n-1}+j}^{j+1}}.$$

This completes the construction of $(\nu_n)$.



We set $x_n = \tilde{x}_{\nu_n}$ and observe that the matrix $(b_n^k)$ represents $(x_n)$ and it is regular. Set

$$\xi_{p_{n-1}+k} = \frac{1}{b_{p_{n-1}+k}^k} \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}}$$

$$\text{for } k = 1, 2, \ldots, p_n - p_{n-1};\ n = 1, 2, \ldots$$

and define the block basic sequence $(y_n)$ by

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} \xi_i x_i, \qquad n = 1, 2, \ldots$$

We will complete the proof by showing that the space $Y$ generated by $(y_n)$ is of type $D_1$. The basis $(y_n)$ is represented by the matrix $(a_n^k)$ where

$$a_n^k = \xi_{q_n^k} b_{q_n^k}^k, \qquad q_n^k = \max\{q:\ \xi_q b_q^k = \max_{p_{n-1} < i \leqslant p_n} \xi_i b_i^k\}, \qquad n, k = 1, 2, \ldots$$

The main step in the proof is to show that, for all $n$,

$$q_n^k = p_{n-1} + k \quad \text{for } k = 1, 2, \ldots, p_n - p_{n-1}. \tag{3}$$

Fix $n$ and suppose that $1 \leqslant k \leqslant p_n - p_{n-1}$. Clearly $p_{n-1} < q_n^k \leqslant p_n$ so we may consider any $l \neq k$ with $1 \leqslant l \leqslant p_n - p_{n-1}$.

If $1 \leqslant l < k$, then by the regularity of $(b_n^k)$ we obtain

$$\begin{aligned}
\xi_{p_{n-1}+l} b_{p_{n-1}+l}^k &= \frac{b_{p_{n-1}+l}^k}{b_{p_{n-1}+l}^l} \prod_{j=1}^{l-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \\
&= \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k \frac{b_{p_{n-1}+l}^k}{b_{p_{n-1}+l}^l} \prod_{j=l}^{k-1} \frac{b_{p_{n-1}+j}^{j}}{b_{p_{n-1}+j}^{j+1}} \\
&= \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k \prod_{j=l}^{k-1} \frac{b_{p_{n-1}+l}^{j+1}}{b_{p_{n-1}+l}^{j}} \prod_{j=l}^{k-1} \frac{b_{p_{n-1}+j}^{j}}{b_{p_{n-1}+j}^{j+1}} \\
&\leqslant \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k \prod_{j=l}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \prod_{j=l}^{k-1} \frac{b_{p_{n-1}+j}^{j}}{b_{p_{n-1}+j}^{j+1}} \\
&= \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k,
\end{aligned}$$

which shows that $q_n^k \geqslant p_{n-1} + k$.

If $k < l \leqslant p_n - p_{n-1}$ we obtain in a similar manner,

$$\xi_{p_{n-1}+l} b_{p_{n-1}+l}^k = \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k \prod_{j=k}^{l-1} \frac{b_{p_{n-1}+l}^{j}}{b_{p_{n-1}+l}^{j+1}} \prod_{j=k}^{l-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} < \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k,$$

which shows that $q_n^k \leqslant p_{n-1} + k$ so (3) is established.

Thus we have, for each $n$,

$$(4)\qquad a_n^k = \xi_{p_{n-1}+k} b_{p_{n-1}+k}^k = \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \quad \text{for } k = 1, 2, \ldots, p_n - p_{n-1},$$

and therefore

$$\frac{a_n^k}{a_n^{k+1}} = \frac{b_{p_{n-1}+k}^k}{b_{p_{n-1}+k}^{k+1}} \quad \text{for all } k = 1, 2, \ldots, p_n - p_{n-1};\ n = 1, 2, \ldots$$

Since $(b_n^k)$ is regular and $(p_n)$ is increasing, it follows that given $k$, $\left(\frac{a_n^k}{a_n^{k+1}}\right)$ is decreasing in $n$ for $p_n - p_{n-1} \geqslant k$, that is, since $\lim (p_n - p_{n-1}) = \infty$, for $n$ sufficiently large. Hence we adjust the norms as we did in the lemma so that the matrix becomes regular and for each $k$, only finitely many of the $a_n^k$ are changed. For $k = 1$, no $a_n^1$ need be changed.

Finally we observe that by our convention, $a_n^1 = 1$ and given $k$ it follows that for $n$ sufficiently large,

$$\frac{(a_n^k)^2}{a_n^{k+1}} = \frac{\prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}}}{\prod_{j=1}^{k} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}}} \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} = \frac{b_{p_{n-1}+k}^k}{b_{p_{n-1}+k}^{k+1}} \prod_{j=1}^{k-1} \frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \leqslant 1$$

because of (3). This completes the proof.

Remark. V. P. Zaharjuta has shown [8] that every continuous linear map from a $D_2$ space to a $D_1$ space is compact. Since embeddings of nuclear Fréchet spaces cannot be compact, it follows from this and Theorem 1 that *every nuclear Fréchet space contains a subspace which contains no subspace of type $D_2$* (we do not have to exclude $\omega$ since all its subspaces are again $\omega$ and hence not $D_2$).

Thus we see an important difference between spaces of type $D_1$ and $D_2$ in that the former appear in every subspace of every nuclear Fréchet space and the latter are excluded from at least one subspace of every space. It seems reasonable to consider which of these two forms of behavior are exhibited by other classes of nuclear Fréchet spaces. The next result answers the question for power series spaces.

THEOREM 2. *Every nuclear Fréchet space contains a subspace which contains no power series spaces.*

Proof. We begin with a nuclear Fréchet space $E$. If $E$ is isomorphic to $\omega$ we are finished. If not, we can apply Theorem 1 to obtain a subspace $E_1$ of type $D_1$. By Zaharjuta's theorem, $E_1$ contains no power series space of finite type. If $E_1$ contains no power series space of infinite type, we are finished. If it does, then it in fact contains a power series space of infinite type, $\Lambda_\infty(\alpha)$ with $\alpha_n < \alpha_{n+1}$ and



$$\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty.$$

It suffices then to construct in $\Lambda_\infty(\alpha)$ a subspace $Y$ which contains no infinite type power series space.

We make exactly the same construction as in the proof of Theorem 1, taking $\nu_n = n$. We do not get (2) but we do get (3) and hence (4). Thus we obtain $Y$ as the subspace generated by the basic sequence $(y_n)$ in $\Lambda_\infty(\alpha)$, and $(y_n)$ is represented by $(a_n^k)$. Since $b_n^k = c_{\nu_n}^k = c_n^k = k^{\alpha_n}$, equation (4) has the simpler form,

$$a_n^k = \prod_{j=1}^{k-1} \left(\frac{j+1}{j}\right)^{\alpha_{p_{n-1}+j}}, \qquad \frac{a_n^k}{a_n^{k+1}} = \left(\frac{k}{k+1}\right)^{\alpha_{p_{n-1}+k}}$$

$$\text{for } k = 1, \ldots, p_n - p_{n-1};\ n = 1, 2, \ldots$$

At this point it is possible to prove directly that $Y$ contains no subspace isomorphic to a power series space of infinite type, but the details are rather cumbersome. V. P. Zaharjuta has suggested the following simple argument based on the theory of $L_f(b, r)$ spaces, which we include with his permission.

We define

$$\tilde{a} = (\tilde{a}_n^k) \quad \text{where } \tilde{a}_n^k = e^{\alpha_{p_{n-1}+k}}, \quad n, k = 1, 2, \ldots$$

Since $\lim\limits_n \frac{\alpha_{n+1}}{\alpha_n} = \infty$, it follows that for each $k$,

$$\lim_{n\to\infty} \frac{a_n^k}{\tilde{a}_n^k} = \lim_{n\to\infty} \frac{\tilde{a}_n^k}{a_n^{k+2}} = 0.$$

Indeed,

$$\frac{a_n^k}{\tilde{a}_n^k} = \frac{\prod_{j=1}^{k-1} \left(\frac{j+1}{j}\right)^{\alpha_{p_{n-1}+j}}}{e^{\alpha_{p_{n-1}+k}}} \leqslant \frac{\left(\prod_{j=1}^{k-1} \frac{j+1}{j}\right)^{\alpha_{p_{n-1}+k-1}}}{e^{\alpha_{p_{n-1}+k}}} = \frac{k^{\alpha_{p_{n-1}+k-1}}}{e^{\alpha_{p_{n-1}+k}}},$$

which goes to 0 as $n$ goes to infinity; and

$$\frac{\tilde{a}_n^k}{a_n^{k+2}} = \frac{e^{\alpha_{p_{n-1}+k}}}{\prod_{j=1}^{k+1} \left(\frac{j+1}{j}\right)^{\alpha_{p_{n-1}+j}}} \leqslant \frac{e^{\alpha_{p_{n-1}+k}}}{\left(\frac{k+2}{k}\right)^{\alpha_{p_{n-1}+k+1}}}$$

which also goes to 0 as $n$ goes to infinity. Thus, we may conclude that $Y$ is isomorphic to $K(\tilde{a})$.

Next we define $f\colon (0,\infty)\to(0,\infty)$ by

$$f(x) = \alpha_n\left(\frac{x}{e^n}\right)^{\log\frac{\alpha_{n+1}}{\alpha_n}} \quad \text{for} \quad n\leqslant \log x\leqslant n+1 \quad (\alpha_0 = 1),\ n = 1,2,\ldots$$

and extend it to $(-\infty,\infty)$ by setting $f(-x) = -f(x)$. Now we can easily arrange our original choice of $(\alpha_n)$ so that $\left(\frac{\alpha_{n+1}}{\alpha_n}\right)$ is an increasing function. It then follows that $f$ is an increasing odd function which is logarithmically convex on $[0,\infty)$.

Next we check that $f$ is rapidly increasing ([4], p. 77) for which it suffices to show that for some $\tau > 0$ we have

$$\overline{\lim_{x\to\infty}}\frac{f(\tau x)}{f(x)} = \infty.$$

Choosing $\tau = e$ and letting $x = e^n$, we obtain

$$\lim_{n\to\infty}\frac{f(e^{n+1})}{f(e^n)} = \frac{\alpha_{n+1}}{\alpha_n} = \infty.$$

Now if we set $b = (b_n)$ where $b_n = e^{p_{n-1}}$ then, by definition, $L_f(b,\infty) = K(\bar a)$ where $\bar a_n^k = e^{f(c_n^k b_n)}$ so

$$\bar a_n^k = e^{f(e^{p_{n-1}+k})} = e^{\alpha_{p_{n-1}+k}} = \tilde a_n^k.$$

Thus, $Y$ is isomorphic to $L_f(b,\infty)$.

On the other hand, if we take $g$ to be the identity function, it is well known that any infinite type power series space $\Lambda_\infty(\beta)$ is isomorphic to $L_g(\beta,\infty)$. Since $f = g^{-1}f$ is rapidly increasing, it follows ([8], Theorem 3) that every continuous linear operator from $L_g(\beta,\infty)$ to $L_f(b,\infty)$ is compact. In particular, $\Lambda_\infty(\beta)$ cannot be isomorphic to a subspace of $Y$.

We turn now to the question of how different the sets of subspaces of two nuclear Fréchet spaces can be. The next result, using methods similar to what has preceeded, show that they must have something in common.

THEOREM 3. *If $E$ and $F$ are nuclear Fréchet spaces not isomorphic to $\omega$, then they contain a common subspace.*

Proof. In view of the lemma we can assume that $E$ and $F$ have regular bases represented by matrices $(c_n^k)$ and $(\bar c_n^k)$ respectively which satisfy

$$(5)\qquad \begin{aligned} &0 < c_n^k \leqslant c_n^{k+1}, \quad \lim_n \frac{c_n^k}{c_n^{k+1}} = 0 \text{ monotonically},\\ &0 < \bar c_n^k \leqslant \bar c_n^{k+1}, \quad \lim_n \frac{\bar c_n^k}{\bar c_n^{k+1}} = 0 \text{ monotonically}.\end{aligned}$$



Let $(p_n)$ be any strictly increasing sequence of indices with $p_0 = 0$ and $\lim(p_n - p_{n-1}) = \infty$. If $(\nu_n)$ and $(\bar\nu_n)$ are strictly increasing sequences of indices we set

$$b_n^k = c_{\nu_n}^k, \quad \bar b_n^k = \bar c_{\bar\nu_n}^k, \quad n,k = 1,2,\ldots$$

We will now choose $(\nu_n)$ and $(\bar\nu_n)$ so that for each $n$,

$$(6)\qquad \prod_{j=1}^{k-1}\frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \leqslant \prod_{j=1}^{k-1}\frac{\bar b_{p_{n-1}+j}^{j+1}}{\bar b_{p_{n-1}+j}^{j}} \leqslant \prod_{j=1}^{k}\frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}},$$
$$k = 1,\ldots,p_n - p_{n-1}.$$

For each $n$, we will construct $\nu_{p_{n-1}+1},\ldots,\nu_{p_n}$ and $\bar\nu_{p_{n-1}+1},\ldots,\bar\nu_{p_{n-1}}$. We begin by setting $\nu_1 = 1$ and $\nu_{p_{n-1}+1} = \nu_{p_{n-1}}+1$ for $n > 1$. Notice that for any $n$, if $k = 1$ then (6) holds because of (5) and our convention. Now suppose that $\nu_{p_{n-1}+1},\ldots,\nu_{p_{n-1}+k-1}$ and $\bar\nu_{p_{n-1}+1},\ldots,\bar\nu_{p_{n-1}+k-2}$ have been chosen so that (6) holds with $k$ replaced by $k-1 < p_n - p_{n-1}$. We shall choose $\nu_{p_{n-1}+k}$ and $\bar\nu_{p_{n-1}+k-1}$ so that (6) holds as written. The desired inequalities can be written as:

$$\prod_{j=1}^{k-1}\frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}} \leqslant \frac{\bar c_{\bar\nu_{p_{n-1}+k-1}}^{k}}{\bar c_{\bar\nu_{p_{n-1}+k-1}}^{k-1}}\prod_{j=1}^{k-2}\frac{\bar b_{p_{n-1}+j}^{j+1}}{\bar b_{p_{n-1}+j}^{j}} \leqslant \frac{c_{\nu_{p_{n-1}+k}}^{k+1}}{c_{\nu_{p_{n-1}+k}}^{k}}\prod_{j=1}^{k-1}\frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}}.$$

In the above statement, all quantities involving $b$ and $\bar b$ have already been chosen and for each quantity involving $c$ or $\bar c$ the subscript is to be selected. Obviously we can apply (5) to select, first $\bar\nu_{p_{n-1}+k-1}$ so that the first inequality holds and then $\nu_{p_{n-1}+k}$ so that the second holds. This can also be done so that $(\bar\nu_n)$ and $(\nu_n)$ are both strictly increasing.

Now we can proceed exactly as in Theorem 1 to construct subspaces of $E$ and $F$ which have bases represented by the matrices $(a_n^k)$ and $(\bar a_n^k)$ respectively where for each $n$ and $k = 1,\ldots,p_n - p_{n-1}$ we have,

$$a_n^k = \prod_{j=1}^{k-1}\frac{b_{p_{n-1}+j}^{j+1}}{b_{p_{n-1}+j}^{j}}, \qquad \bar a_n^k = \prod_{j=1}^{k-1}\frac{\bar b_{p_{n-1}+j}^{j+1}}{\bar b_{p_{n-1}+j}^{j}}.$$

Since $\lim(p_n - p_{n-1}) = \infty$, it follows then from (6) that for each $k$,

$$a_n^k \leqslant \bar a_n^k \leqslant a_n^{k+1}$$

for $n$ sufficiently large. This implies that the two matrices determine isomorphic nuclear Fréchet spaces and the proof is completed.

Remark. In considering the classes of nuclear Fréchet spaces $D_1$, $D_2$ and power series spaces we have seen that in each case *either every nuclear Fréchet space other than $\omega$ contains a member of the class ($D_1$) or every nuclear Fréchet space contains a subspace which contains no member of the class ($D_2$ and power series spaces)*. It is an immediate consequence of Theorem 3 that the dichotomy is always valid. It may be interesting to consider other classes of nuclear Fréchet spaces and determine which of the two alternatives hold. For example, it is again immediate from Theorem 3 that *the first alternative holds for the class of all subspaces of a single space* (other than $\omega$). On the other hand, our final result shows that if a class is too small it will satisfy the second alternative.

PROPOSITION. *If $\mathscr{E}$ is an at most countable collection of nuclear Fréchet spaces not including $\omega$, then every nuclear Fréchet space contains a subspace which contains no member of $\mathscr{E}$.*

Proof. In view of the above remark, it suffices to construct a nuclear Fréchet space which contains no member of $\mathscr{E}$. If $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$, then we can apply Theorem 1 to obtain, for each $i$, a $D_1$ space $F_i$ contained in $E_i$. For each $i$ the space $F_i$ has a basis $(x_n(i))_n$ which is represented by the matrix $(a_n^k(i))$. Next we choose an increasing sequence of positive numbers $(t_n)$ such that

$$\lim_n \frac{t_n}{a_n^k(i)} = \infty \quad \text{for each } k, i \text{ and } \sum_n \frac{1}{t_n} < \infty.$$

Let $a_n^k = (t_n)^{k-1}$, $k, n = 1, 2, \ldots$, and let $E$ be the Köthe sequence space determined by the matrix $(a_n^k)$. Then $E$ is a nuclear Fréchet space of type $D_1$. Moreover, if $K(F_i)$ is the sequence space determined by the basis $(x_n(i))_n$, that is, $K(F_i) = \{\xi = (\xi_n): \sum_n \xi_n x_n(i)$ converges in $F_i\}$, then it follows from the Dragilev theory (see, e.g., [3] 1.10 or [4]) that $F_i$ is isomorphic to a subspace of $E$ if and only if $K(F_i) \subset E$ (as sets of sequences). If this were true then it would follow that for each $i$ and $k$ there exists $j$ and $M$ such that $a_n^k \leqslant M a_n^j(i)$ for all $n$. Choosing $k = 1$ gives a contradiction. Hence $E$ does not contain any $F_i$ so it cannot contain any $E_i$.

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