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A simple diophantine condition in harmonic analysis *

by

RON C. BLEI (Genova)**

Abstract. We prove that if $E \subset \Gamma$ satisfies a simple diophantine condition, then $L_E^\infty(G) = C_E(G)$. In certain settings, not resorting to Drury's theorem, we show the abundance of non-Sidon sets that satisfy our condition. The "Drury free" proof allows us to extend a previous result, that "every non-Sidon set contains a non-Sidon sup-norm partitioned set."

In what follows below, Γ is a discrete abelian group, and $\Gamma^\wedge = G$. Throughout our paper, we do not lose any generality by assuming that Γ is countable. As usual, Z will denote the additive group of integers, and T will denote the cricle group. We refer to [8] for standard notation and facts.

$E \subset \Gamma$ is a Sidon set if there exists $a > 0$ such that $\|p\|_\infty \geqslant a\sum|\hat{p}(\gamma)|$, for all trigonometric polynomials $p \in C_E(G)$, where $C_E(G) = \{f \in C(G)\colon \hat{f}(\gamma) = 0 \text{ for } \gamma \notin E\}$. The Sidon constant of E is the supremum of all such a's. It easily follows that if E is a Sidon set, then $L_E^\infty(G) = C_E(G)$. Non-Sidon sets $E \subset Z$ such that $L_E^\infty(T) = C_E(T)$ were constructed first by Rosenthal [7] — sets E such that $L_E^\infty(G) = C_E(G)$ will be called R-sets — and in [1] we proved that if $E \subset \Gamma$ is a non-Sidon set, then there exists a non-Sidon R-subset of E.

Definition. $E \subset \Gamma$ is said to be a *sup-norm partitioned set* if there exists a family of finite, mutually disjoint sets, $\{F_j\}$, such that $\bigcup_j F_j = E$ and

$$\oplus_{l^1} C_{F_j}(G) \approx C_E(G).$$

$\{F_j\}$ is said to be a *sup-norm partition* for E. In [7] and [1], the existence of non-Sidon R-sets followed as a corollary to the existence of non-Sidon sup-norm partitioned sets. In particular,

Theorem A ([1]). *Every non-Sidon subset of Γ contains a non-Sidon sup-norm partitioned set.*

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Let D be a dense countable subgroup of G, and let $\varphi_D\colon \Gamma \to \hat{D}$ be the natural injective map: $(\varphi_D(\gamma), d) = (\gamma, d)$. Of course, φ_D is simply the canonical map: $\Gamma \to \tilde{\Gamma}/D^{\perp}$, where $\tilde{\Gamma}$ is the Bohr compactification of Γ, and $D^{\perp}$ is the annihilator of D in $\tilde{\Gamma}$. We note that φ_D preserves Sidon constants and sup-norm partitions (see Lemma 2.2 of [1]), and recall that the sup-norm partitions of Theorem A were constructed as *subsets* of $E \subset \Gamma$, where E satified the following diophantine condition: There exists $D \subset G$ as above so that $\overline{\varphi_D(E)}$ is a countable set with one limit point in $\hat{D}$ (we shall henceforth ignore the case where E is finite). The abundance of non-Sidon sets that satisfied the above condition in [1] strongly depended, via Lemma 2.3 of [1], on "the finite union of Sidon sets is a Sidon set ([2])". At the outset, when $G = T$, $\otimes Z_{p_j}$, or a countable product thereof, we prove a similar basic Lemma 1.1 below without resorting to Drury's theorem, and extend Theorem A (Theorem A′) in that setting. Then (for any G), without appealing to sup-norm partitions, we prove that if $E \subset \Gamma$ satisfies our condition, then E *itself* is an R-set (Theorem B). For example, if $(n_j) \subset Z$ is so that $n_j a \to x \in T \pmod{2\pi}$, where $a/2\pi$ is irrational, then (n_j) is an R-set. We do not know whether such sets in Z can be sup-norm partitioned. However, when $G = \otimes Z_p$, where p is any prime integer, and $E \subset \oplus Z_p$ satisfies our diophantine condition with respect to some $D \subset \otimes Z_p$, then E *itself* can be sup-norm partitioned (Theorem C). We conclude by listing some open problems.

1. Let (p_j) be any sequence of distinct prime integers tending to infinity; $\bigoplus_j Z_{p_j}$ denotes the algebraic sum of the Z_{p_j}'s, and $\bigotimes_j Z_{p_j}$ $(= (\bigoplus_j Z_{p_j})\hat{})$ denotes the topological product of the Z_{p_j}'s. In what follows below $G = T$, $\otimes Z_{p_j}$, or any countable product thereof.

LEMMA 1.1. *There exists $D \subset G$, a dense countable subgroup of G, with the property that if $E \subset \Gamma$ is a non-Sidon set, then there exists a non-Sidon set $F \subset E$ so that $\varphi_D(E)$ accumulates at exactly one point.*

Proof. We prove the lemma first in the case $G = T$. We inject $\oplus Z_{p_j}$ into T: $\oplus Z_{p_j} \ni \chi \to \sum_j \chi(j)/2\pi p_j$, where $\chi(j)$ is the jth coordinate of χ, and addition is performed $\bmod 2\pi$. We can then write without ambiguity $\oplus Z_{p_j} \subset T$. Clearly, $\oplus Z_{p_j}$ is dense in T, and the map

$$\varphi_{\oplus Z_{p_j}} = \varphi\colon Z \to \otimes Z_{p_j}$$

is realized as follows: For $n \in Z$, $\varphi(n)(j)$ is the remainder after dividing by p_j. Let E be a non-Sidon set in Z. First we observe that if $E \subset Z$ is a non-Sidon set, and N is a given positive integer, then there exists $0 \leqslant r < N$ so that $E \cap (NZ + r)$ is non-Sidon. This follows from the elementary fact that the characteristic function of $NZ + r$, for any N and r, is a Fourier-

Stieltjes transform. Let $0 \leqslant r_1 < p_1$ be the integer such that

$$E_1 = E \cap (p_1 Z + r_1)$$

is non-Sidon. We proceed to define r_n and E_n for $n > 1$: Let $0 \leqslant r_n < p_n$ be so that

$$E_n = E_{n-1} \cap (p_n Z + r_n)$$

is non-Sidon. Now, let $F_1 \subset E_1$ be an arbitrary finite set. For $n > 1$, let $F_n \subset E_n \setminus \bigcup_{j=1}^{n-1} F_j$ be a finite set so that the Sidon constant of $F_n < 1/n$. Clearly, $F = \bigcup_{n=1}^{\infty} F_n$ is a non-Sidon set, *and* $\varphi(F)$ accumulates at the point $\chi = (r_j)_{j=1}^{\infty} \in \otimes Z_{p_j}$ and only there.

In the case where $G = \otimes Z_{p_j}$, we set $D = \bigoplus_{j=1}^{\infty} Z_{p_j} \subset \bigotimes_{j=1}^{\infty} Z_{p_j}$, and repeat the above argument with $\bigoplus_{j=N}^{\infty} Z_{p_j}$ in place of $p_N Z$. ■

Aside of freeing Theorem A in the above setting of its dependency on Drury's theorem, the proof of Lemma 1.1 allows us to replace Sidonicity in our theorem by a large class of interpolation properties that exactly depend on equivalence of norms, e.g., norms that majorize the L^∞-norm or the L^1-norm. For example, we recall that $E \subset \Gamma$ is a $\Lambda(p)$ set for $1 < p < \infty$ if $L^1_E(G) = L^p_E(G)$ (for instance, see 5.7.7 of [8]), and we deduce

THEOREM A′. *Every non-$\Lambda(p)$ subset of Γ contains a non-$\Lambda(p)$ sup-norm partitioned set.*

Sketch of proof. In order to apply the arguments of [1], we need to replace in the statement of Lemma 1.1 "non-Sidon" by "non-$\Lambda(p)$". To do this, it suffices to prove the following claim: Let E be a non-$\Lambda(p)$ set, and K a finitely indexed subgroup of Γ; then, there exists a coset $r + K$ so that $r + K \cap E$ is not a $\Lambda(p)$ set. Let $\{r_i\}_{i=1}^{M}$ be the coset representatives in Γ/K. Let $h \in L^1_E$, and write $h = \sum_{i=1}^{M} \mu_{r_i} * h$, where $\hat{\mu}_{r_i} = \chi_{E \cap (r_i + K)}$. If $E \cap (r_i + K)$ is $\Lambda(p)$ for each r_i, then $\mu_{r_i} * h \in L^p$ for each r_i, and therefore $h \in L^p_E$, and claim follows. We now recall the construction of the non-Sidon (non-Helson) sup-norm partitioned set of Theorem 1.2 in [1]: Since we had a non-Sidon convergent sequence, at each step of our induction, we were at complete liberty to select an appropriate block with a Sidon constant as small as we liked. Now, as we have a non-$\Lambda(p)$ convergent sequence, we repeat the inductive procedure of [1], where at the jth step the chosen block will have $\Lambda(p)$ constant smaller than $1/j$. ■

Remarks. a. Theorems of the Theorem A type are deduced in our setting without any reference to Riesz products.

b. The above methods fail for compact, torsion free groups, e.g. $Z(p^\infty)\hat{}$. For example, as we do not know whether the finite union of

$\Lambda(2)$ sets is a $\Lambda(2)$ set, we do not know how to deduce a completely general Theorem A′ for $\Lambda(2)$ sets (see proof of 2.3 in [1]).

2. We now consider any compact abelian group, G.

THEOREM B. *Let $E \subset \Gamma$ be so that for some countable dense subgroup $D \subset G$, $\overline{\varphi_D(E)}$ is a countable set with one limit point. Then, $L_E^\infty(G) = C_E(G)$.*

LEMMA 2.1. *$L_E^\infty(G) = C_E(G)$ if and only if $L_E^\infty(G)$ is separable.*

Proof. Let $f \in L_E^\infty(G)$, and assume that $\{\psi_j\}$ is dense in $L_E^\infty(G)$. Let $\varepsilon > 0$, and set

$$E_j = \{g \in G\colon \ \|f_g - \psi_j\|_\infty \leqslant \varepsilon/2\}$$

(f_g denotes the translate of f by g).

For all j, E_j is measurable, and since $\bigcup_j E_j = G$, there exists j_0 so that $m(E_{j_0}) > 0$ (m = Haar measure on G), and therefore $E_{j_0} - E_{j_0} \supset \mathcal{O}$, an open neighborhood of 0 in G. It now easily follows from the translation invariance of $L_E^\infty(G)$ that for $g \in \mathcal{O}$, $\|f_g - f\|_\infty \leqslant \varepsilon$. Since a bounded and measurable function f is continuous (in $L^\infty(G)$ sense) if and only if $\|f_g - f\|_\infty \to 0$ as $g \to 0$ (cf. [3]), the lemma follows. ∎

Notation and remark. For $E \subset \Gamma$ as in statement of Theorem B, we set

$$A(E, \Gamma) = L^1(G)^\wedge / \{\hat{f} \in L^1(G)^\wedge : \hat{f} = 0 \text{ on } E\},$$

and

$$A(\overline{\varphi_D(E)}, \hat{D}) = l^1(D)^\wedge / \{f \in l^1(D)^\wedge : f = 0 \text{ on } \varphi_D(E)\}.$$

Fixing $D \subset G$, for the sake of simplicity, we refer to φ, $A(E)$, and $A(\overline{\varphi(E)})$. Letting $x_0 \in \hat{D}$ be the limit point of $\varphi(E)$, we set

$$A_0(\overline{\varphi(E)}) = \{f \in A(\overline{\varphi(E)}) : f(x_0) = 0\}.$$

We note that, since $\overline{\varphi(E)}$ is a countable set, the dual space of $A(\overline{\varphi(E)})$ is identified with $C_{\overline{\varphi(E)}}(\tilde{D})$ with the L^∞-norm, where $\tilde{D}$ is the Bohr compactification of D, i.e., with almost periodic functions on D with spectrum in $\overline{\varphi(E)}$ (see [5] or VI.5.22 of [4]). We also note that $A(E)^* = L_E^\infty(G)^\wedge$ (an easy application of Parseval's formula).

LEMMA 2.2. *$A(E)$ and $A_0(\overline{\varphi(E)})$ are isometric in the natural way:*

$$A(E) \ni \lambda \leftrightarrow \lambda \circ \varphi^{-1} \in A_0(\overline{\varphi(E)}).$$

Proof. Since finitely supported functions are norm dense in $A(E)$ and $A_0(\overline{\varphi(E)})$, it suffices to prove that

$$\|h\|_{A(E)} = \|h \circ \varphi^{-1}\|_{A(\overline{\varphi(E)})},$$

where h is a finitely supported function on E. But,

$$\|h\|_{A(E)} = \sup_{\psi \in C_E(G)} |(h, \psi)| / \|\psi\|_\infty,$$

and

$$\|h \circ \varphi^{-1}\|_{A(\overline{\varphi(E)})} = \sup_{\psi \in C_{\overline{\varphi(E)}}(\tilde{D})} |(h \circ \varphi^{-1}, \psi)| / \|\psi\|_\infty.$$

Therefore, it suffices to check that if $\{a_i\}_{i=1}^N$ is any finite set of complex numbers, and $\{\lambda_i\}_{i=1}^N$ is any finite subset of E, then

$$\sup_{g \in G} \Big|\sum_{i=1}^N a_i(\gamma_i, g)\Big| = \sup_{y \in \tilde{D}} \Big|\sum_{i=1}^N a_i(\varphi(\gamma_i), y)\Big|.$$

The above equality follows from the definition of φ and the density of D in both $\tilde{D}$ and G.

The proof of Theorem B is now a simple application of the preceding two lemmas: By Lemma 2.2 $L_E^\infty(G)$ is isomorphic to a quotient of $C_{\overline{\varphi(E)}}(\tilde{D})$, and hence separable; by Lemma 2.1, $L_E^\infty(G) = C_E(G)$. ∎

Remarks. a. The map φ, in fact, induces a ring isometry between $C(G)$ and $C_{\varphi(\Gamma)}(\tilde{D})$. By duality arguments, we conclude that $L^1(G)$ is isometric to $L^1_{\varphi(\Gamma)}(\tilde{D})$; therefore we achieve a natural isometry between $L^p(G)$ and $L^p_{\varphi(\Gamma)}(\tilde{D})$, for all $p \geqslant 1$.

b. Since the union of an R-set with a finite set is again an R-set, Theorem B remains valid if we require that $\overline{\varphi(E)}$ be countable with at most finitely many limit points. Furthermore, merely requiring that $\overline{\varphi(E)}$ be countable, we still obtain that E is an R-set if we insist that $\varphi(E)$ contain no limit points of $\varphi(E)$.

In general, confronting the task of sup-norm partitioning a convergent sequence, $\varphi(\gamma_j) \to x_0$ in $\hat{D}$, we note that the interaction between the "arithmetic" structure of the sequence and its "speed" of convergence plays a decisive role. The Cantor group's convenient algebraic structure, however, allows a natural approach to the problem in $\oplus Z_p$:

THEOREM C. *If $E \subset \oplus Z_p$ is such that for some $D \subset \otimes Z_p$, $\overline{\varphi_D(E)}$ is a countable set with one limit point, then E can be sup-norm partitioned (p is any positive prime integer).*

We first establish a general lemma:

LEMMA 2.3. *Let Γ be a discrete (not necessarily countable) abelian group. Let $\{F_j\}_{j=1}^\infty$ be a family of finite, mutually disjoint and mutually independent sets ($0 \notin F_j$), i.e., whenever $\gamma_i \in F_{j_i}$, $i = 1, \ldots, N$, $F_{j_i} \neq F_{j_{i'}}$, if $i \neq i'$, then $\{\gamma_i\}_{i=1}^N$ is an independent set* (in the sense of 5.11 of [8]). *Then, $\{F_j\}$ is a sup-norm partition for $\bigcup_j F_j$.*

Proof. Without loss of generality we can assume that $\Gamma = gp(\bigcup F_j)$ (group generated by $\bigcup F_j$). By our independence condition we have

$$\Gamma = \bigoplus_j gp(F_j),$$

and therefore

$$\Gamma^{\wedge} = G = \bigotimes_j gp(F_j).$$

It easily follows that if $f_i \in C_{F_i}(G)$, $i = 1, \ldots, M$, and $g_1, \ldots, g_M \in G$, then there exists $g_0 \in G$ so that

$$\sum_{j=1}^{M} f_j(g_j) = \sum_{j=1}^{M} f_j(g_0).$$

Let $\{g_j\}_{j=1}^M \subset G$ be so that for each j, $\|f_j\|_\infty = |f_j(g_j)|$. Setting $f_j(g_j) = a_j + b_j i$, we may assume without loss of generality that

$$\sum_{j\in P} a_j \geqslant (1/4) \sum_{j=1}^{M} \|f_j\|_\infty,$$

where $P = \{j\colon a_j \geqslant 0\}$. For each j, $0 \notin F_j$, i.e. $\int_G f_j = 0$, and therefore $\mathrm{Re} f_j$ and $\mathrm{Im} f_j$ must both assume positive and negative values; in particular, for each $k \in \sim P$ find $g'_k \in G$ so that $\mathrm{Re} f_k(g'_k) > 0$. It then follows that

$$\Big|\sum_{j\in P} f_j(g_j) + \sum_{j\notin P} f_j(g'_j)\Big| \geqslant (1/4) \sum \|f_j\|_\infty,$$

and we deduce

$$\Big\|\sum_{j=1}^{M} f_j\Big\|_\infty \geqslant (1/4) \sum \|f_j\|_\infty. \blacksquare$$

Proof of Theorem C. We first observe that any countable dense subgroup of $\otimes Z_p$ is isomorphic to $\oplus Z_p$. Therefore, without loss of generality, we assume that $D = \oplus Z_p$, and thus $\hat{D} = \otimes Z_p$. We endow $\otimes Z_p$ with the discrete topology, and note that for any $\omega \in \otimes Z_p$, $\varphi(E)$ can be sup-norm partitioned if and only if $\varphi(E) - \omega$ can be so partitioned. Therefore, we may assume that the limit point of $\varphi(E)$ is 0.

Recalling that φ preserves sup-norm partitions in both directions, by the above lemma, we need to show that $\varphi(E)$ can be partitioned into finite, mutually disjoint and mutually independent blocks: Let

$$S_1 = \{\omega \in \varphi(E)\colon \omega(1) \neq 0\}$$

and, for any $N > 1$,

$$S_N = \{\omega \in \varphi(E) \setminus \bigcup_{j=1}^{N-1} S_j\colon \omega(N) \neq 0\}.$$

Clearly, $\{S_N\}_{N=1}$, so defined, satisfies our requirements. ■

Remark. The independence condition in Lemma 2.3 is sharp. A sequence of disjoint lacunary blocks of integers (degree of lacunarity as high as one wants), $\{F_j\}$, can be constructed in such a way that $\bigcup F_j$ cannot be sup-norm partitioned: Let $(\varepsilon_j)_{j=1}^\infty$ be a positive monotone sequence such that $\sum \varepsilon_j = 1$. Let $(h_j)_{j=1}^\infty$ be a sequence of trigonometric polynomials:

$$h_j(0) = 1 = \|h_j\|_\infty, \quad \text{and} \quad \sup_{x\notin(-\varepsilon_j, \varepsilon_j)} |h_j(x)| < \varepsilon_j.$$

We can multiply each h_j by a character without disturbing the above properties, and therefore may assume that for each j

$$\mathrm{Min}\{|n|\colon n \in \mathrm{spect}\, h_j\}/\mathrm{Max}\{|n|\colon n \in \mathrm{spect}\, h_{j-1}\}$$

is as large as we want. We first show that $\{\mathrm{spect}\, h_j\}_{j=1}^\infty$ is not a partition for $\bigcup_j \{\mathrm{spect}\, h_j\}$. Let $n_1 = 1$, and choose $\delta_1 > 0$ so that $|h_1 - h_2| < \varepsilon_2$ on $(-\delta_1, \delta_1)$. Proceeding by induction, we let $n_k > n_{k-1}+1$ be so that $(-\varepsilon_{n_k}, \varepsilon_{n_k}) \subset (-\delta_{k-1}, \delta_{k-1})$, and choose $\delta_k > 0$ so that $|h_{n_k} - h_{n_k+1}| < \varepsilon_{n_k+1}$ on $(-\delta_k, \delta_k)$. It follows that, for any $M > 0$, and any $x \in T$,

$$\Big|\sum_{j=1}^{M} (h_{n_j} - h_{n_j+1})(x)\Big| \leqslant 4.$$

If $\{E_j\}_{j=1}^\infty$ is a sup-norm partition for $\bigcup_j \mathrm{spect}\, h_j$, we observe that whenever (N_k) is a monotone sequence of integers tending to infinity $\{\bigcup_{j=N_k}^{N_{k+1}-1} E_j\}_{k=1}^\infty$ is also a partition for $\bigcup E_j$. We may therefore assume that $\mathrm{spect}\, h_{n_j} \subset E_j$, for an appropriate (n_j), and repeat the above argument to reach a contradiction.

3. Open questions

a. Are there sets satisfying the condition of Theorem B, which cannot be sup-norm partitioned?

b. In the above work, we deduced an interpolation property from a topological "thinness" property (in the Bohr compactification of Γ). Can an R-set, or a sup-norm partitioned set in Γ be dense in $\tilde{\Gamma}$?

c. Suppose that $E \subset Z$ satisfies the diophantine condition of Theorem B. It is an easy exercise to see that if F is any subset of E, then $\bar{F} \cap Z = F$ ($\bar{F}$ = closure of F in the Bohr compactification of Z). It then follows from a result of Y. Meyer (Théorème 2 of [6]) that E is a Riesz set, i.e., $L^1_E(T) = M_E(T)$.

(i) Does the above hold for $E \subset \Gamma$, and Γ any discrete abelian group?

(ii) Does there exist an R-set which is not a Riesz set?

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Every nuclear Fréchet space with a regular basis has the quasi-equivalence property

by

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Abstract. The following theorem is proved: *If X is a nuclear Fréchet space with a reqular basis (x_n) and if (y_n) is another basis for X, then the bases (x_n) and (y_n) are quasi-equivalent.*

M. M. Dragilev has shown in [3] that nuclear Fréchet spaces in the classes (d_1) and (d_2) have the quasi-equivalence property. His results and techniques were reformulated and extended by C. Bessaga in [1]. B. S. Mitiagin has shown in [4] that nuclear centers of Hilbert scales have the quasi-equivalence property, and V. P. Zaharjuta extended this in [7] by replacing the hypothesis of nuclearity with the Schwartz condition, and finally Mitiagin [9] established this property for the centers of arbitrary Hilbert scales. Also Zaharjuta recently obtained the quasi-equivalence property for spaces which are products of a (d_1) and (d_2) space in [8]. However, the general problem of quasi-equivalence for nuclear Fréchet spaces remains.

In this paper we prove that any nuclear Köthe space with a regular basis has the quasi-equivalence property. The essential idea of the proof is that the diametral dimension $\delta(E)$ (as defined in [2]) distinguishes regular bases.

1. Definitions. For two sequences a and b, $a\cdot b$ will denote the sequence $(a_n b_n)$, and if B is a collection of sequences, $a\cdot B = \{a\cdot b\colon b\in B\}$. A *Köthe space* is the Fréchet space of sequences

$$\lambda = \bigcap \frac{1}{a^k}\cdot l_1 = \Big\{t\colon \forall k,\ \|t\|_k = \sum_{n=1}^{\infty} |t_n|\, a_n^k < +\infty\Big\},$$

with the topology generated by the norms $\|\cdot\|_k$, $k = 1, 2, \ldots$ We assume that for all k, n, $0 < a_n^k \leqslant a_n^{k+1}$. It is known that λ is nuclear if and only if for all k there exists m such that $\sum_n (a_n^k/a_n^m) < +\infty$, and that λ is a Schwartz space if and only if for all k there exists m such that $a_n^k/a_n^m \to 0$. If λ is