Clearly, $A \in \mathscr{B}_0$. Using (1), we define inductively an increasing sequence of the indices (n_i) so that for every sequence of signs (a_i)

$$P\big(A \cap (a_1 \varepsilon_{n_1} = 1)\big) > 2^{-2},$$

$$(2) \quad P\Big(A \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big)$$
$$> 2^{-1}\Big(2^{-k} P^{-1}\Big(A \cap \bigcap_{i=1}^{k-1} (a_i \varepsilon_{n_i} = 1)\Big)\Big) P\Big(A \cap \bigcap_{i=1}^{k-1} (a_i \varepsilon_{n_i} = 1)\Big) = 2^{-k-1}$$
$$\text{for } k = 1, 2, \ldots$$

Let us put $\varepsilon_i' = \varepsilon_i$ for $i = n_j$ $(j = 1, 2, \ldots)$ and $\varepsilon_i' = -\varepsilon_i$ otherwise. Let

$$A' = \Big\{\omega \in \Omega \colon \sup_n \Big\|\sum_{i=1}^{n} \varepsilon_i'(\omega) x_i\Big\| < M\Big\}.$$

Since the sequences (ε_i) and (ε_i') are equidistributed, it follows from (2) that

$$(3) \qquad P\Big(A \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big) = P\Big(A' \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i}' = 1)\Big) > 2^{-k-1}$$

for every sequence of signs (a_i) and for $k = 1, 2, \ldots$ Fix now k and the signs $a_1, a_2, \ldots, a_k$. Since $P\Big(\bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big) = 2^{-k}$, it follows from (2) and (3) that there exists an $\omega \in \Omega$ which belongs to the intersection $A \cap A' \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)$. Thus

$$\Big\|\sum_{i=1}^{k} a_i x_{n_i}\Big\| = \Big\|2^{-1}\Big(\sum_{j=1}^{n_k} \varepsilon_j(\omega) x_j + \sum_{j=1}^{n_k} \varepsilon_j'(\omega) x_j\Big)\Big\| < M.$$

Since the positive integer k and the signs $a_1, a_2, \ldots, a_k$ have been fixed arbitrary, the last inequality implies that the series $\sum_{i=1}^{n} x_{n_i}$ is weakly unconditionally convergent, while $\inf \|x_{n_i}\| > 0$. Thus, by a result of [1], E contains a subspace isomorphic to c_0.

This completes the proof of the Proposition.

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Projections on Banach spaces with symmetric bases

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Abstract. Let P be a bounded linear projection on a Banach space X with a symmetric basis. Then either PX or $(I-P)X$ contains a subspace E which is complemented in X and such that E is isomorphic to X. As a consequence, Pełczyński's complementably universal space for all Banach spaces with unconditional bases is primary.

A Banach space X is called *prime* (resp., *primary*) if for every bounded linear projection P on X with $\dim PX = \infty$, PX (resp., PX or $(I-P)X$) is isomorphic to X. Clearly, every prime space is primary. It is well known that c_0 and l_p, $1 \leqslant p \leqslant \infty$, are prime spaces ([5], [10]). However, it is an open question whether there are other prime Banach spaces. Recently, Lindenstrauss and Pełczyński ([8]) have shown that $C[0,1]$ is primary. For other information on prime and primary Banach spaces, we refer the reader to [6].

A basis $\{x_n\}$ in a Banach space X is called *symmetric* if every permutation $\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a basis of X, equivalent to $\{x_n\}$. It is well known that in c_0 and l_p, $1 \leqslant p < \infty$, all symmetric basic sequences are equivalent (cf. [1]). There are two important classes of Banach spaces with symmetric bases: the Orlicz sequence spaces ([9]) and the Lorentz sequence spaces ([1], [3]). In this note we show that if P is a bounded linear projection on a Banach space X with a symmetric basis, then either PX or $(I-P)X$ contains a subspace E which is complemented in X and is isomorphic to X. As a consequence, Pełczyński's complementably universal space ([11]) for all Banach spaces with unconditional bases is primary. This indicates that it is conceivable that all Banach spaces with symmetric bases are primary and strengthens the conjecture of Lindenstrauss and Tzafriri ([9]) that there are new examples of prime Banach spaces among the minimal Orlicz sequence spaces.

Let X be a Banach space with a symmetric basis. Then there exists an equivalent symmetric norm (cf. [12]) in X. Throughout this paper, we shall assume that X is equipped with the symmetric norm and that every projection is a bounded linear projection. We shall write $X \sim Y$ (resp., $\{x_n\} \sim \{y_n\}$) to mean that X is isomorphic to Y (resp., $\{x_n\}$ is equivalent to $\{y_n\}$). For the terminology on bases, we follow Singer's book [12].

THEOREM. *Let X be a Banach space with a symmetric basis $\{x_n\}$. If P is a projection on X, then either PX or $(I-P)X$ contains a subspace E which is isomorphic to X and complemented in X.*

Proof. Since the space l_1 is prime, it remains to prove the theorem when X is not isomorphic to l_1. We may assume that $\|x_n\| = 1$, $n = 1, 2, \ldots$ Let $P(x_n) = \sum_{i=1}^{\infty} a_{n,i}x_i$, $n = 1, 2, \ldots$ We claim that $\lim_{n\to\infty} a_{n,i} = 0$ for each $i = 1, 2, \ldots$ If not, then there exist i_0 and $\delta > 0$ such that $|a_{n_k,i_0}| \geqslant \delta$ for some $n_1 < n_2 < \ldots$ For any $\sum_{n=1}^{\infty} a_n x_n \in X$, then $x = \sum_{i=1}^{\infty}\big(\sum_{k=1}^{\infty} \operatorname{sgn} a_{n_k,i_0}|a_k|\, a_{n_k,i}\big)x_i$ is convergent in X and

$$P(x) = \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{\infty} \operatorname{sgn} a_{n_k,i_0}|a_k|\, a_{n_k,i}\Big)x_i.$$

Hence

$$\|P(x)\| \geqslant \sum_{k=1}^{\infty} \operatorname{sgn} a_{n_k,i_0}|a_k|\, a_{n_k,i_0} = \sum_{k=1}^{\infty} |a_k|\, |a_{n_k,i_0}| \geqslant \delta \sum_{k=1}^{\infty} |a_k|.$$

Therefore $\{x_n\}$ is equivalent to the unit vector basis of l_1, which is a contradiction. Now we consider two cases.

Case I. There exists $\varepsilon > 0$ such that $\inf_{1\leqslant j<\infty}\sup_{1\leqslant i<\infty} |a_{n_j,i}| \geqslant \varepsilon$ for some $n_1 < n_2 < \ldots$ Then $\inf_j \|P(x_{n_j})\| \geqslant \inf_j \sup_i |a_{n_j,i}| \geqslant \varepsilon$ and $\lim_{j\to\infty} f_i(Px_{n_j}) = \lim_{j\to\infty} a_{n_j,i} = 0$, $i = 1, 2, \ldots$, where $\{f_i\}$ is the sequence of biorthogonal functionals of $\{x_i\}$. By Theorem 3, [2], there exist $p_1 < p_2 < \ldots$ and a subsequence of $\{P(x_{n_j})\}$, call it $\{P(x_{n_j})\}$ again, such that

$$\sum_{j=1}^{\infty} \|y_j - P(x_{n_j})\| < \tfrac{1}{2}\varepsilon,$$

where $\{y_j = \sum_{i=p_j+1}^{p_{j+1}} a_{n_j,i}x_i\}$ is equivalent to $\{P(x_{n_j})\}$. Since $\sup_{1\leqslant i<\infty} |a_{n_j,i}| \geqslant \varepsilon$ and $\|y_j - P(x_{n_j})\| < \frac{1}{2}\varepsilon$, it follows that

$$\sup_{p_j+1\leqslant i\leqslant p_{j+1}} |a_{n_j,i}| \geqslant \tfrac{1}{2}\varepsilon \quad \text{for each } j = 1, 2, \ldots$$

Hence $\{y_j\}$ is a bounded block basic sequence of $\{x_n\}$ and by Proposition 4; [1], $\{y_j\}$ dominates $\{x_n\}$. On the other hand, if $\sum_{n=1}^{\infty} a_n x_n$ is convergent in X then $\sum_{j=1}^{\infty} a_j x_{n_j}$ converges. Hence $\sum_{j=1}^{\infty} a_j P(x_{n_j})$ converges and therefore $\sum_{j=1}^{\infty} a_j y_j$ is convergent. Thus $\{x_n\}$ dominates $\{y_j\}$ and so $\{x_n\} \sim \{y_j\}$. Now, by Remark 1, [3], there exists a projection Q from X onto $[y_j]$, the closed linear subspace in X spanned by $\{y_j\}$. We define $Q\colon X \to [y_j]$ by

$$Q\Big(\sum_{n=1}^{\infty} b_n x_n\Big) = \sum_{n=1}^{\infty} \frac{b_{i_n}}{a_{i_n}} y_n$$

where $p_n + 1 \leqslant i_n \leqslant p_{n+1}$ have been chosen to satisfy $|a_{i_n}| \geqslant \frac{1}{2}\varepsilon$ for $n = 1, 2, \ldots$ By Theorem 2, [2], and by switching to a subsequence if necessary, we conclude that $[P(x_{n_j})]$ is complemented in X. Finally, since $\{P(x_{n_j})\} \sim \{y_j\} \sim \{x_n\}$, $[P(x_{n_j})]$ is isomorphic to X. Thus, in this case, PX contains a subspace which is isomorphic to X and complemented in X.

Case II. If $\inf_{1\leqslant n<\infty}\sup_{1\leqslant i<\infty} |a_{n,i}| = 0$, then since

$$(I-P)x_n = \sum_{\substack{i=1\\ i\neq n}}^{\infty} a_{n,i}x_i + (1-a_{n,n})x_n, \qquad n = 1, 2, \ldots,$$

there exist n_0 and $\varepsilon > 0$ such that $|1-a_{n,n}| \geqslant \varepsilon$ for all $n \geqslant n_0$. Now, proceeding as in Case I, we conclude that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $[(I-P)x_{n_j}]$ is isomorphic to X and complemented in X. Q.E.D.

Remark 1. If P is a projection on a Banach space X with a symmetric basis, then PX need not have a subspace which is isomorphic to X. For example, it is known ([3]) that in every infinite-dimensional subspace of a Lorentz sequence space $d(a, p)$ there is a subspace which is isomorphic to l_p and is complemented in $d(a, p)$ while $d(a, p)$ is never isomorphic to a subspace of l_p.

COROLLARY 1. *Let X be a Banach space with a symmetric basis. If P is a projection on X such that either PX or $(I-P)X$ is isomorphic to its Cartesian square then either PX or $(I-P)X$ is isomorphic to X. Therefore, if PX has symmetric basis then either PX or $(I-P)X$ is isomorphic to X.*

Proof. We may assume that there is a subspace W in PX such that $PX \sim X \oplus W$.

Case I. $PX \sim PX \oplus PX$. Then $X \sim PX \oplus (I-P)X \sim PX \oplus PX \oplus (I-P)X \sim PX \oplus X \sim X \oplus W \oplus X \sim PX$.

Case II. $(I-P)X \sim (I-P)X \oplus (I-P)X$. Then $PX \sim X \oplus W \sim X \oplus \oplus X \oplus W \sim PX \oplus PX \oplus (I-P)X \oplus (I-P)X \oplus W \sim PX \oplus PX \oplus (I-P)X \oplus W \sim PX \oplus X \oplus W \sim PX \oplus PX$. Hence, by Case I, $PX \sim X$. Q.E.D.

Remark 2. There exists a projection P on a Banach space X with a symmetric basis such that PX is not isomorphic to $PX \oplus PX$. Indeed, if E is a Banach space with unconditional basis such that E is not isomorphic to $E \oplus E$ ([4]), then by [7] there exists a Banach space X with a symmetric basis such that E is complemented in X.

COROLLARY 2. *Let X be a Banach space with a symmetric basis. If $X \sim c_0 \oplus Y$ or $X \sim l_p \oplus Y$, $1 \leqslant p < \infty$ then $X \sim Y$.*

Proof. This follows immediately from Corollary 1 and the fact ([10]) that if Y is a complemented subspaces in c_0 or l_p, $1 \leqslant p < \infty$, then Y is isomorphic to c_0 or l_p, $1 \leqslant p < \infty$. Q.E.D.

COROLLARY 3. *Let X be a Banach space with a symmetric basis. Then X is primary if and only if for every projection P on X, X is isomorphic to $X \oplus PX$.*

Proof. If X is primary and P is projection on X then either $X \sim PX \sim X \oplus X \sim X \oplus PX$ or $X \sim (I-P)X \sim PX \oplus (I-P)X \sim PX \oplus X$. Conversely, suppose $X \sim X \oplus PX$ for every projection P on X. If $PX \sim X \oplus W$, then $PX \sim X \oplus X \oplus W \sim X \oplus PX \sim X$. Similarly, if $(I-P)X$ contains a subspace which is isomorphic to X and complemented in X then $(I-P)X \sim X$. Thus X is primary. Q.E.D.

Remark 3. By Corollary 1, it is easily seen that there exists a Banach space with symmetric basis which is not primary if and only if there exist Banach spaces X and Y which do not have symmetric bases such that $X \oplus Y$ has a symmetric basis. Notice that if X and Y are Banach spaces with symmetric bases such that X is not isomorphic to a complemented subspace of Y and Y is not isomorphic to a complemented subspace of X then $X \oplus Y$ does not have a symmetric basis. For examples, let $d(a, p)$ be a Lorentz sequence space. If $1 \leqslant p < q < \infty$, then $l_p \oplus l_q$, $l_p \oplus d(a, q)$ and $d(a, p) \oplus d(a, q)$ do not have symmetric bases.

COROLLARY 4. *Let X be a Banach space with a symmetric basis $\{x_n\}$ and let*

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} x_i \Big/ \Big\| \sum_{i=p_n+1}^{p_{n+1}} x_i \Big\|, \quad n = 1, 2, \ldots,$$

be a block basic sequence of $\{x_n\}$. If P is a projection from X onto $[y_n]$, then X is isomorphic either to $[y_n]$ or $(I-P)X$.

Proof. Let N be the set of natural numbers and let $N = \bigcup_{i=1}^{\infty} N_i$, $N_i \wedge N_j = \emptyset$ for all $i \neq j$ and $\overline{\overline{N}} = \overline{\overline{N_i}}$, $i = 1, 2, \ldots$ For each $N_i = \{(i, j)\}_{j=1,2,\ldots}$, let

$$z_{i,j} = \sum_{k=p_j+1}^{p_{j+1}} x_{i,k} \Big/ \Big\| \sum_{k=p_j+1}^{p_{j+1}} x_{i,k} \Big\|, \quad j = 1, 2, \ldots$$

Since $\{x_n\}$ is symmetric, for each $i = 1, 2, \ldots$, $\{z_{i,j}\}_{j=1,2,\ldots} \sim \{y_j\}$. Let $Z = [z_{i,j}]_{i,j=1,2,\ldots}$. Then Z is complemented in X (cf. [12], p. 588). It is easy to show that $Z \sim Z \oplus Z \sim Z \oplus [y_n]$. Now $[z_{i,1}]_{i=1,2,\ldots}$ is complemented in Z and, by [1], Proposition 3, $\{z_{i,1}\} \sim \{x_n\}$. Hence Z contains a complemented subspace which is isomorphic to X. By Corollary 1, we conclude that $Z \sim X$. Hence $X \sim Z \sim Z \oplus [y_n] \sim X \oplus [y_n]$. Thus, by Corollary 3, X is isomorphic either to $[y_n]$ or to $(I-P)X$. Q.E.D.

COROLLARY 5. *Let X be a Banach space with a symmetric basis. If X is isomorphic to $(X \oplus X \oplus \ldots)_E$ where E is one of the spaces c_0 or l_p, $1 \leqslant p < \infty$, then X is primary.*

Proof. Let P be a projection on X. By the standard decomposition method of Pełczyński [10],

$$X \sim (X \oplus X \oplus \ldots)_E \sim (PX \oplus PX \oplus \ldots)_E \oplus \big((I-P)X \oplus (I-P)X \oplus \ldots\big)_E$$
$$\sim PX \oplus (PX \oplus PX \oplus \ldots)_E \oplus \big((I-P)X \oplus (I-P)X \oplus \ldots\big)_E \sim PX \oplus X.$$

By Corollary 3, we conclude that X is primary. Q.E.D.

COROLLARY 6. *The Pełczyński's complementably universal space U for the family of all Banach spaces with unconditional bases is primary.*

Proof. By [7], U has a symmetric basis and by [12], p. 550, $U \sim (U \oplus U \oplus \ldots)_{l_p}$, $1 \leqslant p < \infty$. Hence by Corollary 4, U is primary. Q.E.D.

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