[10] G. Köthe, *Topological vector spaces*, Berlin–Heidelberg–New York, 1969.

[11] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector-valued coefficients*, to appear.

[12] M. Loève, *Probability theory*, Princeton, New Jersey–Toronto–New York–London, 1963 (3rd edition).

[13] G. Nordlander, *On sign-independent and almost sign-independent convergence in normed linear spaces*, Arkiv för Mat. 4, 21 (1961), pp. 287–296.

[14] Y. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i. Primenen 1 (1956), pp. 177–238.

[15] B. Rosen, *On the asymptotic distribution of sums of independent identically distributed random variables*, Arkiv för Mat. 24 (1961), pp. 323–332.

Received October 2, 1973 (740)

STUDIA MATHEMATICA, T. LII. (1974)

On Banach spaces containing c_0

A supplement to the paper by J. Hoffmann-Jørgensen "Sums of independent Banach space valued random variables"

by

S. KWAPIEŃ (Warszawa)

Abstract. It is proved that a Banach space E does not contain subspaces isomorphic to c_0 if and only if the almost surely boundedness of sums of independent, symmetric E-valued random variables implies the almost surely convergence of the sums.

We shall prove the following result conjectured by Hoffmann-Jørgensen in the preceding paper [2].

THEOREM. *For every Banach space E the following conditions are equivalent*:

(i) *E does not contain subspaces isomorphic to the space c_0 of all scalar-valued sequences convergent to zero,*

(ii) *$L_1(E)$ does not contain subspaces isomorphic to c_0,*

(iii) *the almost surely boundedness of sums of independent, symmetric, E-valued random variables implies the almost surely convergence of the sums.*

It has been proved in [2] that to prove the theorem it is enough to establish the following

PROPOSITION. *Let E be a Banach space. Let (ε_i) be a Bernouilli sequence on a probability space $(\Omega, \mathscr{B}, P)$, i.e., a sequence of independent random variables such that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$ for all i. Let (x_i) be a sequence in E, such that $\inf_i \|x_i\| > 0$ and $P\left(\sup_n \left\|\sum_{i=1}^n \varepsilon_i x_i\right\| = \infty\right) = 0$. Then E contains a subspace isomorphic to c_0.*

Proof. Let $\mathscr{B}_0$ denote the σ-subfield of $\mathscr{B}$ generated by all ε_i's. Then it is easy to see that for $B \in \mathscr{B}_0$

(1) $$\lim_i P\big(B \cap (\varepsilon_i = 1)\big) = \lim_i P\big(B \cap (\varepsilon_i = -1)\big) = 2^{-1} P(B).$$

Pick $M < \infty$ so that $P(A) > 2^{-1}$ where

$$A = \left\{\omega \in \Omega\colon \sup_n \left\|\sum_{i=1}^n \varepsilon_i(\omega) x_i\right\| < M\right\}.$$

Clearly, $A \in \mathscr{B}_0$. Using (1), we define inductively an increasing sequence of the indices (n_i) so that for every sequence of signs (a_i)

$$P\big(A \cap (a_1 \varepsilon_{n_1} = 1)\big) > 2^{-2},$$

$$(2)\qquad P\Big(A \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big)$$
$$> 2^{-1}\Big(2^{-k} P^{-1}\Big(A \cap \bigcap_{i=1}^{k-1} (a_i \varepsilon_{n_i} = 1)\Big)\Big) P\Big(A \cap \bigcap_{i=1}^{k-1} (a_i \varepsilon_{n_i} = 1)\Big) = 2^{-k-1}$$
$$\text{for } k = 1, 2, \ldots$$

Let us put $\varepsilon_i' = \varepsilon_i$ for $i = n_j$ $(j = 1, 2, \ldots)$ and $\varepsilon_i' = -\varepsilon_i$ otherwise. Let

$$A' = \Big\{\omega \in \Omega \colon \sup_n \Big\| \sum_{i=1}^{n} \varepsilon_i'(\omega) x_i \Big\| < M\Big\}.$$

Since the sequences (ε_i) and (ε_i') are equidistributed, it follows from (2) that

$$(3)\qquad P\Big(A \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big) = P\Big(A' \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i}' = 1)\Big) > 2^{-k-1}$$

for every sequence of signs (a_i) and for $k = 1, 2, \ldots$ Fix now k and the signs $a_1, a_2, \ldots, a_k$. Since $P\Big(\bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)\Big) = 2^{-k}$, it follows from (2) and (3) that there exists an $\omega \in \Omega$ which belongs to the intersection $A \cap A' \cap \bigcap_{i=1}^{k} (a_i \varepsilon_{n_i} = 1)$. Thus

$$\Big\| \sum_{i=1}^{k} a_i x_{n_i} \Big\| = \Big\| 2^{-1} \Big(\sum_{j=1}^{n_k} \varepsilon_j(\omega) x_j + \sum_{j=1}^{n_k} \varepsilon_j'(\omega) x_j \Big) \Big\| < M.$$

Since the positive integer k and the signs $a_1, a_2, \ldots, a_k$ have been fixed arbitrary, the last inequality implies that the series $\sum_{i=1}^{n} x_{n_i}$ is weakly unconditionally convergent, while $\inf \|x_{n_i}\| > 0$. Thus, by a result of [1], E contains a subspace isomorphic to c_0.

This completes the proof of the Proposition.

References

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151–164.

[2] J. Hoffmann-Jørgensen, Sums of independent Banach space valued random variables, ibidem 52 (1974), pp. 159–186.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WARSAW

Received December 9, 1973 (767)

STUDIA MATHEMATICA, T. LII. (1974)

Projections on Banach spaces with symmetric bases

by

P. G. CASAZZA (Huntsville, Alabama) and BOR-LUH LIN (Iowa City, Iowa)

Abstract. Let P be a bounded linear projection on a Banach space X with a symmetric basis. Then either PX or $(I-P)X$ contains a subspace E which is complemented in X and such that E is isomorphic to X. As a consequence, Pełczyński's complementably universal space for all Banach spaces with unconditional bases is primary.

A Banach space X is called *prime* (resp., *primary*) if for every bounded linear projection P on X with $\dim PX = \infty$, PX (resp., PX or $(I-P)X$) is isomorphic to X. Clearly, every prime space is primary. It is well known that c_0 and l_p, $1 \leqslant p \leqslant \infty$, are prime spaces ([5], [10]). However, it is an open question whether there are other prime Banach spaces. Recently, Lindenstrauss and Pełczyński ([8]) have shown that $C[0,1]$ is primary. For other information on prime and primary Banach spaces, we refer the reader to [6].

A basis $\{x_n\}$ in a Banach space X is called *symmetric* if every permutation $\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a basis of X, equivalent to $\{x_n\}$. It is well known that in c_0 and l_p, $1 \leqslant p < \infty$, all symmetric basic sequences are equivalent (cf. [1]). There are two important classes of Banach spaces with symmetric bases: the Orlicz sequence spaces ([9]) and the Lorentz sequence spaces ([1], [3]). In this note we show that if P is a bounded linear projection on a Banach space X with a symmetric basis, then either PX or $(I-P)X$ contains a subspace E which is complemented in X and is isomorphic to X. As a consequence, Pełczyński's complementably universal space ([11]) for all Banach spaces with unconditional bases is primary. This indicates that it is conceivable that all Banach spaces with symmetric bases are primary and strengthens the conjecture of Lindenstrauss and Tzafriri ([9]) that there are new examples of prime Banach spaces among the minimal Orlicz sequence spaces.

Let X be a Banach space with a symmetric basis. Then there exists an equivalent symmetric norm (cf. [12]) in X. Throughout this paper, we shall assume that X is equipped with the symmetric norm and that every projection is a bounded linear projection. We shall write $X \sim Y$ (resp., $\{x_n\} \sim \{y_n\}$) to mean that X is isomorphic to Y (resp., $\{x_n\}$ is